A CHARACTERIZATION OF THE SUBGROUPS OF THE ADDITIVE RATIONALS

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1. Introduction. In the class of abelian groups every element of which (except the identity) has infinite order, the subgroups of the additive group of rational numbers have the simplest structure. These rational groups are the groups of rank one, or generalized cyclic groups, an abelian group G being said to have rank one if for any pair of elements, $a \neq 0$, $b \neq 0$, in G, there exist integers m, n, such that $ma = nb \neq 0$. Although many of the properties of these groups are known [1], it seems worthwhile to give a simple characterization from which their properties can easily be derived. This characterization is given in Theorems 1 and 2 of §2, and the properties of the rational groups are obtained as corollaries of these theorems in §3. In §4, all rings which have a rational group as additive group are determined.

Let $p_1, p_2, \cdots, p_j, \cdots$ be an enumeration of the primes in their natural order; and associate with each p_j an exponent k_j, where k_j is a nonnegative integer or the symbol ∞. We consider sequences i; $k_1, k_2, \cdots, k_j, \cdots$, where i is any positive integer such that $(i, p_j) = 1$ if $k_j > 0$, and define $(i; k_1, k_2, \cdots, k_j, \cdots) = (i; k_j)$ to be the set of all rational numbers of the form ai/b, where a is any integer and b is an integer such that $b = \prod'_{p_j} p_j^{n_j}$ with $n_j \leq k_j$. Then each sequence determines a well-defined set of rational numbers. The symbol $\prod'$ designates a product over an arbitrary subset of the primes that satisfy whatever conditions are put on them; $\prod$ designates a product over all primes that satisfy the given conditions.

2. Characterization of the rational groups. We show that the nontrivial subgroups of R are exactly the subsets $(i; k_j)$ defined in the introduction.

THEOREM 1. *The set $(i; k_j)$ is a subgroup of R^+, the additive group of rational numbers. We have $(i; k_j) = (i'; k'_j)$ if and only if $i = i'$, $k_j = k'_j$ for all j.*

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Proof. If $ai/b \in (i\,; k_j)$, $ci/d \in (i\,; k_j)$, then $b = \Pi'_{p_j} p_j^{n_j}$, $d = \Pi'_{p_j} p_j^{m_j}$, and $[b, d] = \Pi'_{p_j} p_j^{s_j}$, where $s_j = \max\,(n_j, m_j) \le k_j$. Writing $[b, d] = bb' = dd'$, we have

$$\frac{ai}{b} - \frac{ci}{d} = \frac{b'ai}{[b,d]} - \frac{d'ci}{[b,d]} = \frac{(b'a - d'c)i}{[b,d]} \in (i;k_j).$$

It is clear that different sequences determine different subgroups.

In the sequel we need the following properties of a subgroup $G \neq 0$ of R^+.

(1) Every $\zeta \in G$ has the form $\zeta = ai/b$, $(ai, b) = 1$, where i is the least positive integer in G.

For every ζ we have $\zeta = m/b$, where $(m, b) = 1$; and if i is the least positive integer in G, then $m = ai + r$ and $m - ai \in G$ imply $r = 0$.

(2) If $ai/b \in G$, $i \in G$, and $(a, b) = 1$, then $i/b \in G$.

For there exist integers k, l such that $ka + lb = 1$ and

$$\frac{i}{b} = \frac{(ka + lb)i}{b} = \frac{kai}{b} + li \in G.$$

(3) If $ai/b \in G$ where i is the least positive integer in G, and $(a, b) = 1$, then $(i, b) = 1$.

By (2), $i/b \in G$; and if $(i, b) \neq 1$, $h/b' \in G$ with $h < i$. Then $b'(h/b') = h \in G$.

We assume in the proof of the remaining properties that the elements of G are written in the canonical form ai/b with $(ai, b) = 1$ and i the least positive integer in G.

(4) If $ai/bc \in G$, then $i/b \in G$.

For $cai/bc = ai/b \in G$ and $i/b \in G$ by (2).

