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THE STRUCTURE OF A SPECIAL CLASS OF WEIGHTED TRANSLATION SEMIGROUPS

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A special class of weighted translation semigroups $\{S_t\}$ on $\mathscr{L}^2(\mathscr{R}_+)$ is studied. The weakly closed algebra $\mathscr{A}$ generated by the semigroup is maximal abelian and the spectra of elements of $\mathscr{A}$ are studied. It is shown that each densely defined linear transformation commuting with $\mathscr{A}$ is closable and that every transitive algebra containing $\mathscr{A}$ is weakly dense in the full algebra of operators on $L^2(\mathscr{R}_+)$.

1. Introduction. A weighted translation semigroup $\{S_t\}$ with symbol ϕ is defined on $L^2(\mathscr{R}_+)$ by

$$(S_tf)(x)=\begin{cases}\dfrac{\phi(x)}{\phi(x-t)}f(x-t) & \text{for} \quad 0\leqq t\leqq x\\ 0 & \text{for} \quad t>x\end{cases}$$

where ϕ is a continuous, complex-valued function on $\mathscr{R}_+$ such that $\phi(x)\neq 0$ for x in $\mathscr{R}_+$. These semigroups were studied in [2] and [3]. In [3] strongly continuous subnormal weighted translation semigroups are characterized as those for which ϕ^2 is a Laplace–Stieltjes Transform of a probability measure. In [4] a more general type of weighted translation semigroup is studied.

To insure the strong continuity of $\{S_t\}$ we assume that $\sup_{x\in\mathscr{R}_+}|\phi(x+t)/\phi(x)|\leqq Me^{wt}$ for all t and some constants M and w [2, Lemma 2.1]. Two weighted translation semigroups with symbols ϕ and ρ are unitarily equivalent if and only if $|\phi/\rho|$ is constant [2, Theorem 2.5]. Thus without loss of generality we assume that ϕ is positive-valued and that $\phi(0)=1$.

Throughout the paper unless otherwise indicated we shall assume further that $\int_0^x(\phi(x)/\phi(t)\phi(x-t))^2dt$ is bounded and shall say that ϕ is of *bounded kernel type*. For such a ϕ and for each f in $L^2(\mathscr{R}_+)$ we define

$$A_f=\int_0^\infty\frac{f(t)}{\phi(t)}S_tdt. \tag{1}$$

In §2 we show that $\{A_f: f \in L^2(\mathcal{R}_+)\}$ is a subalgebra of $B(L^2)$, the full algebra of operators on $L^2(\mathcal{R}_+)$. We denote $\{A_f: f \in L^2(\mathcal{R}_+)\}$ by $\mathcal{A}_0$ and its closure in the weak operator topology by $\mathcal{A}$. In Theorem 2.6 we show that $\mathcal{A}$ is a maximal abelian algebra and that $\mathcal{A}_0$ is a proper ideal of $\mathcal{A}$.

In §3 we establish a basic relation between the multiplicative linear functionals on $\mathcal{A}$ and the elements of $L^2(\mathcal{R}_+)$ of the form $e^{\lambda t}/\phi(t)$. This relation enables us in Theorem 3.5 to determine completely the spectrum of each element of $\mathcal{A}_0$. In §4 it is shown that any densely defined linear transformation commuting with $\mathcal{A}$ is closable. This result enables us to apply Arveson's Density Theorem to show that if $e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)$ for some λ, then any transitive subalgebra of $B(L^2)$ which contains $\mathcal{A}$ is weakly dense in $B(L^2)$. Finally in §5 certain function theoretic considerations related to ϕ are investigated. It is shown in Corollary 5.5 that if the associated semigroup is hyponormal then ϕ is not of bounded kernel type.

Throughout the paper the following notation is used: $H = \{\lambda : e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)\}$, $E = \{g : g(t) = e^{\lambda t}/\phi(t), \lambda \in H\}$ and $\alpha(\phi) = \sup\{\operatorname{Re}\lambda : \lambda \in H\}$ where $\alpha(\phi) = -\infty$ if H is empty. G will denote the infinitesimal generator of the semigroup $\{S_t\}$.

2. Basic facts about $\mathcal{A}$.

In this section we shall show that each A_f is a bounded linear operator on $L^2(\mathcal{R}_+)$, that the mapping $f \to A_f$ of $L^2(\mathcal{R}_+)$ onto $\mathcal{A}_0$ is a continuous linear mapping, that $\mathcal{A}_0$ is an algebra, and that $\mathcal{A}$ is a maximal abelian algebra.

