ON THE THEORY OF SPACES Λ

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1. Introduction. In this paper we discuss properties of the spaces $\Lambda(\phi, p)$, which were defined for the special case $\phi(x) = \alpha x^{\alpha-1}$, $0 < \alpha \leq 1$, in our previous paper [8]. A function $f(x)$, measurable on the interval $(0, l)$, $l < +\infty$ belongs to the class $\Lambda(\phi, p)$ provided the norm $\|f\|$, defined by

$$(1.1) \qquad \|f\| \equiv \left\{\int_0^l \phi(x)\, f^*(x)^p dx\right\}^{1/p},$$

is finite. Here $\phi(x)$ is a given nonnegative integrable function on $(0, l)$, not identically 0, and $f^*(x)$ is the decreasing rearrangement of $|f(x)|$, that is, the decreasing function on $(0, l)$, equimeasurable with $|f(x)|$. (For the properties of decreasing rearrangements see [5, 12, 7, and 8].) We write also $\Lambda(\alpha, p)$ instead of $\Lambda(\phi, p)$ with $\phi(x) = \alpha x^{\alpha-1}$, and $\Lambda(\phi)$ instead of $\Lambda(\phi, 1)$. We shall also consider spaces $\Lambda(\phi, p)$ for the infinite interval $(0, +\infty)$. In §2 we give some simple properties of the spaces Λ, and show in particular that $\Lambda(\phi, p)$ has the triangle property if and only if $\phi(x)$ is decreasing. In §3 we discuss the conjugate spaces $\Lambda^*(\phi, p)$, and show that the spaces $\Lambda(\phi, p)$ are reflexive. In §4 we give a generalization of the spaces $\Lambda(\phi, p)$, and characterize the conjugate spaces in case $p = 1$. In §5 we give applications; we prove that the Hardy-Littlewood majorants $\theta(x, f)$ of a function $f \in \Lambda(\phi, p)$ or $f \in \Lambda^*(\phi, p)$ also belong to the same class. We give sufficient conditions for an integral transformation to be a linear operation from one of these spaces into itself, and apply them to solve the moment problem for the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$.

2. Properties of spaces $\Lambda(\phi, p)$. We shall establish the following result.

THEOREM 1. *The norm* $\|f\|$ *defined by* (1.1) *has the triangle property if and only if* $\phi(x)$ *is equivalent to a decreasing function; in this case* $f, g \in \Lambda(\phi, p)$

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implies $f + g \in \Lambda(\phi, p)$.

Proof. (a) Suppose that $\|f\|$ has the triangle property. Let $\delta > 0$, $h > 0$, $a > 0$, and $a + 2h \le l$. Set

$$f(x) = \begin{cases} 1 + \delta & \text{on } (0, a + h) \\ 1 & \text{on } (a + h, a + 2h) \\ 0 & \text{on } (a + 2h, l), \end{cases} \qquad g(x) = \begin{cases} 1 & \text{on } (0, h) \\ 1 + \delta & \text{on } (h, a + 2h) \\ 0 & \text{on } (a + 2h, l); \end{cases}$$

then

$$(f + g)^*(x) = \begin{cases} 2 + 2\delta & \text{on } (0, a) \\ 2 + \delta & \text{on } (a, a + 2h) \\ 0 & \text{on } (a + 2h, l). \end{cases}$$

We have $\|f\| = \|g\|$; hence the inequality $\|f + g\| \le \|f\| + \|g\|$ is equivalent to

$$\left\{(2 + 2\delta)^p \int_0^a \phi(x)\,dx + (2 + \delta)^p \int_a^{a+2h} \phi(x)\,dx\right\}^{1/p}$$

$$\le 2\left\{(1 + \delta)^p \int_0^{a+h} \phi(x)\,dx + \int_{a+h}^{a+2h} \phi(x)\,dx\right\}^{1/p},$$

or to

$$(2 + \delta)^p \int_a^{a+2h} \phi(x)\,dx \le (2 + 2\delta)^p \int_a^{a+h} \phi(x)\,dx + 2^p \int_{a+h}^{a+2h} \phi(x)\,dx,$$

and thus to

$$\frac{(1 + \delta)^p - (1 + \frac{1}{2}\delta)^p}{(1 + \frac{1}{2}\delta)^p - 1} \int_a^{a+h} \phi(x)\,dx \ge \int_{a+h}^{a+2h} \phi(x)\,dx. \tag{2.1}$$

If $\Phi(x)$ is the integral of ϕ over $(0, x)$, we obtain from (2.1), making $\delta \longrightarrow 0$,

$$\Phi(a + h) \ge \tfrac{1}{2}\left[\Phi(a) + \Phi(a + 2h)\right];$$

that is, $\Phi(x)$ is concave, and thus $\phi(x)$ is equivalent to a decreasing function.

(b) Suppose that ϕ is decreasing. Instead of (2.1) we can now write

$$\|f\| = \sup_{\phi_r} \left\{\int_0^l \phi_r |f|^p\,dx\right\}^{1/p}, \tag{2.2}$$

the supremum being taken over all possible rearrangements ϕ_r of ϕ. It follows from (2.2) that $f, g \in \Lambda(\phi, p)$ implies $f + g \in \Lambda(\phi, p)$ and $\|f + g\| \leq \|f\| + \|g\|$.