(5) If ai/b, $ci/d \in G$, and if $(b, d) = 1$, then $i/bd \in G$. For by (2) we have
For by (2) we have

$$\frac{i}{bd} = \frac{(kb + ld)i}{bd} = \frac{ki}{d} + \frac{li}{b} \in G.$$

THEOREM 2. *If $G \neq 0$ is a subgroup of R^+, then there exists a sequence $(i\,; k_1, k_2, \cdots, k_j, \cdots)$ such that $G = (i\,; k_j)$.*

Proof. By (1), every $\zeta \in G$ has the form $\zeta = ai/b$, $(ai, b) = 1$, where i is the least positive integer in G. We write all elements of G in this form. If, for every l, there exist $ai/b \in G$ such that $p_j^l \mid b$, let $k_j = \infty$. If not, let $k_j = \max k$ such that $p_j^k \mid b$ for some $ai/b \in G$. Since $(ai, b) = 1$, we have $(i, p_j) = 1$ if $k_j > 0$. By the definition of i and k_j, G is contained in $(i; k_j)$. Now every element of $(i; k_j)$ has the form $ai/(p_1^{n_1} \cdots p_r^{n_r})$, where $n_j \leq k_j$ and $(a, p_1^{n_1} \cdots p_r^{n_r}) = 1$. By (4) and the definition of k_j, G contains every $i/p_j^{n_j}$ with $n_j \leq k_j$, and by (5), G contains $ai/(p_1^{n_1} \cdots p_r^{n_r})$. Hence $G = (i; k_j)$.

3. Properties of the rational groups. In this section, properties of the rational groups are obtained as corollaries of the theorems of §1.

COROLLARY 1. *The group* $(i; k_j)$ *is a subgroup of* $(i'; k'_j)$ *if and only if* $k_j \leq k'_j$ and $i = mi'$.

COROLLARY 2. *The group* $(i; k_j)$ *is cyclic if and only if* $k_j < \infty$ *for all* j *and* $k_j = 0$ *for almost all* j.

Proof. If $(i; k_j)$ is cyclic, it is generated by ai/b with $(ai, b) = 1$. Since every element of $(i; k_j)$ has the form nai/b, we have $a = 1$ and $b = \prod_{k_j > 0} p_j^{k_j}$. Conversely $(i; k_j)$ contains $i/\prod_{k_j>0} p_j^{k_j}$, and this element generates $(i; k_j)$.

COROLLARY 3. *We have* $(i; k_j) \cong (i'; k'_j)$ *if and only if both* $k_j = k'_j$ *for almost all* j, *and, whenever* $k_j \neq k'_j$, *both are finite. Every isomorphism between* $(i; k_j)$ *and* $(i'; k'_j)$ *is given by*

$$\frac{ai}{b} \longleftrightarrow \frac{mai'}{nb},$$

where

$$m = \left(\prod_{k_j = k'_j = \infty}{}' p_j^{a_j} \right) \left(\prod_{\substack{k_j \geq k'_j \\ k_j \text{ finite}}} p_j^{k_j - k'_j} \right),$$

$$n = \left(\prod_{k_h = k'_h = \infty}{}' p_h^{b_h} \right) \left(\prod_{\substack{k'_h \geq k_h \\ k'_h \text{ finite}}} p_h^{k'_h - k_h} \right).$$

Proof. If $(i; k_j) \cong (i'; k'_j)$, then $i \longrightarrow mi'/n$ with $(mi', n) = 1$. If $\eta \longrightarrow i'$, then $m\eta \longrightarrow mi'$ and $ni \longrightarrow mi'$, so that $m\eta = ni$, or $\eta = ni/m$.