LEMMA 2.1. *$\|A_f\| \leqq \rho \|f\|$ for all f in $L^2(\mathcal{R}_+)$ where*

$$\rho = \sup_x \int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)}\right)^2 dt.$$

Proof. Let $g \in L^2(\mathcal{R}_+)$. To see that A_f is well-defined we note that

$$(2)\qquad \begin{aligned}(A_f g)(x) &= \int_0^\infty \frac{f(t)}{\phi(t)} (S_t g)(x) dt \\ &= \int_0^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) g(x-t) dt\end{aligned}$$

and the integral exists since f and g are square integrable and ϕ is continuous and nonzero. We note further that

$$|(A_fg)(x)|^2 \leqq \int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)}\right)^2 dt \int_0^x |f(t)g(x-t)|^2 dt.$$

Therefore

$$\|A_fg\|^2 \leqq \rho^2 \int_0^\infty \int_0^x |f(t)g(x-t)|^2 dt dx = \rho^2 \|f\|^2 \|g\|^2$$

so that A_f is a bounded linear operator on $L^2(\mathcal{R}_+)$ and $\|A_f\| \leqq \rho \|f\|$.

LEMMA 2.2. (i) $A_{\alpha f+\beta g} = \alpha A_f + \beta A_g$,
(ii) $A_f g = A_g f$, *and*
(iii) $A_f A_g = A_{A_f g}$
for all f *and* g *in* $L^2(\mathcal{R}_+)$ *and all complex numbers* α *and* β.

Proof. (i) and (ii) follow immediately from equation (2). To prove (iii) we let $f, g, h \in L^2(\mathcal{R}_+)$ and note that

$$
\begin{aligned}
(A_fA_gh)(x) &= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t)(A_gh)(x-t)dt \\
&= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) \int_{s=0}^{x-t} \frac{g(s)\phi(x-t)}{\phi(s)\phi(x-t-s)} h(x-t-s)dsdt \\
&= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) \int_{s=t}^{x} \frac{g(s-t)\phi(x-t)}{\phi(s-t)\phi(x-s)} h(x-s)dsdt \\
&= \int_{s=0}^x \left[\int_{t=0}^s \frac{\phi(x)}{\phi(t)\phi(s-t)\phi(x-s)} f(t)g(s-t)h(x-s)dt\right] ds \\
&= \int_{s=0}^x \frac{\phi(x)}{\phi(s)\phi(x-s)} \left[\int_{t=0}^s \frac{\phi(s)}{\phi(t)\phi(s-t)} f(t)g(s-t)dt\right] h(x-s)ds \\
&= \int_{s=0}^x \frac{\phi(x)}{\phi(s)\phi(x-s)} (A_fg)(s)h(x-s)ds \\
&= (A_{A_fg}h)(x).
\end{aligned}
$$

Thus (iii) holds for all f and g.

It now follows immediately from Lemma 2.2 that $\mathcal{A}_0$ is a commutative algebra and from Lemma 2.1 that the mapping $f \to A_f$ is continuous. An easy computation shows that this mapping is one-to-one. We state these results in the following theorem.

THEOREM 2.3. *$\mathcal{A}_0$ is a commutative algebra of operators on $L^2(\mathcal{R}_+)$ and the mapping $f \to A_f$ is a continuous, one-to-one, linear mapping of $L^2(\mathcal{R}_+)$ onto $\mathcal{A}_0$.*

It follows from the Open Mapping Theorem and Theorem 2.3 that $\mathcal{A}_0$ is closed in the uniform topology if and only if the mapping $f \to \mathcal{A}_f$ is bicontinuous. It is an open question whether or not $\mathcal{A}_0$ is closed in the uniform operator topology.

LEMMA 2.4. *If T is an operator on $L^2(\mathcal{R}_+)$ which is in the commutant of $\mathcal{A}_0$, then $TA_f = A_{Tf}$ for each f in $L^2(\mathcal{R}_+)$.*

Proof. Let f and g be elements of $L^2(\mathcal{R}_+)$. Then $TA_fg = TA_gf = A_gTf = A_{Tf}g$. Consequently $TA_f = A_{Tf}$ as desired.

LEMMA 2.5. $\{S_t\} \subset \mathcal{A} - \mathcal{A}_0$.

Proof. Let $f_n = n\phi\psi[r, r+1/n]$ where $\psi[a, b]$ is the characteristic function of $[a, b]$. Then $f_n \in L^2(\mathcal{R}_+)$ and $A_{f_n} = n\int_r^{r+1/n} S_t dt$ which, because of the strong continuity of S_t, converges strongly to S_r. Consequently $S_r \in \mathcal{A}$. To see that $S_r \notin \mathcal{A}_0$ we assume the contrary: $S_r = \int_0^\infty (f(t)/\phi(t))S_t dt$ for some f in $L^2(\mathcal{R}_+)$. Consequently, $S_r^*(g/\phi) = \int_0^\infty (\overline{f(t)}/\phi(t))S_t^*(g/\phi)dt$ for each g in $L^2(\mathcal{R}_+)$ of compact support. Thus $g(x+r) = \int_0^\infty (\overline{f(t)}/\phi(t))g(x+t)dt$. If we define $K(y, s) = \overline{(f(s-y+r)/\phi(s-y+r))}$ for $y \geqq r$ and 0 otherwise, we arrive at the integral equation $g(y) = \int_0^\infty K(y, s)g(s)ds$. Since the identity is not an integral operator on $L^2(\mathcal{R}_+)$ [5, p. 87], we arrive at a contradiction and our proof is complete.