It is now easy to see that, for $\phi(x)$ decreasing, $\Lambda(\phi, p)$ is a Banach space; the completeness may be proved by usual methods (compare [8]). In general, $\Lambda(\phi, p)$ is not uniformly convex. Suppose, for instance, that there is a sequence $\delta_n \longrightarrow 0$ such that

$$\Phi(2\delta_n)/\Phi(\delta_n) \longrightarrow 1 . \tag{2.3}$$

This condition is satisfied, for example, if $\phi(x) = x^{-1} \,|\log x|^{-p}$, $p > 1$. We take $f_n(x) = h_n$ on $(0, 2\delta_n)$, $f_n(x) = 0$ on $(2\delta_n, l)$; we take $g_n(x) = h_n$ on $(0, \delta_n)$, $g_n(x) = -h_n$ on $(\delta_n, 2\delta_n)$, and $g_n(x) = 0$ on $(2\delta_n, l)$; and we choose h_n so that

$$\|f_n\|^p = \|g_n\|^p = h_n^p\, \Phi(2\delta_n) = 1 .$$

Then we have

$$\tfrac{1}{2}\{f_n(x) + g_n(x)\} = \begin{cases} h_n & \text{on } (0, \delta_n) , \\ 0 & \text{elsewhere} , \end{cases}$$

and $(1/2)(f_n - g_n)^*(x)$ is the same function. Therefore

$$\left\| \frac{f_n + g_n}{2} \right\|^p = \left\| \frac{f_n - g_n}{2} \right\|^p = h_n^p\, \Phi(\delta_n) \longrightarrow 1 ,$$

and so $\Lambda(\phi, p)$ is not uniformly convex. In case of the spaces $\Lambda(\alpha, p)$, the problem remains open.

The remarks made above apply also to the spaces $\Lambda(\phi, p)$ in case of the infinite interval $(0, +\infty)$. We assume in this case that $\int_0^l \phi(x)\, dx < +\infty$ for any $l < +\infty$; the additional hypothesis on $f \in \Lambda(\phi, p)$ is that the rearrangement $f^*(x)$ exists, which is the case if and only if any set $[\,|f(x)| \geq \epsilon]$, $\epsilon > 0$, has finite measure. The completeness of $\Lambda(\phi, p)$ in this case follows from the fact that the set of such f is a closed linear subset of the Banach space of all f for which (2.2) is finite. If

$$\int_0^{+\infty} \phi(x)\, dx = +\infty , \tag{2.4}$$

this subspace coincides with the whole space. Condition (2.4) is in particular satisfied if $\phi(x) = \alpha x^{\alpha - 1}$.

3. Reflexivity of the spaces $\Lambda(\phi, p)$. We shall first give some definitions and lemmas which will be useful in the sequel. If $g(x)$, $g_1(x)$ are two positive functions defined on $(0, l)$, $0 < l \le +\infty$, we write $g \prec g_1$, if for all finite $0 \le x \le l$ we have

$$\int_0^x g(t)\,dt \le \int_0^x g_1(t)\,dt.$$

Integration by parts readily yields:

LEMMA 1. *If $g \prec g_1$, and f is positive and decreasing on $(0, l)$, then*

$$\int_0^l gf\,dx \le \int_0^l g_1 f\,dx. \tag{3.1}$$

LEMMA 2. *If $g \prec g_1$, and g, g_1 are positive and decreasing, then also $\psi(g) \prec \psi(g_1)$ for any convex increasing positive function, in particular for $\psi(u) = u^p$, $p \ge 1$.*

For the proof, let $f(x) = \{\psi(g_1(x)) - \psi(g(x))\}/\{g_1(x) - g(x)\}$ if $g(x) \ne g_1(x)$, and let $f(x)$ be equal to one of the derivates of $\psi(u)$ at $u = g(x)$ if $g(x) = g_1(x)$. Then $f(x)$ is the slope of the chord of the curve $v = \psi(u)$ on the interval (u, u_1), $u = g(x)$, $u_1 = g_1(x)$. The slope decreases as both u, u_1 decrease. Therefore $f(x)$ is decreasing and positive. Applying Lemma 1, we obtain

$$\int_0^l f(x)[g(x) - g_1(x)]\,dx \le 0,$$

which proves our assertion.

THEOREM 2. *Suppose that $f(x)$, $g(x)$ are positive and decreasing on $(0, l)$, and $f \in \Lambda(\phi, p)$, $p > 1$. Then*

$$\int_0^l fg\,dx \le \|f\|_\Lambda \inf_{\phi D \succ g} \left\{ \int_0^l \phi D^q\,dx \right\}^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.2}$$

where infimum is taken for all decreasing positive $D(x)$ for which $\phi D \succ g$. Moreover, this infimum is equal to the supremum of $\int_0^l fg\,dx$ for all positive decreasing f with $\|f\| \le 1$, if there is a function D with $\phi D \succ g$ and $\int \phi D^q\,dx < +\infty$, and is to $+\infty$ if there is no such D.

This theorem is due to I. Halperin. For the proofs, see a paper of Halperin appearing in the Canadian Journal of Mathematics and, for a simpler proof, [10].

Inequality (3.2) is a combination of (3.1) and the usual Hölder inequality. For if $g_1 = \phi D \succ g$, then

$$\int_0^l fg\,dx \le \int_0^l fg_1\,dx = \int_0^l \phi^{1/p} f \phi^{1/q} D\,dx \tag{3.3}$$

$$\le \|f\| \left\{\int_0^l \phi D^q\,dx\right\}^{1/q}.$$

Here and in the next section, the following theorem will be useful:

THEOREM 3. *Suppose that X is a normed linear space of measurable functions $f(x)$ on $(0,l)$, $0 < l < +\infty$, with the properties*: (i) *X contains all constants*; (ii) *if f_1 is measurable and $|f_1(x)| \le |f(x)|$, $f \in X$, then $f_1 \in X$ and $\|f_1\| \le \|f\|$*; (iii) *if $f \in X$ and f_e denotes the characteristic function of the set e, then $\|f f_e\| \to 0$ as* meas $e \to 0$.

Let Y consist of all measurable functions g for which $\int_0^l fg\,dx$ exists for all $f \in X$. Then

$$F(f) = \int_0^l fg\,dx, \qquad g \in Y, \tag{3.4}$$

is the general form of a linear functional on X, and its norm is equal to

$$\|g\| \equiv \sup_{\|f\| \le 1} \int_0^l fg\,dx < +\infty.$$

Proof. (a) Let $g \in Y$; then $\int_0^l f|g|\,dx$ exists for all $f \in X$, and $\|g\| = \sup \int_0^l f|g|\,dx$, where f runs through all *positive* $f \in X$ with $\|f\| \le 1$. If $\|g\| = +\infty$, there is a sequence $f_n \ge 0$, $\|f_n\| \le 1$ such that $\int f_n|g|\,dx > n^3$. Then $f = \Sigma n^{-2} f_n \in X$, and therefore $\int_0^l f|g|\,dx$ must exist. However $\int f|g|\,dx \ge n^{-2} \int f_n|g|\,dx \ge n$, which is a contradiction. Hence $\|g\| < +\infty$ for $g \in Y$. We see now that for $g \in Y$, $\int fg\,dx$ is a linear functional with norm $\|g\|$.