Hence $ni/m \longrightarrow i'$. We write

$$m = p_{\alpha_1}^{a_1} \cdots p_{\alpha_r}^{a_r}\,, \qquad a_l > 0\,, \quad n = p_{\beta_1}^{b_1} \cdots p_{\beta_s}^{b_s}\,, \quad b_m > 0\,;$$

then for $n_j \leq k_j$ we have

$$\frac{i}{p_j^{n_j}} \longrightarrow \frac{p_{\alpha_1}^{a_1} \cdots p_{\alpha_r}^{a_r}\, i'}{p_j^{n_j} p_{\beta_1}^{b_1} \cdots p_{\beta_s}^{b_s}}\,;$$

while for $n'_j \leq k'_j$ we have

$$\frac{p_{\beta_1}^{b_1} \cdots p_{\beta_s}^{b_s}\, i}{p_j^{n'_j} p_{\alpha_1}^{a_1} \cdots p_{\alpha_r}^{a_r}} \longrightarrow \frac{i'}{p_j^{n'_j}}\,.$$

We have the following alternatives with consequences which follow from (3):

I. $\quad j = \alpha_l \quad : \quad n_j - k'_j \leq a_l \leq k_j - n'_j$

II. $\quad j = \beta_m \quad : \quad n'_j - k_j \leq b_m \leq k'_j - n_j$

III. $\quad \left.\begin{matrix} j \neq \alpha_l \\ j \neq \beta_m \end{matrix}\right\} \quad : \quad n_j \leq k'_j\,, \quad n'_j \leq k_j$

It follows that $k_j = \infty$ implies $k'_j = \infty$ and conversely. With both k_j and k'_j finite we may choose $n_j = k_j$ and $n'_j = k'_j$ and we have:

I. $\quad j = \alpha_l \quad : \quad a_l = k_j - k'_j$

II. $\quad j = \beta_m \quad : \quad b_m = k'_j - k_j$

III. $\quad \left.\begin{matrix} j \neq \alpha_l \\ j \neq \beta_m \end{matrix}\right\} \quad : \quad k_j = k'_j$

We have $k_j = k'_j$ if and only if $j \neq \alpha_l$, $j \neq \beta_m$. In particular, we have $k_j = k'_j$ for almost all j. If $k_j > k'_j$, then $j = \alpha_l$ and $a_l = k_j - k'_j$. If $k'_j > k_j$, then $j = \beta_m$ and $b_m = k'_j - k_j$.

Now $i \longrightarrow mi'/n$ implies $ai/b \longrightarrow ami'/bn$, so that the only isomorphisms between $(i\,; k_j)$ and $(i'\,; k'_j)$ are those described in the corollary. Incidentally, we have derived necessary conditions for the relation $(i\,;\,k_j) \cong (i'\,;\,k'_j)$.

With the necessary conditions satisfied, we check that the given correspondence actually is an isomorphism. These conditions imply that the correspondence is single-valued with a single-valued inverse from $(i\,;\,k_j)$ onto $(i'\,;\,k'_j)$. It is clear that addition is preserved.

COROLLARY 4. *The group* $(i\,;\,k_j)$ *admits only the identity automorphism if and only if* k_j *is finite for all* j.

Proof. If k_j is finite for all j, we have by Corollary 3, with $k_j = k'_j$ for all j, that $m = n = 1$. Conversely, if any $k_j = \infty$, then the correspondence of Corollary 3 gives us nontrivial automorphisms.

The multiplicative group of the field of rational numbers, $R^{\times}$, is a direct product of the infinite cyclic subgroups of $R^{\times}$ generated by the prime numbers p_k for all k. Such a subgroup consists of the elements $p_k\,,\, p_k^2, \cdots, 1, 1/p_k\,,\, 1/p_k^2, \cdots$.

COROLLARY 5. *The group of automorphisms of* $(i\,;\,k_j)$ *is isomorphic to the direct product of all of the infinite cyclic subgroups of* $R^{\times}$ *generated by those primes* p_k *for which* $k_j = \infty$.

Proof. By Corollary 3, there is a (1–1) correspondence between the automorphisms of $(i\,;\,k_j)$ and the rational numbers M/N with $(M, N) = 1$, where M and N are arbitrary products of those primes for which $k_j = \infty$. This correspondence clearly preserves multiplication and the set of all rationals M/N has the stated structure as a group with respect to multiplication.

COROLLARY 6. *For any two subgroups* $(i\,;\,k_j)$ *and* $(i'\,;\,k'_j)$ *of* R^{+}, *the set* T *consisting of all ordinary products of an element of* $(i\,;\,k_j)$ *with an element of* $(i'\,;\,k'_j)$ *is again a subgroup of* R^{+}.

