CAPACITY, VIRTUAL MASS, AND GENERALIZED SYMMETRIZATION

L. E. PAYNE AND ALEXANDER WEINSTEIN

1. **Introduction.** A body of revolution B can be symmetrized with respect to its axis of symmetry in a number of ways. One of these is the Schwarz symmetrization, which preserves the volume of B. Another is the Steiner symmetrization of the meridian section of B, which preserves the area of this section but in general decreases the volume. The influence of the Schwarz symmetrization on the capacity has been investigated by G. Pólya and G. Szegő, [1]. More recently P. R. Garabedian and D. C. Spencer [2] discussed the same question for the virtual mass of bodies of revolution. In the present paper we shall study by a different and simpler method the behavior of the capacity and virtual mass under a more general type of symmetrization, which includes the Schwarz and Steiner symmetrizations as particular cases.

2. **Definitions.** Let the (x, y)-plane be the meridian plane of B, the x-axis being the axis of symmetry. The part of the meridian section of B which lies in the upper half plane $y \geq 0$ is denoted by D. The complement of D in the half plane is designated as E. We assume that D is simply connected and that E is a connected domain. The boundary of D consists in general of a segment of the x-axis and a line L. We exclude the case where L is a closed curve and lies entirely above the x-axis, as is the case in which B is a torus. We assume L to have at most a finite number of angular points.

We shall use in this paper some recent results of axially symmetric potential theory in n-dimensional space. This theory which is of mathematical interest in itself will be used here mainly as a tool to obtain results for bodies of revolution in three dimensions.

Let us henceforth consider our (x, y)-plane as the meridian plane of a body of revolution $B[n]$ in n-dimensions, $n = 3, 4, 5, \cdots$. We assume that $B[n]$ has the same meridian section D as our three-dimensional body $B = B[3]$. All quantities considered hereafter are defined in the meridian plane and therefore are functions of x and y only. Actually we shall never use $B[n]$ but only its meridian section.

Let $\phi(x, y)$ be an axially symmetric potential function defined for $y \geq 0$ and let $\psi(x, y)$ be the corresponding stream function. We have then the generalized Stokes-Beltrami equations

$$(1) \qquad y^{n-2} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad y^{n-2} \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x} .$$

To emphasize the dependence of ϕ and ψ on n we shall often use the notations $\phi[n]$ and $\psi[n]$.

The volume $V[n]$ of $B[n]$ is given by

$$(2) \qquad V[n] = \omega_{n-1} \iint_D y^{n-2} \, dx \, dy,$$

where $\omega_h = 2\pi^{h/2} / \Gamma(h/2)$. We introduce the capacity $C[n]$ of $B[n]$ by the formula

$$(3) \qquad C[n] = \frac{\omega_{n-1}}{\omega_n} \iint_E y^{n-2} (\operatorname{grad} \phi[n])^2 \, dx \, dy,$$

where $\phi[n]$ is a potential which assumes the value unity on L and vanishes as $(x^2 + y^2)^{-(n-2)/2}$ at infinity. It is obvious that (3) reduces to the classical definition of the capacity for $n = 3$.

We define also the virtual mass $M[n]$ of $B[n]$ by the formula

$$(4) \qquad M[n] = \omega_{n-1} \iint_E y^{-(n-2)} (\operatorname{grad} \psi[n])^2 \, dx \, dy.$$

The function $\psi[n]$ in (4) assumes the value $y^{n-1}/(n-1)$ on L and vanishes at infinity like $y^{n-1} (x^2 + y^2)^{-n/2}$. Our definition of the virtual mass generalizes that of P. R. Garabedian and D. C. Spencer [2].

3. **The correspondence principle and the fundamental formula.** We use here a relationship due to A. Weinstein [4],

$$(5) \qquad \psi[n] = y^{n-1} (n-1)^{-1} \phi[n + 2].$$

This equation shows that to each stream function $\psi[n]$ corresponds a well-defined potential $\phi[n + 2]$. In particular to the stream function $\psi[n]$ in formula (4) corresponds a potential $\phi[n + 2]$ which assumes the value unity on L and vanishes as $(x^2 + y^2)^{-n/2}$ at infinity. In other words $\phi[n + 2]$ is the electrostatic potential of $B[n + 2]$. The substitution of (5) into (4) leads after an

elementary integration by parts to the fundamental formula

$$(6) \qquad M[n] + V[n] = \pi^{n/2} \left[(n-1) \, \Gamma\!\left(\frac{n}{2} + 1\right) \right]^{-1} C[n+2],$$

which we shall employ here in the study of the behavior of $M[n]$.

