DAVIS'S CANONICAL PENCILS OF LINES

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1. **Introduction.** The purpose of the present paper is to contribute to the study of two pencils of lines which appear in the development of the theory of conjugate nets on an analytic nonruled surface in ordinary space. The surface is herein referred to its asymptotic curves as parametric.

In his Chicago doctoral dissertation, *Contributions to the theory of conjugate nets*, W. M. Davis defined and studied several canonical configurations, considering the conjugate net as parametric. Among these configurations there are two pencils of lines, called Davis's first and second canonical pencils, which are studied in this paper.

Investigation is made of certain polar relations of the lines of the two pencils with respect to the bundle of quadrics each of which has contact of at least the third order with both curves of a conjugate net at a point. Certain loci which arise in the relation of the two pencils of lines to a pencil of conjugate nets at a point on a surface are studied. A generalization of Davis's canonical quadric is made, and some properties of the resulting one-parameter family of quadrics are demonstrated. Theorems relating to all the lines of one or both of the canonical pencils are shown to include theorems previously proved about certain particular lines.

2. **Analytic basis.** Let the projective homogeneous coordinates x_1, x_2, x_3, x_4 of a point P_x [3, pp. 89, 105-113, 115-120, 180-190], on an analytic nonruled surface S in ordinary space be given by the parametric vector equation $x = x(u, v)$. Suppose the surface to be referred to its asymptotic net. Then P_x satisfies two partial differential equations of the form

$$(2.1)\qquad \begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \end{aligned} \qquad (\theta = \log \beta\gamma).$$

Let x_1, x_2, x_3, x_4, the local point coordinates of P_x, be referred to the tetrahedron of reference whose vertices are the points x, x_u, x_v, x_{uv}.

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A conjugate net N_λ on an integral surface of equations (2.1) may be represented by a curvilinear differential equation of the form

$$dv^2 - \lambda^2 \, du^2 = 0 \qquad (\lambda \neq 0), \tag{2.2}$$

and the associate conjugate net by a similar equation of the form

$$dv^2 + \lambda^2 \, du^2 = 0 \qquad (\lambda \neq 0). \tag{2.3}$$

The class of all conjugate nets on the surface S each of which has the property that at each point of the surface its two tangents form with the tangents of the net (2.2) the same ratio constitutes a pencil of nets whose equation may be written

$$dv^2 - \lambda^2 h^2 \, du^2 = 0 \qquad (\lambda h \neq 0). \tag{2.4}$$

In equations (2.2), (2.3), and (2.4), the coefficient λ is a function of u and v, while h is a constant independent of u and v.

The bundle (linear two-parameter family) of quadrics each of which has contact of at least the third order with both curves of the conjugate net (2.2) at the point P_x has been shown [5, pp. 700, 701] to have the following equation:

$$\begin{aligned} (2.5) \quad & A\left(x_2 x_3 - x_1 x_4 - \frac{\gamma\lambda^2 x_2 x_4}{3} - \frac{\beta x_3 x_4}{3\lambda^2}\right) \\ & + B\left[\left(\theta_u - \frac{\lambda_u}{\lambda} + \gamma\lambda^2\right) x_3 x_4 - \lambda^2 \left(\theta_v + \frac{\lambda_v}{\lambda} + \frac{\beta}{\lambda^2}\right) x_2 x_4 - \lambda^2 x_2^2 + x_3^2\right] \\ & + C x_4^2 = 0 . \end{aligned}$$

If $B = 0$, equation (2.5) reduces to the equation of a pencil of quadrics having second order contact with the surface S at P_x:

$$A\left(x_2 x_3 - x_1 x_4 - \frac{1}{3}\gamma\lambda^2 x_2 x_4 - \frac{\beta x_3 x_4}{3\lambda^2}\right) + C x_4^2 = 0. \tag{2.6}$$

3. **The lines $l_1(k)$ and $l_2(k)$.** Two types of congruences, Γ_1 and Γ_2, will be referred to in this paper [3, pp. 150-155]. A line l_1 at a point P_x on the surface may be regarded as determined by the points x: $(1, 0, 0, 0)$ and y: $(0, -a, -b, 1)$, and also by the planes $x_3 + b\, x_4 = 0$, $x_2 + a\, x_4 = 0$. If a and b are

functions of u and v, the lines l_1 constitute a congruence Γ_1. The reciprocal line l_2 is the polar line of the line l_1 with respect to the quadric of Lie and may be regarded as determined by the points $\rho:(-b, 1, 0, 0)$ and $\sigma:(-a, 0, 1, 0)$, and also by the planes $x_4 = 0$, $x_1 + bx_2 + ax_3 = 0$. As in the case of the lines l_1, the lines l_2 constitute a congruence Γ_2 if a and b are functions of u and v.

