ON A CHARACTERISTIC PROPERTY OF FINITE SETS

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1. Introduction. There are several equivalent definitions of finite sets [2], [5]. The purpose of this note is to give an equivalent property of finite sets in terms of ramifications of sets.

DEFINITION 1. A partially ordered set $S \equiv (S; \leq)$ is said to be *ramified* or to satisfy the *ramification condition* ([3], pp. 69, 127; cf. 4) provided that for every $x \in S$ the set $(-\infty, x)$ of all $y \in S$ satisfying $y < x$ is totally ordered (that is, contains no distinct noncomparable points). If the points of a ramified set $(S; \leq)$ are the same as these of a set M, one says that $(S; \leq)$ is a *ramification of* M.

DEFINITION 2. A *chain* (*anti-chain*) of a partially ordered set $(S; \leq)$ is any subset of S containing no distinct incomparable (comparable) points. Every set containing a single point is considered both as chain and as anti-chain.

DEFINITION 3. For a partially ordered set $(S; \leq) = S$, we denote by

$$O(S) \quad \text{or} \quad OS \tag{1}$$

the system of all *maximal* chains contained in S; analogously,

$$\bar{O}(S) \quad \text{or} \quad \bar{O}S \tag{2}$$

denotes the system of all maximal anti-chains of S.

THEOREM. *In order that a nonvoid set S be finite, it is necessary and sufficient that for every ramification $T(S)$ of S the relations*

$$M \in O\,T(S), \quad A \in \bar{O}\,T(S) \tag{3}$$

imply

$$M \cap A \neq \Lambda \qquad (\Lambda \equiv \text{vacuous set}). \tag{4}$$

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2. **The condition is necessary.** Otherwise, there would be a finite set S, a ramification $T(S)$, a set $M \in OT(S)$, and a set $A \in \overline{O}T(S)$, such that

$$M \cap A = \Lambda . \tag{5}$$

Now, A is a maximal anti-chain of $T(S)$; consequently, for every $x \in T(S)$ there is a point $a(x) \in A$ such that the set $\{x, a(x)\}$ is a chain of $T(S)$. (Otherwise, the set $A \cup \{x\}$ would be an anti-chain greater than the maximal anti-chain A.)

In particular, for any $x \in M \in O\,T(S)$, the points x, $a(x)$ are comparable. We say that

$$x < a(x). \tag{6}$$

Since M is a maximal chain of the ramified set $T(S)$, M is an initial portion of $T(S)$; that is, M, which contains the point x contains also every point of $T(S)$ preceding x. In particular, if (6) did not hold then M would contain also $a(x) \leq x$; consequently, $a(x) \in M \cap A$, contrary to the assumption (5).

Thus if (5) held then for every $x \in M$ one would have (6); but M, as a non-void subset of the finite set $T(S)$, would have a terminal point, say l; l would be a final point of $T(S)$, too, contrary to the relation (6) for $x = l$. Thus the relation (5) is not possible.

3. **The condition is sufficient.** If for every ramification $T(S)$ the relations (3) imply (4), then the set S is finite. Otherwise, the set S would be infinite; consequently, there would be a one-to-one correspondence ϕ of the set N of all natural numbers into S. Now, let us define the ordering $(S; \leq)$ by transplantation of a certain order of the set N. We shall order N according to the scheme [1],

$$\begin{tikzcd}[column sep=small, row sep=small]
1 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & 5 \arrow[r] \arrow[d] & 7 \arrow[r] \arrow[d] & \cdots \\
2 & 4 & 6 & 8 & \cdots .
\end{tikzcd}$$

That is, the set $2N-1$ of all $2n-1$ $(n \in N)$ is ordered as in the natural order; for every $n \in N$, the set of numbers preceding $2n$ consists of the numbers $2\nu - 1$ $(\nu = 1, 2, \cdots, n)$; all other couples of natural integers are incomparable, by definition. In the ramified set N_0 so obtained one sees that $2N \in \overline{O}N_0$, that $2N-1 \in ON_0$, and that the sets $2N$, $2N-1$ are disjoint. Now, the set S being infinite by hypothesis, there is a one-to-one mapping ϕ of $N = N_0$ into S.

[1] For the definition of schemes or diagrams of partly ordered sets see Birkhoff [1, p.6].

That enables us to define the order in S by transporting the order of N_0 into S so that, on the one hand, the mapping ϕ is a similitude between N_0 and $\phi N_0 \subseteq S$, and so that, on the other hand, no point of ϕN_0 is comparable to any point of $S \setminus \phi N_0$, and so that $S \setminus \phi N_0$ contains no comparable couple of distinct points.

It is obvious that the set $(S; \leq)$ is ramified, that the set $\phi(2N-1)$ is a maximal chain of $(S; \leq)$, and that the set $A = \phi(2N) \cup (S \setminus \phi N)$ is a maximal anti-chain of $(S; \leq)$.

According to (4), the set $A \cap \phi(2N-1)$ would be nonvacuous, contrary to the fact that the sets A, $\phi(2N-1)$ are disjoint.

Thus, the proof of the theorem is completed.

4. **Observation.** We observe that the condition of ramification in the statement of the theorem is essential. Namely, if we consider the partially ordered set $S_1 = \{1, 2, 3, 4, 5\}$ with the diagram

$$\begin{tikzcd} 5 & 4 \\ & 3 \arrow[u] \arrow[ul] \\ 1 \arrow[uu] & 2 \arrow[u] \end{tikzcd}$$

it is obvious that $\{2, 3, 5\}$ is a maximal chain of S, that $\{1, 4\}$ is a maximal anti-chain of S, and that the set $\{2, 3, 5\}$ does not intersect the set $\{1, 4\}$.

5. **Questions.** In connection with the statement of the theorem it is interesting to consider the following two questions:

QUESTION 1. Is there a partially ordered nonvacuous set S such that there is no maximal anti-chain $A \in \bar{O}S$ satisfying $A \cap M \neq \Lambda$ for every maximal chain $M \in OS$?

QUESTION 2. Is there a partially ordered nonvacuous set S such that there is no maximal chain $M \in OS$ satisfying $M \cap A \neq \Lambda$ for every maximal anti-chain $A \in \bar{O}S$?

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