An immediate consequence of Lemma 2.5 is that the weakly closed algebra $\mathcal{A}_1$ generated by $\{S_t\}$ is a subalgebra of $\mathcal{A}$. Since each element of $\mathcal{A}_0$ is obviously in $\mathcal{A}_1$, we see that $\mathcal{A}_1 = \mathcal{A}$; that is, the weakly closed algebra generated by the semigroup $\{S_t\}$ is the same as the weakly closed algebra generated by $\{A_f\}$.

THEOREM 2.6. *$\mathcal{A}$ is a maximal abelain algebra and $\mathcal{A}_0$ is a proper ideal of $\mathcal{A}$.*

Proof. That $\mathcal{A}$ is abelian follows from the fact that $\mathcal{A}_0$ is abelian. Thus by Lemma 2.4 if $T \in \mathcal{A}$, then $TA_f = A_{Tf} \in \mathcal{A}_0$, proving that $\mathcal{A}_0$ is an ideal of $\mathcal{A}$. Lemma 2.5 assures us that $\mathcal{A}_0$ is proper and that $I \in \mathcal{A}$. Choose a net g_λ such that A_{g_λ} converges weakly to the identity operator I. Then since $TA_{g_\lambda} = A_{Tg_\lambda}$, we have $T = \lim A_{Tg_\lambda}$ and hence $T \in \mathcal{A}$, proving that each element of the commutant of $\mathcal{A}$ is an element of $\mathcal{A}$. The proof that $\mathcal{A}$ is maximal abelian is complete.

We observe that no A_f is invertible since $\mathcal{A}_0$ is a proper ideal of the maximal abelian algebra $\mathcal{A}$. We shall study in more detail the spectral properties of elements of $\mathcal{A}_0$ in the next section.

3. Spectral properties of $\mathcal{A}$.

In this section we first characterize certain multiplicative linear functionals on $\mathcal{A}$ and then use this information to study the spectra of elements of $\mathcal{A}$. In particular we are able to show in Corollary 3.4 that whenever $g_\lambda \in L^2(\mathcal{R}_+)$ where $g_\lambda(x) = e^{\lambda x}/\phi(x)$, then g is an eigenvector for each element of $\mathcal{A}^*$. For an element A_f of $\mathcal{A}_0$ we then show in Theorem 3.5 that the eigenvalues corresponding to the g_λ together with the real number 0 make up the entire spectrum of A_f^*.

THEOREM 3.1. *If m is a multiplicative linear functional on $\mathcal{A}$, then there exists a unique g in $L^2(\mathcal{R}_+)$ such that*

(i) $m(A_f) = \langle f, g\rangle$ *and*

(ii) $A_f^*g = \langle g, f\rangle g$ *for all f in $L^2(\mathcal{R}_+)$.*

Conversely, if m and g satisfy (i) *and* (ii) *and $g \neq 0$, then*

(iii) $A^*g = (\langle g, Ag\rangle/\|g\|^2)g$ *for all A in $\mathcal{A}$*

and m can be extended to a multiplicative linear functional K on $\mathcal{A}$ such that

(iv) $K(A) = \langle Ag, g\rangle/\|g\|^2$ *for all A in $\mathcal{A}$.*

Proof. Assume that m is a multiplicative linear functional on $\mathcal{A}$ and define $L(f) = m(A_f)$ for each f in $L^2(\mathcal{R}_+)$. It follows from Theorem 2.3 that L is a continuous linear functional on $L^2(\mathcal{R}_+)$. By the Riesz Representation Theorem there exists a unique element g of $L^2(\mathcal{R}_+)$ such that $L(f) = \langle f, g\rangle$ for all f. Consequently $m(A_f) = \langle f, g\rangle$ for all f in $L^2(\mathcal{R}_+)$.

Assuming now that m is multiplicative, we have for all f and h in $L^2(\mathcal{R}_+)$ $\langle h, \langle g, f\rangle g\rangle = \langle f, g\rangle\langle h, g\rangle = m(A_f)m(A_h) = m(A_fA_h) = m(A_{A_fh}) = \langle A_fh, g\rangle = \langle h, A_f^*g\rangle$. Consequently $A_f^*g = \langle g, f\rangle g$ as desired.

Assume now that m and g satisfy (i) and (ii) and that $g \neq 0$. Reversing the computation in the preceding paragraph, we conclude that m is a multiplicative linear functional on $\mathcal{A}_0$. We shall construct a multiplicative linear extension of m on $\mathcal{A}_0$. Let $A \in \mathcal{A}$

and $A_{f_\lambda} \to A$ weakly. Then $\langle g, Ag\rangle = \lim\langle g, A_{f_\lambda}g\rangle = \lim\langle A^*_{f_\lambda}g, g\rangle = \lim\langle g, f_\lambda\rangle\|g\|^2$ by (ii). Consequently $\lim\langle g, f_\lambda\rangle = \langle g, Ag\rangle/\|g\|^2$ and for each h in $L^2(\mathcal{R}_+), \langle g, Ah\rangle = \lim\langle g, A_{f_\lambda}h\rangle = \lim\langle\langle g, f_\lambda\rangle g, h\rangle = \langle g, h\rangle\langle g, Ag\rangle/\|g\|^2$. Thus $A^*g = (\langle g, Ag\rangle/\|g\|^2)g$, proving the final assertion. We now define $K(A) = (\langle Ag, g\rangle/\|g\|^2)$ for each A in $\mathcal{A}$. A straightforward computation shows that K is a multiplicative linear functional on $\mathcal{A}$ and that K is an extension of m.