The lines l_1 and l_2 to be considered here are those for which

$$a = \frac{1}{2}\left(\theta_v + \frac{\lambda_v}{\lambda} + k\,\frac{\beta}{\lambda^2}\right),\quad b = \frac{1}{2}\left(\theta_u - \frac{\lambda_u}{\lambda} + k\gamma\lambda^2\right),$$

where k is a parameter independent of u and v. These values of a and b are hereinafter denoted by a_k and b_k, respectively. The lines l_1 and l_2 characterized by a_k and b_k are denoted by $l_1(k)$ and $l_2(k)$. Two lines $l_1(k)$ or $l_2(k)$ are called *associate lines* if they are characterized by values of k which are numerically equal but opposite in sign. The pair of associate lines $l_1(1)$ and $l_1(-1)$ are the *axis* and the *associate axis*, respectively, of the conjugate net (2.2), whereas $l_2(1)$ and $l_2(-1)$ are the *associate ray* and the *ray*, respectively. The lines $l_2(5/3)$ and $l_2(-5/3)$ are the *principal join* and the *associate principal join*, respectively. The lines $l_1(0)$ and $l_2(0)$ are the *cusp-axis* and the *flex-ray* of the pencil of conjugate nets (2.4), while the lines $l_1(\infty)$ and $l_2(\infty)$ are *Davis's first* and *second canonical tangents*, respectively.

In asymptotic parameters, Davis's canonical plane [2, pp. 17, 18] may be represented by the equation [5, pp. 703, 706]:

$$\gamma\lambda^2 x_2 - \frac{\beta x_3}{\lambda^2} - \frac{1}{2}\left[\frac{\beta}{\lambda^2}\left(\theta_u - \frac{\lambda_u}{\lambda}\right) - \gamma\lambda^2\left(\theta_v + \frac{\lambda_v}{\lambda}\right)\right]x_4 = 0. \tag{3.1}$$

Substitution of the coordinates of the points x: $(1, 0, 0, 0)$ and y: $(0, -a_k, -b_k, 1)$ in this equation shows that the lines $l_1(k)$ lie in Davis's canonical plane and hence in Davis's first canonical pencil, the pencil of lines in the canonical plane with center at P_x. Likewise, in asymptotic parameters the coordinates of Davis's canonical point are:

$$\begin{aligned} x_1 &= \frac{\beta}{2\lambda^2}\left(\theta_u - \frac{\lambda_u}{\lambda}\right) - \frac{\gamma\lambda^2}{2}\left(\theta_v + \frac{\lambda_v}{\lambda}\right), \\ x_2 &= -\frac{\beta^2}{\lambda^2},\quad x_3 = \gamma\lambda^2,\quad x_4 = 0. \end{aligned} \tag{3.2}$$

Substitution of these coordinates in the equations

$$x_4 = 0, \quad x_1 + b_k x_2 + a_k x_3 = 0$$

shows that the lines $l_2(k)$ pass through the point (3.2) and thus lie in Davis's second canonical pencil, the pencil of lines in the tangent plane at P_x with center at the canonical point.

4. **Polar relationships**. This section deals with some of the polar relationships of the lines $l_1(k)$ and $l_2(k)$ with respect to the bundle of quadrics (2.5) and the pencil of quadrics (2.6).

It is known [5, p. 702] that the locus of the polar line of a line l_2 of the congruence Γ_2 with respect to the quadrics of the bundle (2.5) is a quadric cone. For the lines $l_2(k)$ the equation of this cone may be written in the following form:

$$\lambda^2(x_2 + a_1 x_4)(x_2 + a_{k-2/3} x_4) + (x_3 + b_1 x_4)(x_3 + b_{k-2/3} x_4) = 0. \tag{4.1}$$

Evidently, the axis and the line $l_1(k - 2/3)$ lie on this cone. Thus we have proved the following theorem:

THEOREM 1. *The cone which is the locus of the polar lines of any line $l_2(k)$ with respect to the quadrics of the bundle* (2.5) *is intersected by Davis's canonical plane in the axis of the conjugate net* (2.2) *and in the line $l_1(k - 2/3)$.*

In particular, for $l_2(5/3)$ equation (4.1) may be written

$$\lambda(x_2 + a_1 x_4) = \pm i(x_3 + b_1 x_4).$$

Hence, for the principal join the cone (4.1) is composed of two planes which intersect in the axis of the net N_λ.