Proof. We have $T = (I\,;\,K_j)$, where

$$I = \frac{i i'}{\Pi_{p_j}\, p_j^{s_j}}\,, \qquad K_j = k_j + k'_j - s_j$$

with $s_j = \min(\alpha_j, k'_j) + \min(\alpha'_j, k_j)$, where α_j is the highest power of p_j that divides i, and α'_j the highest power of p_j that divides i'.

COROLLARY 7. *If* $(i\,;\,k_j) \geq (i'\,;\,k'_j)$ *and* $p_j^{l_j}$ *is the maximum power of* p_j *such that* $p_j^{l_j}$ *divides* i'/i, *then the difference group* $(i\,;\,k_j) - (i'\,;\,k'_j)$ *is a direct sum* *of the groups* G_j *where*

(i) G_j *is the cyclic group,*

$$\left\{(i';\ k'_j) + \frac{i'}{p_j^{k_j+l_j}}\right\},$$

if k_j *is finite;*

(ii) G_j *is the group of type* p^∞,

$$p^\infty\left\{(i';k'_j) + \frac{i'}{p_j^{k'_j+1}}\ ,\quad (i';k'_j) + \frac{i'}{p_j^{k'_j+2}}\ ,\ \cdots\right\},$$

if k_j *is infinite and* k'_j *is finite;*

(iii) $G_j = \{0\}$ *if* $k_j = k'_j = \infty$.

4. Rings which have a rational group as additive group. The distributive laws in any ring S with $(i\,;k_j)$ as additive group are used to determine all possible definitions of multiplication in S.

LEMMA. *If* S *is a ring with additive group* $(i\,;k_j)$, *then multiplication in* S *is defined by*

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{ac}{bd}\ (i \times i)\ .$$

Proof. We prove this by showing that

$$bd\left(\frac{ai}{b} \times \frac{ci}{d}\right) = ac\ (i \times i)\ .$$

We have

$$ac\,(i \times i) = ai \times ci \qquad \text{(by the distributive laws in } S\text{)}$$

$$= \left[b\left(\frac{ai}{b}\right)\right] \times \left[d\left(\frac{ci}{d}\right)\right]$$

$$= \Big[\underbrace{\frac{ai}{b} + \cdots + \frac{ai}{b}}_{b \text{ summands}}\Big] \times \Big[\underbrace{\frac{ci}{d} + \cdots + \frac{ci}{d}}_{d \text{ summands}}\Big]$$

whence $ac\,(i \times i)$

$$= \left[\frac{ai}{b} \times \frac{ci}{d}\right] \underbrace{+ \cdots +}_{bd \text{ summands}} \left[\frac{ai}{b} \times \frac{ci}{d}\right] \qquad \text{(by the distributive laws in } S\text{)}$$

$$= (bd)\left[\frac{ai}{b} \times \frac{ci}{d}\right].$$

THEOREM 3. *If there is an infinite number of* k_j *such that* $0 < k_j < \infty$, *then the only ring* S *with* $(i\,;k_j)$ *as additive group is the null ring. If* $0 < k_j < \infty$ *for only a finite number of* k_j, *then* S *is a ring with additive group* $(i\,;k_j)$ *if and only if multiplication in* S *is defined by*

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{acA'\left(\prod_{0<k_j<\infty} p_j^{k_j}\right) i}{bd \prod'_{k_j=\infty} p_j^{n_j}},$$

where A' *and* n_j *are arbitrary.*

Proof. If S is a ring with additive group $(i\,;k_j)$, then $i \times i = Ai/B \in (i;k_j)$, where $(Ai, B) = 1$, $B = \prod' p_j^{n_j}$, $n_j \le k_j$. By the lemma, we have

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{acAi}{bdB}.$$

If $0 < k_r < \infty$, this yields in particular

$$\frac{i}{p_r^{k_r}} \times \frac{i}{p_r^{k_r}} = \frac{Ai}{p_r^{2k_r}B}.$$

Therefore $(p_r, B) = 1$, for otherwise we would have $2k_r + n_r \le k_r$, which is impossible. Hence, $B = \prod' p_j^{n_j}$ is a product of primes for which $k_j = \infty$, and it is necessary that $p_r^{k_r} | A$. If there is an infinite number of primes p_j with $0 < k_j < \infty$, then $A = 0$ and $(ai/b) \times (ci/d) = 0$. This proves the first statement in the theorem.