We have shown that to each multiplicative linear functional on $\mathcal{A}$ there corresponds a unique element g of $L^2(\mathcal{R}_+)$ which is a common eigenvector for the elements of $\mathcal{A}^*$, provided $g \neq 0$. In Theorem 3.3 we shall show that each such function g is necessarily of the form $e^{\lambda t}/\phi(t)$ for some complex number λ.

LEMMA 3.2. *If G is the generator of $\{S_t\}$ and λ is sufficiently large, then $A_f = (\lambda - G)^{-1}$ where $f(t) = e^{-\lambda t}\phi(t)$.*

Proof. Since $\{S_t\}$ is strongly continuous, there exist constants M and w so that $\sup_x |\phi(x+t)/\phi(x)| = \|S_t\| \leqq Me^{wt}$ [2, Lemma 2.1]. Thus $\phi(t) \leqq Me^{wt}$ and for λ sufficiently large $f(t) = e^{-\lambda t}\phi(t) \in L^2(\mathcal{R}_+)$. Then $A_f = \int_0^\infty (f(t)/\phi(t))S_t dt = \int_0^\infty e^{-\lambda t}S_t dt = (\lambda - G)^{-1}$. [6, p. 344].

THEOREM 3.3. *If m is a multiplicative linear functional on $\mathcal{A}$ and g satisfies*

(i) *$m(A_f) = \langle f, g\rangle$ and*

(ii) *$A^*_f g = \langle g, f\rangle g$ for all f in $L^2(\mathcal{R}_+)$,*

then either $g = 0$ or there exists a complex number λ such that $g(x) = e^{\lambda x}/\phi(x)$. Conversely, if $g(x) = e^{\lambda x}/\phi(x)$ and $g \in L^2(\mathcal{R}_+)$, then g satisfies (ii).

Proof. Let g satisfy (i) and (ii). By Lemma 3.2 $(\lambda^* - G^*)^{-1}g = \langle g, f\rangle g$ where $f(t) = e^{-\lambda t}\phi(t) \in L^2(\mathcal{R}_+)$. If $\langle g, f\rangle = 0$, then $g = 0$. Assume that $\langle g, f\rangle \neq 0$. Since $A^*_f g = \langle g, f\rangle g$, we have

$$\langle g, f\rangle g(x) = \frac{e^{\lambda x}}{\phi(x)} \int_x^\infty \frac{\phi(t)}{e^{\lambda t}} g(t)dt. \qquad \text{a.e.} \tag{3}$$

Let $h(x) = \phi(x)g(x)/e^{\lambda x}$ and note that $h \in L^1(\mathcal{R}_+)$ since $\phi(x)/e^{\lambda x} \in L^2(\mathcal{R}_+)$ and $g \in L^2(\mathcal{R}_+)$. We now have $\langle g, f\rangle h(x) = \int_x^\infty h(t)dt$ a.e. Since h is integrable and $\langle g, f\rangle \neq 0$, we can conclude first that h is continuous and secondly that h is differentiable. Thus $\langle g, f\rangle h'(x) =$

$-h(x)$ and $h(x) = Ae^{\beta x}$ or equivalently $g(x) = Ae^{(\lambda+\beta)x}/\phi(x)$. It follows from (3) that $\langle g, f\rangle g(0) = (1/\phi(0)) \int_0^\infty (\phi(t)g(t)/e^{\lambda t})dt = (1/\phi(0))\langle g, f\rangle$. Thus $g(0) = 1/\phi(0)$, so that $A = 1$ and $g(x) = e^{(\lambda+\beta)x}/\phi(x)$, as desired. A straightforward computation shows that if g is of this form, then g satisfies (ii).

As an immediate consequence of Theorems 3.3 and 3.1 we have:

COROLLARY 3.4. *If* $g_\lambda(t) = e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)$, *then* $A^*g_\lambda = (\langle g_\lambda, Ag_\lambda\rangle/\|g_\lambda\|^2)g_\lambda$ *for all* A *in* $\mathcal{A}$.

In the remainder of the paper we let $H = \{\lambda : e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)\}$ and $E = \{g : g(t) = e^{\lambda t}/\phi(t), \lambda \in H\}$. We shall show that both sets are either empty or large: more precisely, either H is empty or H is a closed half-plane and at the same time either E is empty or its linear span is dense in $L^2(\mathcal{R}_+)$.

THEOREM 3.5. $\sigma(A_f) = \{\langle f, g\rangle : g \in E\} \cup \{0\}$.