The polar line of $l_2(k)$ with respect to the pencil of quadrics (2.6) is the line $l_1(k - 2/3)$, whose equations are

$$x_2 + a_{k-2/3} x_4 = 0, \quad x_3 + b_{k-2/3} x_4 = 0.$$

The point through which pass the polar planes of a point P_y not on the tangent plane with respect to all the quadrics of the bundle (2.5) has been shown [5, p. 703] to have the following coordinates:

$$
(4.2)\qquad \begin{aligned}
x_1 &= \lambda^2 y_2^2 + y_3^2 - \frac{1}{3}(\gamma\lambda^2 b_1 + \beta a_1) y_4^2 + \lambda^2 a_{1/3} y_2 y_3 + b_{1/3} y_3 y_4,\\
x_2 &= y_4(y_3 + b_1 y_4), \quad x_3 = \lambda^2 y_4 (y_2 + a_1 y_4), \quad x_4 = 0.
\end{aligned}
$$

It has also been shown that for P_y on the axis of the net N_λ this point is indeterminate and all the polar planes pass through the principal join.

Consider, now, P_y to be any point on a line $l_1(k)$. The coordinates (4.2) may then be written

$$
\begin{aligned}
x_1 &= -\frac{1}{2}(1-k)\left[\frac{\beta}{\lambda} a_{k+2/3} + \gamma\lambda\, b_{k+2/3}\right],\\
x_2 &= \frac{1}{2}(1-k)\gamma\lambda, \quad x_3 = \frac{\beta}{2\lambda}(1-k), \quad x_4 = 0.
\end{aligned}
$$

If $k = 1$, then $l_1(k)$ is the axis, as was discussed in the preceding paragraph. If $k \neq 1$, then the coordinates (4.2) become

$$
(4.3)\qquad \begin{aligned}
x_1 &= -\left[\frac{\beta}{\lambda} a_{k+2/3} + \gamma\lambda\, b_{k+2/3}\right],\\
x_2 &= \gamma\lambda, \quad x_3 = \frac{\beta}{\lambda}, \quad x_4 = 0.
\end{aligned}
$$

This is the point of intersection of the line

$$
\frac{\beta x_2}{\lambda} - \gamma\lambda x_3 = 0, \quad x_4 = 0
$$

and the line $l_2(k + 2/3)$. Hence we have the following result:

THEOREM 2. *The polar planes of the points on a line* $l_1(k)$ *with respect to the quadrics of the bundle* (2.5) *pass through the point* (4.3), *which is the point of intersection of the line* $(\beta/\lambda)x_2 - \gamma\lambda x_3 = 0$, $x_4 = 0$ *and the line* $l_2(k + 2/3)$.

5. Certain loci and an envelope. The associate conjugate tangents intersect any line $l_2(k)$ in the points

$$
(5.1)\qquad \left(-b_k \mp i\lambda a_k,\ 1,\ \pm i\lambda,\ 0\right),
$$

and the conjugate tangents intersect the associate line $l_2(-k)$ in the points

$$(5.2) \qquad \left(- b_{-k} \mp \lambda\, a_{-k},\ 1,\ \pm\, \lambda,\ 0\right).$$

Homogeneous elimination of the parameter h yields, for each of the points (5.1) and (5.2), the equations of a cubic in the tangent plane:

$$(5.3) \qquad l x_2\, x_3 - k\left(\beta x_2^3 + \gamma x_3^3\right) = 0, \quad x_4 = 0,$$

where

$$l = 2\, x_1 + \left(\theta_u - \frac{\lambda_u}{\lambda}\right) x_2 + \left(\theta_v + \frac{\lambda_v}{\lambda}\right) x_3\,.$$

Thus we have proved the following theorem:

THEOREM 3. *The locus of the points of intersection of the associate conjugate tangents with the line $l_2(k)$ and of the points of intersection of the conjugate tangents with the line $l_2(-k)$ is a cubic in the tangent plane with equations* (5.3).