(b) Suppose that $F(f)$ is a given linear functional on X. By (i) and (ii), any characteristic function $f_e(x)$ belongs to X. Define $G(e) = F(f_e)$; since $|G(e) \le \|F\|\,\|f_e\| \to 0$ as meas $e \to 0$, there is an integrable $g(x)$ with $G(e) = \int_e g\,dx$. This means that (3.4) holds for $f = f_e$, and therefore also for all step-functions $\bar{f}$ (which are linear combinations of the f_e). For a bounded f, there is a sequence $\bar{f}_n(x) \to f(x)$ uniformly. As $\|\bar{f}_n - f\| \to 0$, this establishes (3.4) for all bounded f. Now suppose $f \in X$ is such that $fg = |f|\,|g|$. Let $f_n(x) = f(x)$ if $|f(x)| \le n$,

$f_n(x) = 0$ otherwise; then $\|f - f_n\| \longrightarrow 0$ by (iii), and hence $\int_0^l f_n g\,dx = F(f_n)$ has a finite limit. This shows that $\int |f|\,|g|\,dx < +\infty$; therefore $g \in Y$. Repeating the last part of this argument for an arbitrary $f \in A$, we obtain (3.4).

REMARKS. (A) Let X have the additional property: (iv) *$f_n(x) \longrightarrow f(x)$ almost everywhere, $f_n \in X$, and $\|f_n\| \le M$ imply $f \in X$*. Then the existence of $\int fg\,dx$ for all $g \in Y$ implies $f \in X$.

For taking the subsequence $f_n(x) \longrightarrow f(x)$ of (b), we see that $F_n(g) = \int f_n g\,dx$ is a sequence of linear functionals convergent toward $\int fg\,dx$ for any $g \in Y$. Then the norms $\|F_n\| = \|f_n\|$ are uniformly bounded, and using (iv) we obtain $f \in X$.

(B) Since Y is the conjugate space to X, Y is a Banach space, and Y clearly satisfies (ii). Suppose now that X satisfies (i)–(iv) and that Y satisfies (i) and (iii). Then Remark A and Theorem 3 together imply that X is the conjugate space of Y, in other words that any linear functional $F(g)$ in Y is of the form $F(g) = \int fg\,dx$, $f \in X$ and $\|F\| = \|f\|$.

(C) The above results hold for the interval $(0, +\infty)$ if the conditions (i)–(iii) [and eventually (iv)] are true for functions vanishing outside of a finite interval, and also (v) *for any $f \in X$, $\|f - f^l\| \longrightarrow 0$ as $l \longrightarrow \infty$, where f^l is defined by $f^l(x) = f(x)$ on $(0,l)$ and $f^l(x) = 0$ on $(l, +\infty)$.*

Applying these general results to the space $\Lambda(\phi, p)$ in case of a finite interval, we see that (i) and (ii) are satisfied. Condition (iii) follows from

$$\|h_e\|^p \le \int_0^{\text{meas } e} \phi f^{*p}\,dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0,$$

[$h_e(x)$ is the function $f(x) f_e(x)$], and (iv) from (2.2) and Fatou's theorem. We obtain the result that *the space $\Lambda^*(\phi, p)$ conjugate to $\Lambda(\phi, p)$ consists of all measurable functions g such that there is a decreasing positive D with $\phi D \succ g^*$ and $\int_0^l \phi D^q\,dx < +\infty$; further,*

$$\|g\|_{\Lambda^*} = \inf_{\phi D \succ g^*} \left\{ \int_0^l \phi D^q\,dx \right\}^{1/q}. \tag{3.5}$$

For it follows from Theorem 2 that

$$\left| \int_0^l fg\,dx \right| \le \int_0^l f^* g^*\,dx \le \|f\|_\Lambda \|g\|_{\Lambda^*},$$

and that $\|g\|_{\Lambda^*}$ is the supremum of the integral $\int fg\,dx$ for all $\|f\| \leq 1$.

Now if $g(x) = C > 0$ is a constant, we take an $l_1 > 0$ with $\phi(l_1) > 0$ and $C_1 = Cl[l_1\phi(l_1)]^{-1}$. Then $\int_0^{l_1} C_1\phi(x)\,dx \geq Cl$; and if $D(x) = C_1$ on $(0, l_1)$, $D(x) = 0$ on (l_1, l), then $\phi D \succ g$. Therefore Λ^* satisfies (i). Also (iii) holds, for if $h_e(x) = g(x)f_e(x)$, $g \in \Lambda^*$, $g^* \prec \phi D$, then $h_e^* \prec \phi D_1$, where $D_1(x) = D(x)$ on $(0, \text{meas } e)$, $D_1(x) = 0$ on $(\text{meas } e, l)$, and

$$\|h_e\|_{\Lambda^*}^q \leq \int_0^l \phi D_1^q\,dx = \int_0^{\text{meas } e} \phi D^q\,dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0.$$

We have proved the theorem:

THEOREM 4. *The space $\Lambda(\phi, p)$, $p > 1$, is reflexive. Its conjugate is defined by* (3.5).

We now consider the case of an infinite interval and assume $\int_0^\infty \phi\,dx = +\infty$. Then $f \in \Lambda(\phi, p)$ implies $f^*(x) \longrightarrow 0$ for $x \longrightarrow \infty$. If $a > 0$ is fixed and l sufficiently large, then the function $|f^l(x)|$ of (v) will take values $\geq f^*(a)$ only on a set of arbitrarily small measure. In view of (iii), condition (v) will follow for $\Lambda(\phi, p)$, if we can show that the norm of the function $f^*(a + x)$, $0 \leq x < +\infty$, tends to 0 as $a \longrightarrow \infty$, or even if this is true for some sequence $a \longrightarrow \infty$. This norm does not exceed

$$\left\{\int_0^\infty \phi(x) f^*(a + x)^p\,dx\right\}^{1/p} = \left\{\int_0^\infty \phi(x) f^*(x)^p \left[\frac{f^*(x + a)}{f^*(x)}\right]^p dx\right\}^{1/p} \longrightarrow 0,$$

as the integrand has the majorant ϕf^{*p}, and $f^*(x + a)/f^*(x) \longrightarrow 0$ for $a \longrightarrow \infty$.