Proof. In our comments following Theorem 2.6 we observed that $0 \in \sigma_{\mathcal{A}}(A_f)$ whenever $f \in L^2(\mathcal{R}_+)$. By Theorem 2.6 $\mathcal{A}$ is a maximal abelian algebra and hence for each A in $\mathcal{A}$, $\sigma(A) = \sigma_{\mathcal{A}}(A) = \{m(A) : m$ a multiplicative linear functional on $\mathcal{A}\}$. By Theorems 3.1 and 3.3, $m(A_f) = \langle f, g\rangle$ for some g in E provided m is not identically zero on $\mathcal{A}_0$, completing the proof.

We observed in the proof of Theorem 3.5 that $\sigma(A) = \{m(A) : m$ a multiplicative linear functional on $\mathcal{A}\}$ which implies that $\sigma(A) \supset \{\langle Ag, g\rangle/\|g\|^2 : g \in E\} \cup \{m_0(A)\}$ where m_0 is identically zero on $\mathcal{A}_0$. It is not known whether this set is the entire spectrum of A; equivalently, it is not known if m_0 is unique.

COROLLARY 3.6. *Among the conditions*

(i) $\mathcal{A}$ *contains a nonzero quasinilpotent element*;

(ii) $\sigma((\beta - G)^{-1}) = \{0\}$ *for some* β *such that* $(\beta - G)^{-1}$ *is bounded*;

(iii) $\sigma(A_f) = \{0\}$ *for all* f *in* $L^2(\mathcal{R}_+)$;

(iv) $E = \varnothing$;

(v) *the linear span of* E *is not dense in* $L^2(\mathcal{R}_+)$, *the following implications hold*: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).

Proof. (i) $\Rightarrow$ (v). If $\sigma(A) = \{0\}$, then by Theorem 3.1(iii) $A^*g = 0$ for each g in E. Thus if A is nonzero, the linear span of E is not dense in $L^2(\mathcal{R}_+)$ and (v) holds. (v) $\Rightarrow$ (ii). If $\sigma((\beta - G)^{-1}) \neq \{0\}$ for sufficiently large β, then by Theorem 3.5 there exist $g(t) = e^{\lambda t}/\phi(t) \in E$. If the linear span of E is not dense in $L^2(\mathcal{R}_+)$, then there exists a nonzero f such that

$$
\begin{aligned}
0 &= \int \frac{e^{zt}}{\phi(t)} f(t) dt \text{ whenever } \operatorname{Re} z \leqq \operatorname{Re} \lambda \\
&= \int e^{(z-\lambda)t} \frac{e^{\lambda t} f(t)}{\phi(t)} dt \\
&= \int e^{ixt} \frac{e^{\lambda t} f(t)}{\phi(t)} dt \text{ for } z = \lambda + ix, x \text{ real.}
\end{aligned}
$$

Thus the Fourier coefficients of the $L^1(\mathcal{R}_+)$ function $e^{\lambda t} f(t)/\phi(t)$ are zero, implying that $f = 0$ a.e. This contradiction completes the proof that (v) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) by Theorem 3.5; (iv) $\Rightarrow$ (v) trivially.

The following two examples demonstrate the two different types of symbols ϕ: in the first example $\alpha(\phi) > -\infty$ and H is a half plane and in the second example $\alpha(\phi) = -\infty$ and H is empty. Thus in the second example each A_f is quasinilpotent.

EXAMPLE 1. Let $\phi(x) = x + 1$. We shall show that ϕ is of bounded kernel type and $\alpha(\phi) = 0$.

$$
\int_0^x \frac{\phi(x)^2}{\phi(t)^2 \phi(x-t)^2} dt = 2(x+1)^2 \left[\frac{\log(x+1)}{(x+3)^3} + \frac{x}{(x+2)^2(x+1)} \right]
$$

which is bounded on $\mathcal{R}_+$. To see that $\alpha(\phi) = 0$ we note that $\int_0^\infty |e^{\lambda x}/(x+1)|^2 dx$ converges for $\operatorname{Re} \lambda \leqq 0$ and diverges for $\operatorname{Re} \lambda > 0$. Thus by Corollary 3.6 no element of $\mathcal{A}$ is quasinilpotent and by Corollary 3.4 g is a common eigenvector for A^* whenever $g(x) = e^{\lambda x}/\phi(x)$, $\operatorname{Re} \lambda \leqq 0$.

EXAMPLE 2. Let $\phi(x) = e^{-x^2/2}$. We shall show that ϕ is of bounded kernel type and $\alpha(\phi) = -\infty$. Obviously for each complex number λ $e^{\lambda t}/\phi(t) \notin L^2$ so that $\alpha(\phi) = -\infty$. To see that ϕ is of bounded kernel type we compute as follows

$$
\int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)} \right)^2 dt = \int_0^x \frac{e^{-x^2}}{e^{-t^2} e^{-(x-t)^2}} dt
$$

$$= \int_0^x e^{-2t(x-t)}dt$$

$$= e^{-x^2/2}\int_0^x e^{(x-2t)^2/2}dt$$

$$= e^{-x^2/2}\int_{-x}^x \frac{1}{2} e^{\frac{1}{2}s^2}ds$$

$$= e^{-x^2/2}\int_0^x e^{\frac{1}{2}s^2}ds.$$

Thus by l'Hopital's Rule

$$\lim_{x\to\infty}\int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)}\right)^2 dx = \lim_{x\to\infty}\frac{\frac{d}{dx}\int_0^x e^{\frac{1}{2}s^2}dx}{\frac{d}{dx}e^{x^2/2}} = 0.$$

Consequently since $\int_0^x (\phi(x)/\phi(t)\phi(x-t))^2dx$ is continuous and vanishes at ∞, it is bounded.