For $k = 5/3$ and $k = 1$, equations (5.3) represent the principal cubic and the ray-point cubic, respectively [5, p. 705].

It is obvious that for the pencil of lines $l_2(k)$ the equations (5.3) represent a pencil of cubics in the tangent plane. This pencil constitutes a subclass of those studied by Bell [1, p. 401] in the investigation of the locus of the point $K_{\overline{\lambda},j,\overline{k}}$, the point of intersection of the tangent $t_{\overline{\lambda}}$ with the $R_{\overline{\lambda},j,\overline{k}}$-associate of a line l. In the notation of this paper, the point $K_{\overline{\lambda},j,\overline{k}}$ coincides with the point of intersection of $l_2(-k)$ with the tangent $t_{h\lambda}$ if $\overline{\lambda} = h\lambda$, $j = \overline{k} = k/2$, and l is the flex-ray of the pencil (2.4).

A line $l_2(k)$ intersects the cubic (5.3) in the points (5.1) and in the point

$$(5.4) \qquad \left(- b_k\, \gamma\lambda + \frac{a_k \beta}{\lambda},\ \gamma\lambda,\ -\frac{\beta}{\lambda},\ 0\right),$$

which also lies on the line

$$(5.5) \qquad \frac{\beta x_2}{\lambda} + \gamma\lambda\, x_3 = 0, \quad x_4 = 0.$$

The line (5.5) is the R_λ-correspondent [1, pp. 392, 393] of the tangent t_λ of the net (2.2). The line $l_2(\infty)$, Davis's second canonical tangent, intersects the line $l_2(k)$ in the point

$$\left(-\frac{b_k\,\beta}{\lambda^2} - a_k\,\gamma\lambda^2,\ \frac{\beta}{\lambda^2},\ \gamma\lambda^2,\ 0\right). \tag{5.6}$$

The points (5.4) and (5.6) separate harmonically the points (5.1).

Similarly, the line $l_2(-k)$ intersects the cubic (5.3) in the points (5.2) and in a third point which lies on the line

$$\frac{\beta x_2}{\lambda} - \gamma\lambda\, x_3 = 0, \quad x_4 = 0. \tag{5.7}$$

The line $l_2(\infty)$ intersects $l_2(-k)$ in the point

$$\left(-b_{-k}\,\frac{\beta}{\lambda^2} - a_{-k}\,\gamma\lambda^2,\ \frac{\beta}{\lambda^2},\ \gamma\lambda^2,\ 0\right). \tag{5.8}$$

The latter two points separate harmonically the points (5.3).

The lines (5.5) and (5.7) are the tangents of the R_λ-derived conjugate net [1, pp. 392, 393].

Thus we have established the following result.

THEOREM 4. *A line $l_2(k)$ intersects the cubic* (5.3) *in three points, one of which lies on each of the associate conjugate tangents and the third of which lies on the R_λ-correspondent of the tangent t_λ. The two points on the associate conjugate tangents are separated harmonically by the point on the R_λ-correspondent and the point of intersection of $l_2(k)$ by Davis's second canonical tangent.*

A similar statement expresses the relationship among the cubic (5.3), a line $l_2(-k)$, the conjugate tangents, the other tangent of the R_λ-derived conjugate net, and the line $l_2(\infty)$.

It may be pointed out here that since the ray, the associate ray, the flex-ray, and any other $l_2(k)$ are characterized by the values of k: $(-1, 1, 0, k)$, the cross-ratio of these lines, in the order named, is $(1-k)/(1+k)$. Lane has discussed [4, p. 699] this cross-ratio for $k = 5/3$ and $k = -5/3$, for which $l_2(k)$ represents the principal join and the associate principal join, respectively. For the lines $l_2(3)$ and $l_2(-3)$, Davis's line and associate line of collineation, the cross-ratios are $-1/2$ and 2, respectively.