4. Generalized symmetrization. A line $x = $ constant, $y \geq 0$, intersects L in m points $y_1(x) > y_2(x) > y_3(x) > \cdots > y_m(x) > 0$. The number of intersections m usually depends on x. Let us consider the line L_q defined by the equation

$$(7) \qquad y^q(x) = \sum_{k=1}^{m} (-1)^{k-1} \, y_k^q(x),$$

where q is a positive constant not necessarily an integer. The body of revolution $B_q[n]$ with section D_q defined by its profile L_q is said to be obtained by a symmetrization S_q. Let us note that S_{n-1} can be considered as a Schwarz symmetrization of $B[n]$. On the other hand, under S_1 the meridian profile of $B[n]$ undergoes a Steiner symmetrization. Our main results are embodied in the following theorems:

I. $V[n]$ does not increase under S_q for $0 < q \leq n-1$ and does not decrease under S_q for $q \geq n-1$. In particular, $V[n]$ remains invariant under S_{n-1}.

II. $C[n]$ does not increase under S_q for $0 < q \leq n-1$.

III. $M[n]$ does not increase under S_q for $n-1 \leq q \leq n+1$.

Let us observe that by (6) Theorem III follows immediately from I and II. In order to prove Theorems I and II we shall first establish some useful inequalities.

5. Fundamental inequalities. Let $y_1 > y_2 > \cdots > y_m > 0$ and let q and s be two positive numbers. We have then

$$(8) \qquad \left[\sum_{k=1}^{m} (-1)^{k-1} \, y_k^q \right]^{1/q} \leq \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{q+s} \right]^{1/(q+s)} \leq \left[\sum_{k=1}^{m} y_k^q \right]^{1/q}.$$

To prove the second inequality of (8) let us observe that it is sufficient to show that

$$(9) \qquad \left[\sum_{k=1}^{m} y_k^{q+s} \right]^{1/(q+s)} \leq \left[\sum_{k=1}^{m} y_k^{q} \right]^{1/q} .$$

Let us put $y_k^q = a_k$ and $(q + s)/q = r > 1$. Then we need only show that

$$(10) \qquad a_1^r + a_2^r + \cdots + a_m^r \leq (a_1 + a_2 + \cdots + a_m)^r.$$

But this is a classical inequality [5, p. 32]. As to the first part* of (8) we give here a proof communicated to us by H. F. Weinberger [7]. This inequality does not seem to be mentioned in the available literature. Using again the notations in (10) and putting

$$(11) \qquad F(a_1, a_2, \cdots, a_m) = \sum_{k=1}^{m} (-1)^{k-1} a_k^r - \left[\sum_{k=1}^{m} (-1)^{k-1} a_k \right]^r ,$$

we have to prove that, for $a_1 > a_2 > \cdots > a_m > 0$ and $r > 1$,

$$(12) \qquad F(a_1, a_2, \cdots, a_m) \geq 0 .$$

This inequality is obviously true for $m = 1$ and follows immediately if $m = 2$ from inequality (10). Let us therefore assume that (12) holds if we replace m by $m - 2$; this is equivalent to assuming the inequality

$$(13) \qquad F(a_2, a_2, a_3, \cdots, a_m) \geq 0 .$$

We have also

$$(14) \qquad F(a_1, a_2, \cdots, a_m) = F(a_2, a_2, a_3, \cdots, a_m) + \int_{a_2}^{a_1} \frac{\partial F}{\partial a_1} \, da_1 .$$

But from (11) we observe that

$$(15) \qquad \frac{1}{r} \frac{\partial F}{\partial a_1} = a_1^{r-1} - \left[\sum_{k=1}^{m} (-1)^{k-1} a_k \right]^{r-1} ,$$

which shows that $\partial F/\partial a_1$ is nonnegative. Since the same holds by assumption for $F(a_2, a_2, a_3, \cdots, a_m)$ we obtain at once the required inequality (12).

---

*R. Bellman has pointed out that this inequality holds more generally with y^r replaced by an arbitrary continuous convex function $f(y)$ defined for $y \geq 0$.