We note also for this example that since $\|S_t^n\|^{1/n} = e^{-nt^2/2}$, each $S_t (t\neq 0)$ is quasinilpotent. The fact that each A_f is quasinilpotent follows from Corollary 3.6.

Although it appears difficult in general to determine which symbols ϕ are of bounded kernel type, in certain cases one can use information about the set H to show that ϕ is not of bounded kernel type. More precisely, if ϕ is of bounded kernel type, then H is a closed half plane. To see this we argue as follows. Assume $\lambda \in H$. Then $\int_0^\infty (e^{2\mathrm{Re}\,\lambda x}/\phi(x)^2)dx = \int_0^\infty |e^{\lambda x}/\phi(x)|^2dx < \infty$. Consequently if $\mathrm{Re}\, z \leqq \mathrm{Re}\,\lambda$, then $z \in H$, proving that H is a half plane. Now choose β so that $f(t) = e^{-\beta t}\phi(t) \in L^2(\mathcal{R}_+)$ and $(\beta - G)^{-1}$ is bounded. By Lemma 3.2 and Theorem 3.5 $\sigma((\beta - G)^{-1}) = \{1/(\beta-\lambda): \lambda \in H\} \cup \{0\}$. Thus $\{1/(\beta-\lambda): \lambda\in H\}\cup\{0\}$ is compact and it easily follows that H is closed.

In **[3]** it was shown that $\phi(x) = (x+1)^{-1/2}$ is the symbol for a subnormal weighted translation semigroup. Since $\int_0^\infty |e^{\lambda t}/\phi(t)|^2dt$ converges for $\mathrm{Re}\,\lambda < 0$ and diverges otherwise we see that H is not closed and hence ϕ is not of bounded kernel type. At the end of §5 we shall see that no subnormal weighted translation semigroup has symbol of bounded kernel type. Indeed a stronger conclusion is obtained: if $\{S_t\}$ is hyponormal ($S_t^*S_t \geqslant S_tS_t^*$ for each t), then the symbol ϕ of $\{S_t\}$ is not of bounded kernel type.

4. Transitivity. For clarity in this section we shall let $\mathcal{A}_\phi$ denote the weakly closed algebra generated by $\{S_t\}$, where ϕ is the symbol of $\{S_t\}$ and ϕ is of bounded kernel type.

Let T be a linear transformation with domain $D(T) \subseteq X$. We say that T *commutes* with A in $B(X)$ if $AD(T) \subseteq D(T)$ and $ATx = TAx$ for each x in $D(T)$. Also, T commutes with a set of operators S if it commutes with each operator in S. T is said to be *closable* if T has a closed extension.

THEOREM 4.1. *If T is a densely defined linear transformation commuting with $\mathcal{A}_\phi$, then T is closable and TA_h is bounded for every h in $D(T)$.*

Proof. To prove that T is closable we must show that if $\{h_n\}$ is a sequence in $D(T)$ converging to 0 and $\{Th_n\}$ converges to some vector f, then $f = 0$. Note that if u is in $D(T)$ and v is in $L^2(\mathcal{R}_+)$, then $A_u v = A_v u$ is in $D(T)$. Let g be in $D(T)$. Then

$$TA_{h_n}g = A_{h_n}Tg \to 0.$$

But $TA_{h_n}g = TA_g h_n = A_g Th_n \to A_g f$. Thus for every g in $D(T)$, $A_g f = 0$. But since $D(T)$ is dense in $L^2(\mathcal{R}_+)$, $\{A_g \colon g \text{ in } D(T)\}$ is weakly dense in $\mathcal{A}_\phi$. Since I is in $\mathcal{A}_\phi$, we have $f = 0$. Hence T is closable.

Now suppose h is in $D(T)$. Since TA_h commutes with $\mathcal{A}_\phi$, TA_h is closable. But TA_h is everywhere defined, so TA_h is in $B(L^2)$. In fact, since $TA_h f = TA_f h = A_f Th = A_{Th} f$, we have $TA_h = A_{Th}$.

Note that with T as in the above theorem and h in $D(T)$, $(TA_h)^*$ is, of course, bounded. Explicitly, $(TA_h)^* = T^* A_h^*$. To see this let f also be in $D(T)$ and let g be in $L^2(\mathcal{R}_+)$. Then

$$\begin{aligned} \langle Tf, A_h^* g\rangle &= \langle A_h Tf, g\rangle \\ &= \langle TA_h f, g\rangle \\ &= \langle f, (TA_h)^* g\rangle \end{aligned}$$

so that $A_h^* g$ is in $D(T^*)$ and $(TA_h)^* g = T^* A_h^* g$.