The equations of any line $l_2(k)$ for a general net of the pencil (2.4) are

$$x_1 + \frac{1}{2}\left(\theta_u - \frac{\lambda_u}{\lambda} + k\,\gamma\lambda^2 h^2\right) x_2 + \frac{1}{2}\left(\theta_v + \frac{\lambda_v}{\lambda} + \frac{k\,\beta}{\lambda^2 h^2}\right) x_3 = 0, \quad x_4 = 0.$$

By the usual method the envelope of this line is found to be

$$4k^2\ \beta\gamma x_2 x_3 - l^2 = 0, \qquad x_4 = 0, \tag{5.9}$$

where l is defined as for equations (5.3). Equations (5.9) represent a conic in the tangent plane for a particular value of k, and a pencil of conics if k is allowed to vary. Among the conics of this pencil which have been studied before [5, p. 705] are the ray conic and the principal conic.

6. **Generalization of Davis's canonical quadric.** This section is devoted to the study of the equation of a quadric at a point P_x of a conjugate net having the following properties:

1. The quadric has second-order contact with the surface sustaining the net.
2. A line $l_1(k)$ of the first canonical pencil and the line $l_2(-k)$ are reciprocal polars with respect to the quadric.
3. The quadric passes through the point P_y which is the harmonic conjugate of the point P_x with respect to the focal points of the line $l_1(k)$.

The equation of the most general noncomposite quadric surface having second-order contact with a surface (2.1) at P_x can be written [3, p. 142] thus:

$$x_2 x_3 - x_1 x_4 + k_2 x_2 x_4 + k_3 x_3 x_4 + k_4 x_4^2 = 0.$$

Demanding that $l_1(k)$ and $l_2(-k)$ be reciprocal polars with respect to this quadric yields the following result:

$$k_2 = k\gamma\lambda^2, \quad k_3 = k\beta/\lambda^2.$$

Demanding that the quadric pass through the point P_y which is the harmonic conjugate of P_x with respect to the focal points of $l_1(k)$ requires that

$$\begin{aligned} k_4 = &-\frac{1}{2}\theta_{uv} + \frac{1}{4}k\gamma_v\lambda^2 + \frac{1}{4}k\frac{\beta_u}{\lambda^2} + \frac{1}{2}k\gamma\lambda^2\left(\theta_v + 2\frac{\lambda_v}{\lambda}\right) \\ &+ \frac{1}{2}k\frac{\beta}{\lambda^2}\left(\theta_u - 2\frac{\lambda_u}{\lambda}\right) + (k^2-1)\beta\gamma. \end{aligned} \tag{6.1}$$

Hence, if k is allowed to vary, we have the equation of a one-parameter family of quadrics:

$$x_2 x_3 - x_1 x_4 + k\,\gamma\lambda^2 x_2 x_4 + k\,\frac{\beta}{\lambda^2}\,x_3 x_4 + k_4 x_4^2 = 0, \tag{6.2}$$

where k_4 is defined by (6.1).

For $k = 1$ and $k = -1$, equation (6.2) represents Davis's canonical quadric and associate canonical quadric, respectively. For $k = 0$, the line l_1 is the cusp-axis, the line l_2 is the flex-ray, and the quadric (6.2) is the quadric of Darboux for which $k_4 = -(1/2)(2\beta\gamma + \theta_{uv})$. This quadric has been investigated by Môri [6, p. 59]. If $k = \infty$, the line l_1 is the first canonical tangent, and the quadric is composite, the two planes intersecting in the first canonical tangent.

A short computation leads to the following result:

Davis's canonical point and canonical plane are pole and polar with respect to any quadric of the family (6.2).

Two quadrics of this family may be called associate if they are characterized by values of k which are equal numerically but opposite in sign. For $k = 0$, the associate quadrics coincide. For any two associate quadrics for which $k \neq 0$, the plane of the residual conic of intersection has the equation

$$\begin{aligned} &2\gamma\lambda^2 x_2 + 2\frac{\beta}{\lambda^2} x_3 + \left[\frac{1}{2}\gamma_v \lambda^2 + \frac{1}{2}\frac{\beta_u}{\lambda^2}\right. \\ &\left. + \gamma\lambda^2 (\theta_v + 2\lambda_{v\lambda}) + \frac{\beta}{\lambda^2}\left(\theta_u - 2\frac{\lambda_u}{\lambda}\right)\right] x_4 = 0. \end{aligned} \tag{6.3}$$

Hence we have the theorem:

THEOREM 5. *The residual conics of intersection of pairs of associate quadrics of the family of quadrics* (6.2) *form a pencil of conics lying in the plane* (6.3).

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