ON THE BOUNDARY VALUES OF SOLUTIONS OF THE HEAT EQUATION

W. FULKS

1. Introduction. In a recent paper Hartman and Wintner [3] consider solutions of the heat equation

$$(1) \qquad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle R: $0 < x < 1$ $(0 \leq t < k \leq \infty)$. There they obtain necessary and sufficient conditions for a solution of (1) in R to be representable in the form

$$(2) \qquad u(x, t) = \int_{0+}^{1-0} G(x, t; y, s)\, d\, A(y)$$

$$+ \int_0^t G_y(x, t; 0, s)\, d\, B(s) - \int_0^t G_y(x, t; 1, s)\, d\, C(s),$$

the Green's function G being defined by

$$(3) \qquad G(x, t; y, s) = \frac{1}{2}\left[\vartheta_3\left(\frac{x-y}{2}, t-s\right) - \vartheta_3\left(\frac{x+y}{2}, t-s\right)\right]$$

where ϑ_3 is the well known Jacobi theta function. (The first integral in (2) is an absolutely convergent improper Riemann-Stieltjes integral.) They proceed to show that the functions representable in the form (2) exhibit the following behavior at the boundary of R:

$$(4) \qquad \lim_{t \to 0+} u(x, t) = A'(x),$$

$$(5) \qquad \lim_{x \to 0+} u(x, t) = B'(t), \quad \lim_{x \to 1-0} u(x, t) = C'(t)$$

wherever the derivatives in question exist.

In the present note we present an improvement of (5) first given in the author's thesis [2]. The admittedly slight mathematical improvement is physically significant. A solution of (1) which admits the representation (2) gives the

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temperature at time t and position x in an insulated rod of length unity and with a certain initial temperature distribution, given essentially by (4), and imposed end temperatures, given essentially by (5). We note that such solutions are not uniquely determined by (4) and (5).

As x approaches the boundary of R along a line $t = t_0$, it seems intuitively clear that the limit should be independent of values of B (or C) for $t \geq t_0$. Hence the expected result (for the left side of R) would be

$$\lim_{x \to 0+} u(x, t) = B'(t - 0) = \lim_{h \to 0+} \frac{B(t-h) - B(t-0)}{-h}$$

wherever this derivative exists.

2. **Theorem.** For the above improvement it is sufficient to establish the following result.

THEOREM. *If $B(s)$ is of bounded variation on every closed interval of $0 \leq s < k \leq \infty$, then*

$$\lim_{x \to 0+} \int_0^t G_y(x, t; 0, s)\, dB(s) = B'(t-0)$$

wherever this derivative exists.

Proof. Let

$$u(x, t) = \int_0^t G_y(x, t; 0, s)\, dB(s).$$

Then since

$$\vartheta_3\left(\frac{x}{2}, t\right) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(x+2n)^2}{4t}\right]$$

(see, for example, [1, p. 307]), we can write

$$u(x, t) = \frac{1}{2}\, x \pi^{-1/2} \int_0^t (t-s)^{-3/2} \exp\left[\frac{-x^2}{4(t-s)}\right] dB(s)$$

$$+ \frac{1}{2}\, \pi^{-1/2} \int_0^t (t-s)^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (x+2n) \exp\left[\frac{-(x+2n)^2}{4(t-s)}\right] dB(s).$$

Clearly the latter integral vanishes with x. Then denoting the first integral on the right by I and by setting $z = x^2/4$ and $t - s = 1/v$, we get

$$I = \left(\frac{z}{\pi}\right)^{1/2} \int_{v=1/t}^{\infty} e^{-zv}\, v^{3/2}\, dB(t - 1/v).$$