To prove (v) for $\Lambda^*(\phi, p)$, we need a result going beyond Lemma 1, namely that if g and D are decreasing and positive, and $\phi D \succ g$, then there is another such function D_0 for which $\phi D \succ \phi D_0 \succ g$, and that except for certain open intervals I where D_0 is constant, $\int_0^x \phi D_0\,dt = \int_0^x g\,dt$. (This fact is proved in the paper of Halperin, mentioned at the beginning of this section and in [10]). As before, we have to prove that if $g \in \Lambda^*(\phi, p)$ is positive and decreasing, then the norm of the function $h(x) = g(x + a)$, $x \geq 0$, tends to 0 as $a \longrightarrow \infty$ for certain values of a. There is a D with $\phi D \succ g$ and $\int_0^\infty \phi D^q\,dx < +\infty$; and, by Lemma 2, $\int_0^\infty \phi D_0^q\,dx < +\infty$. As $\int_0^\infty \phi\,dx = +\infty$, we deduce that $D_0(x) \longrightarrow 0$ for $x \longrightarrow \infty$. Therefore

$$\int_0^x \phi D_0\,dx = o[\Phi(x)].$$

On intervals I, $\int_0^x \phi D_0\, dt$ is of the form $C\Phi(x) + C_1$, where $\Phi(x) = \int_0^x \phi\, dt$. If an I extends to $+\infty$, we have $C = 0$, that is $\int_0^x \phi D_0\, dt = C_1$ for all large x. and $D_0(x)$ is necessarily 0 for all such x. In this case also $g(x) = 0$ for all large x, and our assertion is trivial. If, on the other hand, there are arbitrarily large values a which do not belong to any I, then we have for these a,

$$\int_0^a \phi D_0\, dt = \int_0^a g\, dt \ .$$

It follows that $\int_0^x \phi D_0\, dt \geq \int_0^x g\, dt$, $x \geq a$, or $\phi(x+a)D_0(x+a) \succ g(x+a)$, and this implies $\phi(x)D_0(x+a) \succ g(x+a)$. Therefore,

$$\|h\|^q \leq \int_0^\infty \phi(x)\, D_0(x+a)^q\, dx = \int_0^\infty \phi(x)\, D_0(x)^q \left[\frac{D_0(x+a)}{D_0(x)}\right]^q dx \longrightarrow 0$$

for $a \longrightarrow \infty$. We obtain in this way:

THEOREM 5. *The space* $\Lambda(\phi, p)$, $p > 1$, $l = \infty$ *is reflexive; its conjugate is given by* (3.5).

4. A generalization. There is an obvious generalization of the spaces $\Lambda(\phi, p)$. Consider a class C of functions $\phi(x) \geq 0$ integrable over $(0, l)$, and let $X(C, p)$ consist of all those functions $f(x)$ for which

$$\|f\| = \sup_{\phi \in C} \left\{\int_0^l \phi\, |f|^p\, dx\right\}^{1/p} < +\infty. \tag{4.1}$$

A special type of these spaces is obtained if C is chosen to consist of all integrable positive functions $\phi(x)$ whose integrals $\phi_1(e)$ satisfy the condition

$$\phi_1(e) \leq \Phi(e)\ , \tag{4.2}$$

where $\Phi(e)$ is a given positive finite set function of measurable sets $e \subset (0, l)$. We may then assume that

$$\Phi(e) = \sup_{\phi_1} \phi_1(e)\ . \tag{4.3}$$

(A full characterization of set functions $\Phi(e)$ which may be represented in form (4.3) by means of a class of positive additive ϕ_1 will be given by the author elsewhere [9].) In particular, let $\phi_0(x)$ be a fixed decreasing positive function, and let $\Phi(e) = \int_0^{\text{meas } e} \phi_0\, dx$; then condition (4.2) is equivalent to the condition

$$\phi^*(x) \leq \phi_0(x) .$$

Therefore, in this case the norm (4.1) is equal to (1.1), and so $X(\Phi, p) = \Lambda(\phi_0, p)$.

For the space $X(\Phi, p)$, the condition $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere if and only if $\Phi(e) > 0$ for any set e of positive measure. Suppose now that $\Phi(e)$, defined by (4.3), vanishes on certain sets e with meas $e > 0$. There is then [2, p. 80, Theorem 15] a least measurable set e_0 which contains any such set e up to a null set; and e_0 is a union of a properly chosen denumerable set of these sets e. Hence $\phi_1(e_0) = 0$, and $\Phi(e_0) = 0$. It is easy to see that in this case $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere on $(0, l) - e_0$, and that the values of $f(x)$ on e_0 have no significance whatsoever for $\|f\|$. Omitting e_0 from $(0, l)$, we do not change the space $X(\phi, p)$, and we obtain a $\Phi(e)$ satisfying the above condition. In the sequel, ϕ is assumed to have this property.

The spaces $X(\Phi, p)$ are normed linear spaces. Their completeness may be proved by usual methods, if for instance $F(e)$ has the property that meas $e \longrightarrow 0$ implies $\Phi(e) \longrightarrow 0$ and if $l < +\infty$.