If $0 < k_j < \infty$ for only a finite number of primes p_j, then

$$A = A' \prod_{0<k_j<\infty} p_j^{k_j}.$$

Together with what has been proved above, this gives

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{acA'\left(\prod_{0<k_j<\infty} p_j^{k_j}\right) i}{bd \prod'_{k_j=\infty} p_j^{n_j}},$$

where A' and $n_j > 0$ are arbitrary integers.

Conversely, this definition of multiplication always makes $(i; k_j)$ a ring. Closure with respect to $\times$ is insured by providing $p_j^{k_j}$ in the numerator when $0 < k_j < \infty$, and the associative and distributive laws are readily verified.

COROLLARY 1. *The set $(i; k_j)$ is a subring of R if and only if there is no k_j such that $0 < k_j < \infty$.*

Proof. Let $(i; k_j)$ be a subring of R and assume that for at least one k_j we have $0 < k_j < \infty$. If $0 < k_j < \infty$ for infinitely many k_j, then $(i; k_j)$ is not a subring of R, since by Theorem 3 it is the null ring. If $0 < k_j < \infty$ for a finite number of k_j, then multiplication in any ring with $(i; k_j)$ as additive group is given by the formula of the theorem. Hence this must reduce to ordinary multiplication for some choice of A' and n_j; that is,

$$\frac{A' \prod_{0<k_j<\infty} p_j^{k_j}}{\prod'_{k_j=\infty} p_j^{n_j}} = i; \quad A' \prod_{0<k_j<\infty} p_j^{k_j} = i \prod'_{k_j=\infty} p_j^{n_j}.$$

By hypothesis, at least one p_j with $k_j > 0$ appears in the left member of the above equality. Since no prime appears in both products, we have $p_j | i$. This contradicts $(i, p_j) = 1$ for $k_j > 0$.

Conversely, let every k_j be either 0 or ∞. By the theorem, we have

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{acA' i}{bd \prod'_{k_j=\infty} p_j^{n_j}},$$

and we may select $A' = i$, $\prod'_{k_j=\infty} p_j^{n_j} = 1$, yielding ordinary multiplication.

COROLLARY 2. *If $(i; k_j)$ is a subring of R, then $(i; k_j)$ is a ring under the* *multiplication*

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{ac}{bd}\left(\frac{ei}{f}\right)$$

for arbitrary $ei/f \in (i\,;\,k_j)$.

Proof. By Corollary 1, we have $k_j = 0$ or $k_j = \infty$, so that every element of $(i\,;\,k_j)$ has the form

$$\frac{A' i}{\prod'_{k_j=\infty} p_j^{n_j}}\;;$$

and by the theorem these are just the multipliers which are used to define multiplication.

COROLLARY 3. *If S is a ring with additive group $(i\,;\,k_j)$, then either S is a null ring or S is isomorphic to a subring of R.*

Proof. If S is not null, the correspondence

$$\frac{ai}{b} \longrightarrow \frac{aA}{bB}, \qquad \text{where} \qquad \frac{ai}{b} \times \frac{ci}{d} = \frac{acAi}{bdB},$$

is (1–1) from S on a subset of R, and

$$\frac{ai}{b} + \frac{ci}{d} = \frac{(da+bc)i}{bd} \longrightarrow \frac{(da+bc)A}{bdB} = \frac{aA}{bB} + \frac{cA}{dB},$$

$$\frac{ai}{b} \times \frac{ci}{d} = \frac{acAi}{bdB} \longrightarrow \frac{acA^2}{bdB^2} = \frac{aA}{bB}\,\frac{cA}{dB}.$$

COROLLARY 4. *All rings with additive group R^+ are isomorphic to R.*

Proof. The correspondence of Corollary 3 clearly exhausts R.

REFERENCE

1. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. 3 (1937), 68-122.

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