If we define

$$\alpha(v) = \begin{cases} \displaystyle\int_{r=a}^{v} r^{3/2}\, dB(t - 1/r) & (v \geq 1/t), \\ \alpha(1/t) & (v < 1/t), \end{cases}$$

where a is a suitable constant, then we have

$$I = \left(\frac{z}{\pi}\right)^{1/2} \int_0^{\infty} e^{-zv}\, d\alpha(v).$$

To evaluate $\lim_{z\to\infty} I$ we apply [4, Theorem 1, p. 181], which states: *If*

$$f(s) = \int_0^{\infty} e^{-st}\, d\alpha(t),$$

then for any $\gamma \geq 0$ *any constant* A *we have*

$$\lim_{s\to 0+} |S^{\gamma} f(s) - A| \leq \lim_{t\to\infty} |\alpha(t)\, t^{-\gamma}\, \Gamma(\gamma + 1) - A|.$$

To this end we evaluate $\lim_{v\to\infty} v^{-1/2}\, \alpha(v)$. Now

$$\begin{aligned} v^{-1/2}\, \alpha(v) &= v^{-1/2} \int_{r=a}^{v} r^{3/2}\, dB(t - 1/r) \\ &= v^{-1/2} \int_a^v r^{3/2}\, d[B(t - 1/r) - B(t - 0)] \\ &= r^{3/2}\, v^{-1/2}[B(t - 1/r) - B(t - 0)]\Big|_a^v \\ &\quad + \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)]\, r^{1/2}\, dr \\ &= \frac{B(t - 1/v) - B(t - 0)}{1/v} - \frac{B(t - 1/a) - B(t - 0)}{v^{1/2}}\, a^{3/2} \\ &\quad + \frac{3}{2} v^{-1/2} \int_a^v [B(t - 0) - B(t - 1/r)]\, r^{1/2}\, dr. \end{aligned}$$

As $v \longrightarrow \infty$ the first expression on the right tends to $-B'(t-0)$, if this derivative exists, and the second vanishes. Now consider the integral term: given $\epsilon > 0$, choose T so large that

$$\left| B'(t-0) - \frac{B(t-0) - B(t-1/r)}{1/r} \right| < \epsilon \text{ if } r > T.$$

Then

$$\frac{3}{2} v^{-1/2} \int_{r=a}^{v} [B(t-0) - B(t-1/r)]\, r^{1/2}\, dr$$

$$= \frac{3}{2} v^{-1/2} \int_{r=a}^{T} [B(t-0) - B(t-1/r)]\, r^{1/2}\, dr$$

$$+ \frac{3}{2} v^{-1/2} \int_{r=T}^{v} \frac{B(t-0) - B(t-1/r)}{1/r}\, r^{-1/2}\, dr.$$

The first integral on the right $\longrightarrow 0$ as $v \longrightarrow \infty$, and

$$\frac{3}{2} v^{-1/2} \int_{T}^{v} \frac{B(t-0) - B(t-1/r)}{1/r}\, r^{-1/2}\, dr$$

$$= 3[B'(t-0) + \eta\,(T, v)]\,(v^{1/2} - T^{1/2})\, v^{-1/2},$$

where $|\eta| < \epsilon$ for all values of $v > T$. Let $v \longrightarrow \infty$, then let $\epsilon \longrightarrow 0$; the right side of the above equation approaches $3B'(t-0)$. Consequently we now have

$$\lim_{v \to \infty} v^{-1/2}\, \alpha(v) = 2\, B'(t-0).$$

By applying the above-mentioned theorem with $\gamma = 1/2$, $A = \pi^{1/2}\, B'(t-0)$, we now obtain

$$\overline{\lim_{z \to 0}} \left| z^{1/2} \int_0^\infty e^{-zv}\, d\alpha(v) - \pi^{1/2}\, B'(t-0) \right|$$

$$\leq \overline{\lim_{v \to \infty}} \left| \frac{1}{2} \pi^{1/2}\, v^{-1/2}\, B(v) - \pi^{1/2}\, B'(t-0) \right| = 0.$$

Hence

$$\lim_{x \to 0+} u(x, t) = \lim_{z \to 0} I = B'(t-0).$$

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UNIVERSITY OF MINNESOTA