The spaces $X(C, p)$ satisfy the conditions (i), (ii), and (iv) of 3 [(iv) follows easily by Fatou's theorem]. Condition (iii) is not fulfilled in general. We can however enforce (iii) by defining the spaces $\Lambda(C, p)$ and $\Lambda(\Phi, p)$ to consist of all those functions $f \in X(C, p)$ or $f \in X(\Phi, p)$, respectively, for which $\|f f_e\| \longrightarrow 0$ with meas $e \longrightarrow 0$ in X. Then the conjugate space $\Lambda^*(C, p)$ and all linear functionals in $\Lambda(C, p)$ are given by Theorem 3. We conclude this section by describing the spaces $\Lambda^*(\Phi, 1)$ more precisely:

THEOREM 6. *If* $f \in \Lambda(\Phi, 1)$, *then*

$$\left| \int_0^l f g \, dx \right| \leq \|f\| \sup_{\Phi(e) > 0} \frac{1}{\Phi(e)} \int_e |g| \, dx , \tag{4.4}$$

and the left integral exists provided the right side is finite; moreover, the supremum $M(g)$ *in the right side is equal to the supremum of* $\int_0^l fg \, dx$ *for all* $f \in \Lambda(\Phi, 1)$ *with* $\|f\| \leq 1$.

Proof. Consider the function $\phi_0(x) = M(g)^{-1} |g(x)|$; then

$$\int_0^l |f| \, |g| \, dx = M(g) \int_0^l \phi_0 |f| \, dx \leq M(g) \|f\|_\Lambda ,$$

since

$$\int_e \phi_0(x)\,dx = M(g)^{-1}\int_e g(x)\,dx \le \Phi(e)\,, \qquad e \subset (0,l)\,.$$

This proves (4.4). On the other hand, if e is an arbitrary subset of $(0,l)$ with $\Phi(e) > 0$, then the function $f(x) = \Phi(e)^{-1} f_e(x)$ sign $g(x)$ has norm 1 in $\Lambda(\phi, 1)$, and

$$\int_0^l f g\,dx = \Phi(e)^{-1}\int_e |g|\,dx\,.$$

Therefore the integral $\int_0^l fg\,dx$ takes values arbitrarily close to $M(g)$.

From Theorems 3 and 6 we deduce that the space $\mathrm{M}(\Phi,1) = \Lambda^*(\Phi,1)$ consists of all $g(x)$ for which

$$\|g\| = \sup_e \left\{\Phi(e)^{-1}\int_e |g(x)|\,dx\right\} < +\infty. \tag{4.5}$$

In particular, the space $\mathrm{M}(\phi)$, conjugate to $\Lambda(\phi)$, is given by

$$\|g\|_{\mathrm{M}(\phi)} = \sup_e \left\{\phi_1(e)^{-1}\int_e |g|\,dx\right\}. \tag{4.6}$$

It is easy to see that the expression (4.6) is the limit, for $p \longrightarrow 1$, of the norm of g in the space $\Lambda^*(\phi, p)$, $p > 1$.

5. Applications. We shall make three applications.

5.1. *Hardy-Littlewood majorants.* We take in this section $l = 1$. We write

$$\theta(x,f) = \sup_{0 \le y \le 1} \frac{1}{y-x}\int_x^y |f(t)|\,dt\,, \tag{5.1}$$

and denote by $\theta_1(x,f)$ and $\theta_2(x,f)$ the supremum of the same expression for $0 \le y < x$ or $x < y \le 1$, respectively. Then

$$\theta(x,f) \le \max\,\{\theta_1(x,f),\ \theta_2(x,f)\}\,. \tag{5.2}$$

On the other hand, it is well known [5, p. 291] that

$$\theta_1^*(x,f) \le \theta(x,f^*) = \frac{1}{x}\int_0^x f^*(t)\,dt\,, \tag{5.3}$$

and this is also true with θ_2 in place of θ_1. From (5.2) we derive, for any $p \ge 1$,

$$\theta^p(x, f) \leq \theta_1^p(x, f) + \theta_2^p(x, f)\,.$$

It follows that

$$\theta^*(x, f)^p \leq (\theta_1^p + \theta_2^p)^* < (\theta_1^p)^* + (\theta_2^p)^* = \theta_1^{*p} + \theta_2^{*p} \leq 2\theta(x, f^*)^p\,;$$

that is,

$$\theta^*(x, f)^p < 2\theta(x, f^*)^p\,. \tag{5.4}$$

We shall make repeated use of the inequality of Hardy [12, p. 72]:

$$\int_0^l x^{s-p}F(x)^p\,dx \leq \left(\frac{p}{p-s-1}\right)^p \int_0^l x^s f(x)^p\,dx\,, \tag{5.5}$$

where $p > 1$, $s < p - 1$, $0 < l \leq +\infty$, and $F(x)$ is the integral of the positive function $f(x)$.

In our present situation it follows from (5.3) and (5.5), if $p > 1$, that

$$\int_0^x \theta(t, f^*)^p\,dt \leq \left(\frac{p}{p-1}\right)^p \int_0^x f^*(t)^p\,dt\,;$$

and, by Lemma 1,

$$\int_0^1 \phi(x)\theta^*(x, f)^p\,dx \leq 2\left(\frac{p}{p-1}\right)^p \int_0^1 \phi(x) f^*(x)^p\,dx\,. \tag{5.6}$$

This is case (i) of the following theorem:

THEOREM 7. (i) *If* $f \in \Lambda(\phi, p)$ *and* $p > 1$, *then also* $\theta(x, f) \in \Lambda(\phi, p)$; (ii) *if* $f^*(x)\log(1/x) \in \Lambda(\phi)$, *then* $\theta(x, f) \in \Lambda(\phi)$; (iii) *if* $f \in \Lambda(\phi)$, *and* $\phi(x)$ *is decreasing with respect to* $x^{-\delta}$ *for some* $\delta > 0$, *then* $\theta(x, f) \in \Lambda(\phi)$.

To prove (ii) we observe that (5.4) with $p = 1$ and Lemma 1 imply

$$\|\theta\|_{\Lambda(\phi)} = \int_0^1 \phi(x)\theta^*(x, f)\,dx \leq 2\int_0^1 \phi(x)\,\frac{1}{x}\,dx \int_0^x f^*(t)\,dt$$

$$= 2\int_0^1 f^*(t)\,dt \int_t^1 \frac{\phi(x)}{x}\,dx \leq 2\int_0^1 \phi(t) f^*(t)\,\log\frac{1}{t}\,dt < +\infty\,.$$

Finally, if the hypothesis of (iii) holds, that is if $\phi(x) = x^{-\delta}D(x)$ with a decreasing positive D, then the preceeding inequality gives

$$\|\theta\| \leq 2 \int_0^1 f^*(t) D(t) \int_t^1 x^{-\delta-1} dx \leq 2\,\delta^{-1} \int_0^1 \phi(t) f^*(t)\, dt\ .$$

THEOREM 8. (i) *If* $f^*(x) \log(1/x) \in \Lambda^*(\phi, p)$, $p \geq 1$, *then* $\theta(x, f) \in \Lambda^*(\phi, p)$; (ii) *if* $f \in \Lambda^*(\alpha, p)$, $p > 1$, *then* $\theta(f) \in \Lambda^*(\alpha, p)$.