6. **The effect of the generalized symmetrization on** $V[n]$. It follows immediately from (2) that

$$(16) \qquad V[n] = \omega_{n-1} (n - 1)^{-1} \int_\alpha^\beta \left[\sum_{k=1}^m (-1)^{k-1} y_k^{n-1} \right] dx,$$

where the integral is taken over the interval (α, β) bounded by the greatest and smallest values of x on L. Let us apply the symmetrization S_q defined by (7). The volume $V_q[n]$ is then given by

$$(17) \qquad V_q[n] = \omega_{n-1} (n - 1)^{-1} \int_\alpha^\beta \left[\sum_{k=1}^m (-1)^{k-1} y_k^q \right]^{n-1/q} dx.$$

By (8) we see that for $q \leq n - 1$ we have

$$(18) \qquad V_q[n] \leq \omega_{n-1} (n - 1)^{-1} \int_\alpha^\beta \left[\sum_{k=1}^m (-1)^{k-1} y_k^{n-1} \right] dx = V[n].$$

On the other hand for $q \geq n - 1$ we have again by (8)

$$(19) \qquad V_q[n] \geq \omega_{n-1} (n - 1)^{-1} \int_\alpha^\beta \left[\sum_{k=1}^m (-1)^{k-1} y_k^{n-1} \right] dx = V[n].$$

The formulas (18) and (19) establish the proof of Theorem 1 of $\S$ 4.

7. **The effect of generalized symmetrization on** $C[n]$. In studying the behavior of $C[n]$ under the symmetrization S_q we shall generalize to a certain extent the procedure given by Pólya and Szegő for the Steiner symmetrization [1, p. 182]. Let us introduce a Cartesian system (x, y, z) and consider a surface $z(x, y)$ defined in a large half circle A enclosing D. We assume $z(x, y)$ to be a function positive throughout A and vanishing on the circular portion of its boundary. The particular function z which we shall consider will assume a constant positive value z_0 in the subdomain D of A. This value will be the maximum of $z(x, y)$ in A. We further assume that $z(x, y)$ is analytic outside D. The surface $z = z(x, y)$ except for its flat portion may also be defined as a surface $y = y(x, z)$ in a certain domain G of the (x, z)—plane. However, $y(x, z)$ may not be a single-valued function of x and z. For this reason we must consider as in [1] the surfaces $y_k(x, z)$ $(k = 1, 2, \cdots, m)$, where

$$y_1(x, z) > y_2(x, z) > \cdots > y_m(x, z) > 0.$$

These surfaces taken together with the flat portion constitute the surface $z(x, y)$.

Let us consider the integral

$$(20) \qquad I = \iint\limits_{G} \sum_{k=1}^{m} y_k^{n-2}(x, z) \left[1 + \left(\frac{\partial y_k}{\partial x} \right)^2 + \left(\frac{\partial y_k}{\partial z} \right)^2 \right]^{1/2} dx\, dz.$$

Let us first apply the symmetrization S_{n-1} by putting

$$(21) \qquad y_*^{n-1} = \sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1},$$

and consider the integral

$$(22) \qquad I_* = \iint\limits_{G} y_*^{n-2} \left[1 + \left(\frac{\partial y_*}{\partial x} \right)^2 + \left(\frac{\partial y_*}{\partial z} \right)^2 \right]^{1/2} dx\, dz.$$

We prove now that

$$(23) \qquad I \geq I_*.$$

In fact by substituting (21) into (22) and computing $\partial y_*/\partial x$ and $\partial y_*/\partial z$ we obtain the formula

$$(24) \quad I_* = \iint\limits_{G} \left\{ \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-1} \right]^{2(n-2)/(n-1)} + \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-2} \frac{\partial y_k}{\partial x} \right]^2 \right.$$
$$\left. + \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{n-2} \frac{\partial y_k}{\partial z} \right]^2 \right\}^{1/2} dx\, dz.$$

According to the inequality (8), I_* will not diminish if we replace the first square bracket in (24) by $\left[\sum_{k=1}^{m} y_k^{n-2} \right]^2$. Upon applying the Minkowski inequality we find that the integrand in I_* is not greater than the integrand in I; this proves formula (23).

Let us observe that

$$(25) \qquad I = \iint\limits_{A} y^{n-2} \left[1 + z_x^2 + z_y^2 \right]^{1/2} dx\, dy - \iint\limits_{D} y^{n-2} dx\, dy,$$

the last integral being the contribution from the flat part of the z surface. We now insert into (25) the expression $z(x, y) = \epsilon\Phi(x, y)$, ϵ being a small positive number and Φ satisfying the same conditions as z. This substitution yields

$$(26) \quad I = \iint_{A-D} y^{n-2} \, dx\,dy + (\epsilon^2/2) \iint_A y^{n-2} (\Phi_x^2 + \Phi_y^2) \, dx\,dy + O(\epsilon^4).$$

According to inequality (23), I does not increase under S_{n-1}. The first integral in (26) is obviously equal to the same integral taken over the symmetrized domain $A_* - D_*$, where $A_* = A$. By letting ϵ tend to zero we conclude in the usual way [1] that the integral

$$\iint_A y^{n-2} (\Phi_x^2 + \Phi_y^2) \, dx\,dy$$

does not increase under S_{n-1}. If we let the radius of the half circle bounding A tend to infinity we obtain the same statement for a function Φ which vanishes at infinity, providing that the integral converges. In particular if we take for Φ a function which is equal to unity in D and equal to the electrostatic potential $\phi[n]$ in E we find that $C[n]$ does not increase under S_{n-1}.