The properties of transformations commuting with the algebra $\mathcal{A}_\phi$ just developed are nicely applicable to the theory of transitive algebras. An algebra $\mathcal{T}$ of operators on X is *transitive* if the only closed subspaces of X invariant under all the operators in $\mathcal{T}$ are $\{0\}$ and X. For general discussions of transitive algebras see [1] and [7, Chapter 8]. The following result is an immediate corollary to Arveson's density theorem.

PROPOSITION. (Arveson). *If $\mathcal{T}$ is a transitive algebra with the* *property that every linear transformation commuting with $\mathcal{T}$ is a multiple of the identity, then $\mathcal{T}$ is weakly dense in $B(X)$.*

Now if T is a closed densely defined linear transformation commuting with the transitive algebra $\mathcal{T}$ and either T or T^* has an eigenvector (other than 0), then T is a multiple of the identity. Since T^* commutes with $\mathcal{T}^* = \{A^*: A \text{ in } \mathcal{T}\}$ and $\mathcal{T}^*$ is transitive if and only if $\mathcal{T}$ is, it suffices to justify the above remark in the case $Tx = \lambda x$, $x \neq 0$. But then for every A in $\mathcal{T}$, $TAx = ATx = \lambda Ax$ so $T - \lambda I = 0$ on $\{Ax: A \text{ in } \mathcal{T}\}$ which is dense in X. But one sees easily that a closed transformation agreeing with a bounded operator on a dense set is in fact that bounded operator, and so $T = \lambda I$.

We now apply these remarks to certain algebras of the form $\mathcal{A}_\phi$. Recall that $\alpha(\phi) = \sup\left\{\lambda \text{ in } \mathcal{R}: \int_0^\infty (e^{2\lambda x}/\phi^2(x))dx < \infty\right\}$.

THEOREM 4.2. *If ϕ is of bounded kernel type and $\alpha(\phi) > -\infty$, then every transitive algebra containing $\mathcal{A}_\phi$ is weakly dense in $B(L^2)$.*

Proof. We have seen that every densely defined linear transformation commuting with $\mathcal{A}_\phi$ is closable. It is easy to show that the minimal closed extension of a closable transformation L commutes with all the operators commuting with L. Let T be a closed linear transformation commuting with $\mathcal{A}_\phi$. Then for each h in $D(T)$, we have seen that $T^*A_h^*$ is in $\mathcal{A}_\phi^*$. Let $g(x) = e^{\alpha(\phi)x}/\phi(x)$. Then g is in $L^2(\mathcal{R}_+)$ and $A_f^*g = (g,f)g$ for each f in $L^2(\mathcal{R}_+)$. Now $T^*A_h^* = (TA_h)^* = A_{Th}^*$, so

$$\langle g, Th\rangle g = T^*A_h^*g = \langle g, h\rangle T^*g.$$

Thus g is an eigenvector for T^* ((g,h) cannot be 0 for all h in the dense set $D(T)$). It follows from the proposition preceding this theorem that every transitive algebra containing $\mathcal{A}_\phi$ is dense.

Question. What about transitivity considerations in the case $\alpha(\phi) = -\infty$?

5. Functional properties of ϕ. We now concentrate on some properties of the function $\phi \to \alpha(\phi)$. Throughout the following discussion we assume that ϕ is in $C^1([a,\infty))$ for some $a \geqq 0$ and that $\phi(x) \neq 0$ for all $x \geqq 0$. Note that Theorem 5.2 is not dependent upon ϕ being of bounded kernel type. Define

$$i(\phi) = \liminf_{t\to\infty} \frac{\phi'(t)}{\phi(t)}$$

$$s(\phi) = \limsup_{t\to\infty} \frac{\phi'(t)}{\phi(t)}.$$

LEMMA 5.1. *If ϕ is of bounded kernel type, then $\alpha(\phi) < \infty$.*

Proof. We have seen that for $\operatorname{Re}\lambda \leqq \alpha(\phi)$ $g_\lambda(x) = e^{\lambda x}/\phi(x)$ is in $L^2(\mathcal{R}_+)$ and for every f in $L^2(\mathcal{R}_+)$ $(f, g_{\bar\lambda}) = \int_0^\infty (e^{\lambda x}/\phi(x))f(x)dx$ which is the Laplace transform of f/ϕ, Lf, evaluated at $-\lambda$. Thus if $\alpha(\phi) = \infty$, Lf is entire for each f in $L^2(\mathcal{R}_+)$. But $\langle f, g_{\bar\lambda}\rangle$ is in the spectrum of A_f so Lf is bounded and entire. By Liouville's Theorem Lf is constant for every f in L^2. But then $f/\phi = 0$ and $f = 0$. Thus $\alpha(\phi) < \infty$.