Proof. (i) Let $p > 1$ [the case $p = 1$, $\Lambda^*(\phi, p) = M(\phi)$ is simpler]. By (5.4), and since $\theta(x, f^*)$ decreases, we have

$$\|\theta(f)\|^q \leq 2^q \|\theta(f^*)\|^q = 2^q \inf_{\phi D \succ \theta(f^*)} \int_0^1 \phi(x) D(x)^q\, dx\ .$$

But by (5.3), we have

$$\int_0^x \theta(u, f^*)\, du = \int_0^x f^*(t)\, dt \int_t^x \frac{du}{u} \leq \int_0^x f^*(t) \log \frac{1}{t}\, dt\ ,$$

which means that $\theta(x, f^*) \prec f^*(x) \log(1/x) = h(x)$; hence

$$\|\theta(f)\|^q \leq 2^q \inf_{\phi D \succ h} \int_0^1 \phi D^q\, dx = 2^q \|h\|^q < +\infty\ .$$

(ii) Let $f \in \Lambda^*(\alpha, p)$; because of (5.4) we may assume that $f = f^*$, that is, that f is positive and decreasing. Suppose $f \prec \phi D$ and $\int_0^1 \phi D^q\, dx < +\infty$ with $\phi(x) = \alpha x^{\alpha - 1}$. Then by (5.3) we have

$$\theta(x, f) = \frac{1}{x} \int_0^x f(t)\, dt \leq \frac{\alpha}{x} \int_0^x t^{\alpha-1} D(t)\, dt$$

$$= \alpha x^{\alpha-1} \frac{1}{x^\alpha} \int_0^x t^{\alpha-1} D(t)\, dt = \phi(x) D_1(x)\ ,$$

say. The function $D_1(x)$ is positive and decreasing, as

$$D_1'(x) = -\alpha x^{-\alpha-1} \int_0^x t^{\alpha-1} D\, dt + x^{-1} D(x)$$

$$\leq -\alpha x^{-\alpha-1} D(x) \int_0^x t^{\alpha-1}\, dt + x^{-1} D(x) = 0\ .$$

Therefore, by Hardy's inequality, we have

$$\|\theta(f)\|^q \leq \alpha \int_0^1 x^{\alpha-1} D_1^q\, dx = \alpha \int_0^1 x^{(1-\alpha)(q-1)} \left\{\frac{1}{x} \int_0^x t^{\alpha-1} D\, dt\right\}^q dx$$

$$\leq C\int_0^1 x^{(1-\alpha)(q-1)+(\alpha-1)q} D(x)^q\,dx = C\int_0^1 x^{\alpha-1} D^q\,dx$$

with some constant C. Thus $\theta(f) \in \Lambda^*$, which proves (ii).

It should be remarked that $f^* \log(1/x)$ behaves very much like $f^* \log^+ f^*$:

(a) *If* $f^* \log(1/x)$ *belongs to* $\Lambda^*(\phi,p)$, $p \geq 1$, *then* $f \log^+ |f|$ *belongs to* $\Lambda^*(\phi,p)$. For if $p > 1$ [the case $p = 1$ is similar but simpler], there is a $D(x)$ with $f^* \log(1/x) \prec \phi D$ and $\int_0^1 \phi D\,dx < +\infty$. Then also $f^*(\delta)\log(1/x) \prec \phi D$ on $(0,\delta)$; in particular,

$$f^*(\delta)\int_0^\delta \log\frac{1}{x}\,dx \leq \int_0^\delta \phi D\,dx \leq 1$$

if δ is small. Therefore $f^*(\delta) \leq \delta^{-1}$ for all small δ, which shows that

$$f\log^+|f| \in \Lambda^*(\phi,p).$$

(b) Now suppose $\phi(x)$ is such that, for some $\delta > 0$, we have $\int_0^1 \phi(x)x^{-\delta}dx < +\infty$. *If* $f\log^+|f|$ *belongs to* $\Lambda^*(\phi,p)$, $p \geq 1$, *then* $f^* \log(1/x)$ *also does*. In fact, by Young's inequality [5, p. 111; or 11, p. 64], for the pair of inverse functions $\phi(u) = \log^+ u$, $\psi(v) = e^v$, we obtain $ab \leq a\log^+ a + e^b$ $(a, b \geq 0)$ and therefore

$$f^* \log\frac{1}{x} \leq \delta^{-1}f^* \log^+(\delta^{-1}f^*) + x^{-\delta} \leq \delta^{-1}f^* \log^+\frac{1}{\delta} + \delta^{-1}f^* \log^+ f^* + x^{-\delta}$$

$$\leq Af^* \log^+ f^* + B + x^{-\delta}$$

for some constants A, B.

It follows from these remarks, that Theorem 7 (ii) may be regarded as a generalization of the theorem of Hardy-Littlewood [12, p. 245] that $f\log^+|f| \in L$ implies $\theta(f) \in L$.