In order to prove that $C[n]$ does not increase under S_q for $0 < q \leq n - 1$ let us observe that under S_q the line L bounding $D[n]$ goes into a line L_q which has by the inequalities (8) the following property: if $q_1 < q_2$ then the domain $D_{q_1}[n]$ bounded by L_{q_1} has no points outside the domain $D_{q_2}[n]$ bounded by L_{q_2}. We denote the capacities corresponding to these domains by $C_{q_1}[n]$ and $C_{q_2}[n]$, respectively. It is a well known property of the ordinary three-dimensional capacity that if one body contains another body the former has the larger capacity. The proof of this statement is based essentially on the variational definition of the capacity. The same property holds obviously for all values of n. We therefore have $C_{q_1}[n] \leq C_{q_2}[n]$. In particular $C_q[n] \leq C_{n-1}[n]$. As we have already proved $C_{n-1}[n] \leq C[n]$ we obtain the result

$$(27) \qquad C_q[n] \leq C[n], \qquad\qquad\qquad 0 < q \leq n - 1,$$

which concludes the proof of Theorem II of § 4. As already mentioned in § 4, Theorem III follows immediately as a corollary of I and II.

8. **Steiner's Symmetrization of the meridian section with respect to the y-axis.** We shall consider briefly a symmetrization of the domain D with respect to the y-axis defined by the classical equation

$$(28) \qquad\qquad 2x = \sum_{k=1}^{m} (-1)^{k \cdot 1} x_k .$$

In a manner similar to that used in §7 we find that $V[n]$ remains invariant and $C[n]$ and $M[n]$ do not increase under such a symmetrization.

9. **Concluding remarks.** All results of §4 can be extended to the case of two dimensional bodies which are symmetric with respect to the x-axis. It should be noted that these results hold for $C[2]$ as long as the radius of A remains finite. It has already been proven [1, 2] that $C[2]$ and $M[2]$ do not increase under S_1 and also that $C[3]$ and $M[3]$ do not increase under S_2. These cases are included in our Theorems II and III. We note also that formula (6) appears in an equivalent form for $n = 2$ and $n = 3$ in papers by G. I. Taylor [6] and M. Schiffer and G. Szegö [3], where $C[4]$ and $C[5]$ are (up to a constant factor) called dipole coefficients. No attempt was made in these papers to study the behavior of the dipole coefficients under symmetrization. However, it was recognized in [3] that they are increasing set functions, a fact which becomes almost obvious in our theory of generalized electrostatics (see §7). Finally let us remark that in §2 we have introduced the (x, y)-plane as the meridian plane of an n-dimensional space. But since all quantities are defined in terms of x and y, the index n appearing in our formulas need not be restricted to integral values. In fact it can easily be seen that all our formulas and results remain valid for all real positive values of n greater than two. For such values of n our results are mathematical statements about certain integrals such as $V[n]$, $C[n]$, and $M[n]$ which are associated with the generalized Stokes-Beltrami equations.

REFERENCES

1. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies No. 27, Princeton University Press, 1951.

2. P. R. Garabedian and D. C. Spencer, *Extremal methods in cavitational flow*, Technical Report No. 3, Applied Mathematics and Statistics Laboratory, Stanford University, 1951.

3. M. Schiffer and G. Szegö, *Virtual mass and polarization*, Trans. Amer. Math. Soc. 67 (1949), 130-205.

4. A. Weinstein, *On cracks and dislocations in shafts under torsion*, Quart. Appl. Math. 10 (1952), 77-81; *The method of singularities in the physical and in the hodograph plane*, Fourth Symp. Appl. Math. (in print).

5. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge, England, 1934.

6. G. I. Taylor, *The energy of a body moving in an infinite fluid with an application to airships*, Proc. Roy. Soc. London Ser. A, 120 (1928), 21-33.

7. H. Weinberger, *An inequality with alterating signs*, Proc. Nat. Acad. Sci., U.S.A. 38 (1952), 611-613.

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS
UNIVERSITY OF MARYLAND