THEOREM 5.2. *If ϕ is in $C^1[a, \infty)$ for some $a \geqq 0$ and $\phi(x) \neq 0$ for all x then $i(\phi) \leqq \alpha(\phi) \leqq s(\phi)$.*

Proof. We prove only the inequality $i(\phi) \leqq \alpha(\phi)$, the other inequality's validity being quite similarly (and symmetrically) ascertained. If $i(\phi) = -\infty$ the inequality holds. Assume first that $i(\phi)$ is finite. Let $\epsilon > 0$ and let $\lambda = i(\phi) - \epsilon$. Then since ϕ'/ϕ is continuous for $t \geqq a$ we have $\phi'(t)/\phi(t) > \lambda + (\epsilon/2)$ for all $t \geqq a$. Let $f(x) = e^{2\lambda x}/\phi^2(x)$. Then $f'(x)/f(x) = 2(\lambda - (\phi'(x)/\phi(x))) < -\epsilon$ for all $x \geqq a$ hence $f(x) \leqq f(a)e^{-\epsilon(x-a)}$ for all $x \geqq a$. Since f is continuous on $[0, a]$, f is in $L^1(\mathcal{R}_+)$. Thus $i(\phi) - \epsilon \leqq \alpha(\phi)$ for all $\epsilon > 0$ and so $i(\phi) \leqq \alpha(\phi)$.

Now, if $i(\phi) = +\infty$ then we have $\lim_{t\to\infty}(\phi'(t)/\phi(t)) = +\infty$. But then we easily see that for *any* λ in $\mathcal{R}$ the function f defined above is in L' and so $\alpha(\phi) = +\infty$.

COROLLARY 5.3. *If* $\lim_{t\to\infty}(\phi'(t)/\phi(t))$ *exists, then* $\alpha(\phi) = \lim_{t\to\infty}(\phi'(t)/\phi(t))$.

In order to see that strict inequalities in the above Theorem 5.2 are possible, even for ϕ of bounded kernel type, note that if h and $1/h$ are bounded continuous functions on $\mathcal{R}_+$ and ϕ is of bounded kernel type, then $h\phi$ also is of bounded kernel type. Moreover, one easily verifies that $\alpha(\phi) = \alpha(h\phi)$. However (assuming h is in $C^1([a, \infty))$) for $\rho = h\phi$, $\rho'/\rho = (h'/h) + (\phi'/\phi)$. If, for example, we let $\phi(x) = x + 1$ and $h(x) = 2 + \sin x$ then all requirements above are satisfied, $\lim_{t\to\infty}(\phi'(t)/\phi(t)) = 0$, $\liminf_{t\to\infty}(h'(t)/h(t)) = -\sqrt{3}/3$, $\limsup_{t\to\infty}(h'(t)/h(t)) = \sqrt{3}/3$, and so $i(\rho) = -\sqrt{3}/3$, $\alpha(\rho) = 0$, and $s(\rho) = \sqrt{3}/3$.

We conclude the paper by showing that the class of weighted translation semigroups with symbol of bounded kernel type is disjoint from a rather large class of weighted translation semigroups, including the hyponormal (and of course subnormal) ones.

LEMMA 5.4. *For ϕ of bounded kernel type in $C^1([a,\infty))$ for some $a>0$, $\alpha(\phi)<s=\sup_{t\geqq a}(\phi'(t)/\phi(t))$.*

Proof. We have already seen that $\alpha(\phi)<\infty$ so the case $s=\infty$ is obvious. Suppose then that $s<\infty$. Then for $t\geqq a$, $\phi'(t)/\phi(t)\leqq s$ so $\phi(t)/\phi(a)\leqq e^{s(t-a)}$ and hence $1/\phi^2(t)\geqq(1/\phi^2(a))e^{2s(a-t)}$. We then have

$$\infty>\int_0^\infty \frac{e^{2\alpha(\phi)t}}{\phi^2(t)}\,dt\geqq\int_0^a \frac{e^{2\alpha(\phi)t}}{\phi^2(t)}\,dt+\left(\int_a^\infty e^{2[\alpha(\phi)-s]t}dt\right)e^{2sa}.$$

Thus $\alpha(\phi)<s$, for otherwise the last integral diverges.

COROLLARY 5.5. *If $\{S_t\}$ is a hyponormal weighted translation semigroup with symbol ϕ in $C^1([a,\infty))$ for some $a\geqq 0$, then ϕ is not of bounded kernel type.*

Proof. In [2] we showed that $\{S_t\}$ is hyponormal if and only if $\log\phi$ is convex. Thus ϕ'/ϕ is an increasing function and so $\lim_{t\to\infty}(\phi'(t)/\phi(t))=\sup_{t\geqq 0}(\phi'(t)/\phi(t))$. By Corollary 5.3 and Lemma 5.4 ϕ cannot be of bounded kernel type.

Note that if $\{S_t\}$ is subnormal, the condition of ϕ being in $C^1([a,\infty))$ holds automatically since ϕ has the form $\phi^2(x)=e^{\alpha x}\int_0^\infty e^{-tx}d\rho(t)$ where ρ is a probability measure.

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