Theorems 7 and 8 have many applications which may be derived in the same way as the corresponding results for the spaces L^p (see [12, p. 246]). As an example, we give the following result. Let $k > 0$, and let $\sigma_n^{(k)}(x,f)$ denote the Cesàro sum of order k of the Fourier series of a function $f(x)$. If $\theta(x,f)$ is taken for the interval $(0,4\pi)$, we have: *if* $f(x)$ *satisfies one of the hypotheses of Theorems 7 or 8, then* $|\sigma_n^{(k)}(x,f)| \leq C_k\theta(x,f)$, $n = 0,1,\cdots$. We may give another formulation of this result. In the spaces $\Lambda(\phi,p)$ and $\Lambda^*(\phi,p)$ we introduce a partial ordering, writing $f_1 \le f_2$ if $f_1(x) \le f_2(x)$ almost everywhere. With this ordering, Λ and Λ^* become Banach lattices for which the order convergence $f_n \longrightarrow f$ is identical with the convergence $f_n(x) \longrightarrow f(x)$ almost everywhere and the existence of a function $h(x)$ of the lattice such that $|f_n(x)| \le h(x)$ almost everywhere. This is an immediate consequence of the fact that the lattices Λ, Λ^* satisfy the condition (ii) of Theorem 3 (see [6, pp. 154-156]). Then the above result implies that $\sigma_n^{(k)} \longrightarrow f$ in order in the corresponding space. Theorems of this section may also be used to obtain analogues of theorems of Hardy [3] and Bellman [1] for spaces Λ and Λ^*; see Petersen [11].

5.2. *Integral transformations.* Let $K(x, t)$ be measurable on the square $0 \le x \le 1$, $0 \le t \le 1$, and let

$$F(x) = \int_0^1 K(x, t)\, f(t)\, dt. \tag{5.7}$$

THEOREM 9. *Suppose that there is a constant M such that*

(i) $$\int_0^1 |K(x, t)|\, dt \le M \quad \textit{almost everywhere};$$

(ii) *for any rearrangement $\phi_r(x)$ of $\phi(x)$, the function $h_r(t) = \int_0^1 \phi_r(x) K(x, t)\, dx$ belongs to $M(\phi)$ and has a norm not exceeding M. Then* (5.7) *is a linear operator of norm $\le M$ mapping $\Lambda(\phi, p)$ into itself. Condition* (ii) *may also be replaced by*

(iii) $$\int_0^1 |K(x, t)|\, dx \le M \quad \textit{almost everywhere}.$$

Proof. Condition (ii) is equivalent to

$$h_r^*(t) \prec M\phi(t). \tag{5.8}$$

Assuming $f \in \Lambda(\phi, p)$, $p > 1$, we have

$$\begin{aligned}\int_0^1 \phi_r(x)\, |F(x)|^p\, dx &\le \int_0^1 \phi_r\, dx \left\{\int_0^1 |K|\, |f(t)|\, dt\right\}^p \\ &\le \int_0^1 \phi_r\, dx \int_0^1 |K|\, |f|^p\, dt \left\{\int_0^1 |K|\, dt\right\}^{p/q} \\ &\le M^{p/q} \int_0^1 |f(t)|^p\, dt \int_0^1 \phi_r(x)\, |K(x, t)|\, dx \\ &\le M^{p/q} \int_0^1 h_r^*(t)\, f^*(t)^p\, dt;\end{aligned}$$

by (5.8) and Lemma 1, this is

$$\leq M^{1+p/q} \int_0^1 \phi(t) f^*(t)^p \, dt = M^p \|f\|^p ,$$

which proves the first part of the theorem. Suppose now that (i) and (iii) hold. Let $\delta > 0$, e an arbitrary set of measure δ, and e_1 a set of measure δ such that $\phi_r(x) \geq \phi(\delta)$ on e_1 and $\phi_r(x) \leq \phi(\delta)$ on the complement Ce_1 of e_1. Then we have

$$\int_e |h_r(t)| \, dt \leq \int_e dt \int_{e_1} |\phi_r(x)| \, |K| \, dx + \int_e \int_{Ce_1}$$

$$\leq M \int_{e_1} |\phi_r(x)| \, dx + \phi(\delta) \int_e dt \int_0^1 |K(x, t)| \, dx$$

$$\leq M \Phi(\delta) + M \delta \phi(\delta) \leq 2M \Phi(\delta) .$$

This shows that the norm of $h_r(t)$ in $M(\phi)$ does not exceed $2M$, and proves (ii).

REMARK. If the conditions of Theorem 9 are satisfied, then

$$G(t) = \int_0^1 K(x, t) g(x) \, dx \tag{5.9}$$

is a linear operator of norm $\leq 2M$ mapping $\Lambda^*(\phi, p)$ into itself.

We have in fact, for $g \in \Lambda^*(\phi, p)$ and $f \in \Lambda(\phi, p)$,

$$\int_0^1 G(t) f(t) \, dt = \int_0^1 g(x) \, dx \int_0^1 K(x, t) f(t) \, dt = \int_0^1 g(x) F(x) \, dx$$

$$\leq \|g\|_{\Lambda^*} \|F\|_\Lambda \leq M \|f\|_\Lambda \|g\|_{\Lambda^*} ,$$

(the integrals evidently exist), and this shows that $G \in \Lambda^*$ and that $\|G\| \leq M \|g\|$.

Theorem 9 is akin to the "convexity theorem" of M. Riesz [12, p.198]. We mention for completeness that there is a generalization of this theorem, in which the different spaces L^p involved are replaced by the spaces $\Lambda(\phi, p)$ *with the same* ϕ. The proof, which follows closely the proof of M. Riesz's theorem in [12], is omitted.

5.3. *Moment problems.* We give an application of Theorem 9 to moment problems of the form

$$\mu_n = \int_0^1 x^n f(x) \, dx, \qquad n = 0, 1, 2, \cdots . \tag{5.10}$$

We shall write

$$\Phi_{n\nu} = \Phi\left(\frac{\nu+1}{n+1}\right) - \Phi\left(\frac{\nu}{n+1}\right), \qquad \Phi(x) = \int_0^x \phi\, dt\,,$$

$$\mu_{n\nu} = \binom{n}{\nu}\Delta^{n-\nu}\mu_\nu = \int_0^1 f(x) p_{n\nu}(x)\, dx\,,$$

$$p_{n\nu} = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}\,, \qquad \nu = 0, 1, \cdots, n\,,$$

and $\mu_{n\nu}^*$ for the decreasing rearrangement of the $|\mu_{n\nu}|$, $\nu = 0, 1, \cdots, n$. Moreover, we set

$$f_n(x) = (n+1)\mu_{n\nu} \qquad \text{for } \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1}\,, \tag{5.11}$$

and obtain

$$f_n(x) = \int_0^1 K_n(x,t) f(t)\, dt\,, \tag{5.12}$$

$$K_n(x,t) = (n+1)\, p_{n\nu}(t), \qquad \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1}\,,$$

For the special case $\phi(x) = \alpha x^{\alpha-1}$ and $p = 1$, the following theorem (with another proof) has been given in [8].

THEOREM 10. *The sequence of real numbers μ_n is a moment sequence of a function of the space $\Lambda(\phi, p)$ or of $\Lambda^*(\phi, p)$ [for the case $\Lambda(\phi, 1)$, we assume $\phi(x) \longrightarrow \infty$ for $x \longrightarrow 0$] if and only if the norms of the functions* (5.11) *are uniformly bounded in this space.*

For the space $\Lambda(\phi, p)$, the condition is

$$\sum_{\nu=0}^{n} \Phi_{n\nu}\mu_{n\nu}^{*p} \le M(n+1)^{-p}\,, \tag{5.13}$$

and for $\Lambda^*(\phi, p)$, $p > 1$,

$$\mu_{n\nu}^* < \Phi_{n\nu} D_{n\nu}\,, \qquad \sum_{\nu=0}^{n} \Phi_{n\nu} D_{n\nu}^q \le M^q, \tag{5.14}$$

with some positive decreasing $D_{n\nu}$, $\nu = 0, 1, \cdots, n$.

Proof. If $f \in \Lambda(\phi, p)$, then condition (5.13) is satisfied by Theorem 9, because the kernel (5.12) satisfies (i) and (iii) with $M = 1$.

Conversely, let $\|f_n\|_\Lambda \leq M$. Since

$$\int_e |f_n(x)|\, dx \leq \phi(\delta)^{-1} \int_0^\delta \phi(x)\, |f_n(x)|\, dx \leq M\phi(\delta)^{-1}, \qquad \text{meas } e = \delta,$$

it follows in case $p = 1$ that the integrals $\int_e |f_n|\, dx$ are uniformly absolutely continuous and uniformly bounded. In case $p > 1$, this follows by Hölder's inequality. We deduce that for a certain subsequence $f_{n_k}(x)$, the integrals $\int_e f_{n_k}(x)\, dx$ converge for any $e = (0, x)$ with x rational; hence they converge for any measurable set $e \subset (0,1)$. We then have

$$\lim_{k\to\infty} \int_e f_{n_k}(x)\, dx = \int_e f(x)\, dx, \tag{5.15}$$

with some $f \in L$. Then also

$$\int_0^1 f_{n_k}\psi\, dx \longrightarrow \int_0^1 f\psi\, dx \tag{5.16}$$

for any bounded ψ. For any such ψ we have, by (3.2),

$$\left|\int_0^1 f\psi\, dx\right| \leq \lim \left|\int_0^1 f_{n_k}\psi\, dx\right| \leq M\|\psi\|_{\Lambda^*};$$

hence this must be true for any ψ in Λ^*. Thus by §3, it follows that $f \in \Lambda(\phi, p)$.

We remark also that it follows easily from (5.16) that we have

$$\int_0^1 f_{n_k}\psi_k\, dx \longrightarrow \int_0^1 f\psi\, dx, \tag{5.17}$$

if the sequence $\psi_k(x)$ is uniformly convergent towards a bounded function $\psi(x)$.

Now let P be the vector space of all polynomials

$$\psi(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with usual addition and scalar multiplication. On P we define an additive and homogeneous functional F by

$$F(\psi) = a_0\mu_0 + a_1\mu_1 + \cdots + a_m\mu_m .$$

Let

$$B_n^\psi(x) = \sum_{\nu=0}^{n} \psi\left(\frac{\nu}{n}\right) p_{n\nu}(x)$$

be the Bernstein polynomial of order n of $\psi(x)$; then it is known [10] that

$$B_n^\psi(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_m^{(n)}x^m ,$$

and that $a_i^{(n)} \longrightarrow a_i$ for $n \longrightarrow \infty$. Hence $F(B_n^\psi) \longrightarrow F(\psi)$. In particular, let $\psi(x) = x^m$. We have

$$\text{(5.18)} \qquad F(B_n^\psi) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m F(p_{n\nu}) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m \mu_{n\nu}$$

$$= \int_0^1 f_n(x) g_n(x)\, dx ,$$

where $\psi_n(x)$ is equal to $(\nu/n)^m$ in the interval $[\nu/(n+1), (\nu+1)/(n+1)]$. As $\psi_n(x) \longrightarrow \psi(x)$ uniformly, we deduce from (5.18) and (5.17) that

$$\int_0^1 f(x)x^m\, dx = \lim F(B_n) = F(\psi) = \mu_m , \qquad m = 0, 1, \cdots .$$

Since $f \in \Lambda(\phi, p)$, this proves that the condition is sufficient in case of the space Λ. The proof for the space $\Lambda^*(\phi, p)$, which is similar, is omitted.

References

1. R. Bellman, *A note on a theorem of Hardy on Fourier constants*, Bull. Amer. Math. Soc. 50 (1944), 741-744.

2. G. Birkhoff, *Lattice theory*, 2nd ed., New York, 1948.

3. G. H. Hardy, *The arithmetic mean of a Fourier constant*, Messenger of Math. 68 (1929), 50-52.

4. ———, *Divergent series*, Oxford, 1949.

5. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, England, 1934.

6. L. Kantorovitch, *Lineare halbgeordnete Räume*, Mat. Sbornik (N.S.) 2 (44) (1937), 121-168.

7. G. G. Lorentz, *A problem of plane measure*, Amer. J. Math. **71** (1949), 417-426.

8. ———, *Some new functional spaces*, Ann. of Math. (2) **51** (1950), 37-55.

9. ———, *Multiply subadditive functions*, to appear in the Canadian J. Math.

10. ———, *Bernstein polynomials*, to appear as a book at the University of Toronto Press.

11. G. M. Petersen, paper not yet published.

12. A. Zygmund, *Trigonometric series*, Warszawa, 1935.

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