EVALUATION OF AN INTEGRAL OCCURRING IN SERVOMECHANISM THEORY

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1. Introduction. In the study of dynamical systems in general, and servomechanisms in particular, it is often required to determine the (constant) coefficients in a linear, ordinary, differential equation in such a way as to minimize an integral involving the square of the difference between the solution of the equation and a known function. The latter may be given in either analytical or numerical form. In the design of a servomechanism the known function is the "input"; the solution of the equation is the "output"; and the coefficients of the equation are the circuit constants to be determined. A similar problem arises in the study of aircraft flight records, in which the known function is any of the dynamic variables used to describe the motion, and the coefficients are the so-called aerodynamic derivatives, the determination of which is the purpose of the flight.

Mathematically similar problems also arise in the analysis of a mixture of radioactive substances or of bacteria. The known function is, say, the total weight of the mixture as a function of time, and the unknown coefficients are the relative weights of the different substances initially present.

All such problems can be solved by the method of least squares, and the procedure always leads, at a certain stage, to the evaluation of an integral of a particular type. This integral has been studied by R. S. Phillips [3, Chap. 7, §7.9], who has given a procedure for its evaluation and a short table of results. The purpose of the present note is to derive a simple, explicit formula for this integral.

2, Evaluation of the integral. The integral to be evaluated is

$$I = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{g(x)}{h(x)\,h(-x)}\, dx, \tag{1}$$

where

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$$i = \sqrt{-1},$$

$$g(x) = \sum_{k=1}^{n} g_k \, x^{2(n-k)},$$

$$h(x) = \sum_{k=0}^{n} a_k \, x^{n-k}, \ a_k \text{ real}, \ a_0 \neq 0.$$

There is no loss of generality in restricting $g(x)$ to contain only even powers, since odd powers would make no contribution to the value of the integral. It is assumed that the zeros of $h(x)$ are all distinct and have their real parts negative. Then the integration can be performed immediately by means of the theory of residues [4, Chap. 6], and the result is

$$I = \sum_{k=1}^{n} A_k, \tag{2}$$

where A_k is the residue of the integrand at x_k, and $h(x_k) = 0$. This expression can be evaluated in terms of the coefficients g_k and a_k by starting with the obvious identity

$$\frac{g(x)}{h(x)\,h(-x)} \equiv \sum_{k=1}^{n} A_k \left(\frac{1}{x - x_k} - \frac{1}{x + x_k} \right).$$

Clearing fractions gives

$$g(x) \equiv \sum_{k=1}^{n} A_k \left[\frac{h(x)}{x - x_k} \, h(-x) + \frac{h(-x)}{-x - x_k} \, h(x) \right]. \tag{3}$$

Since x_k is a zero of $h(x)$, the quantity $h(x)/(x - x_k)$ is a polynomial; in fact,

$$\frac{h(x)}{x - x_k} = \sum_{j=0}^{n-1} x^{n-1-j} \sum_{i=0}^{j} a_i \, x_k^{j-i}.$$

Substitution in (3) gives an identity between two polynomials. Equating coefficients of like powers of x gives a set of simultaneous, linear, algebraic equations for the A_k:

$$(4) \qquad \sum_{k=1}^{n} \alpha_{lk} A_k = (-1)^n g_l/2 \qquad (l = 1, 2, \cdots, n),$$

where

$$\alpha_{lk} = \sum_{i=1}^{n} b_{li} x_k^{i-1} \qquad (l, k = 1, 2, \cdots n),$$

$$b_{li} = \sum_{j=1}^{n} c_{lj} d_{ji} \qquad (l, i = 1, 2, \cdots n),$$

$$c_{lj} = a_{2l-j} \qquad (l, j = 1, 2, \cdots, n),$$

$$d_{ji} = (-1)^j a_{j-i} \qquad (j, i = 1, 2, \cdots, n),$$

with the convention that $a_k = 0$ if $k < 0$ or $k > n$. With $|\alpha_{lk}|$ for the determinant with n rows and n columns having α_{lk} in the lth row and kth column, the rule for multiplying determinants [1, Chap. 8] gives

$$|\alpha_{lk}| = |c_{lj}| \cdot |d_{ji}| \cdot |x_k^{i-1}|.$$

Now,

$$|d_{ji}| = \begin{vmatrix} -a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ -a_2 & -a_1 & -a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \pm a_{n-1} & \pm a_{n-2} & \pm a_{n-3} & \cdots & \pm a_0 \end{vmatrix} = (-1)^{n(n+1)/2} a_0^n$$

and

$$x_k^{i-1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \equiv V_n,$$

where V_n is the well-known Vandermonde determinant.

Hence, writing $C_n \equiv |c_{lj}|$, we have

$$|\alpha_{lk}| = (-1)^{n(n+1)/2}\, a_0^n\, C_n\, V_n,$$

In equation (4), write $\beta_l = (-1)^n\, g_l/2$ for convenience, and subtract $\beta_l I$ from both sides. Recalling equation (2), we see that the resulting system can be put in the form

$$(5) \qquad \begin{cases} I - \sum\limits_{k=1}^{n} A_k = 0 \\ \beta_l I + \sum\limits_{k=1}^{n} (\alpha_{lk} - \beta_l)\, A_k = \beta_l, \qquad (1 \leq l \leq n), \end{cases}$$

a system of $n+1$ equations in the $n+1$ unknowns I, A_1, A_2, $\cdots$, A_n that can be solved directly for I. First consider the determinant, D, of the coefficients in the left members of (5):

$$D = \begin{vmatrix} 1 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} - \beta_1 & \cdots & \alpha_{1n} - \beta_1 \\ \beta_2 & \alpha_{21} - \beta_2 & \cdots & \alpha_{2n} - \beta_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} - \beta_n & \cdots & \alpha_{nn} - \beta_n \end{vmatrix}.$$

Adding the first column to each of the succeeding columns immediately gives the result

$$D = |\alpha_{ij}| = (-1)^{n(n+1)/2}\, a_0^n\, C_n\, V_n\,.$$

Now $V_n \neq 0$, since all the zeros, x_k, of $h(x)$ were assumed to be distinct; and C_n does not vanish, since it is precisely the Hurwitz determinant [2, p. 163] of the polynomial $h(x)$, all the roots of which lie in the left half-plane. Hence $D \neq 0$, and the system (5) can be solved for I directly by Cramer's rule [1, Chap. 8]

$$DI = \begin{vmatrix} 0 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} - \beta_1 & \cdots & \alpha_{1n} - \beta_1 \\ \beta_2 & \alpha_{21} - \beta_2 & \cdots & \alpha_{2n} - \beta_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} - \beta_n & \cdots & \alpha_{nn} - \beta_n \end{vmatrix} .$$

Again adding the first column to each succeeding column gives

$$DI = \begin{vmatrix} 0 & -1 & \cdots & -1 \\ \beta_1 & \alpha_{11} & \cdots & \alpha_{1n} \\ \beta_2 & \alpha_{21} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix} .$$

By the definition of α_{ij}, this can be factored twice to give

$$DI = \frac{M}{a_0} \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \beta_1 & C_{11} & C_{12} & \cdots & C_{1n} \\ \beta_2 & C_{21} & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \beta_n & C_{n1} & C_{n2} & \cdots & C_{nn} \end{vmatrix} ,$$

where

$$M = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & d_{11} & \cdots & d_{1n} \\ 0 & d_{21} & \cdots & d_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & d_{n1} & \cdots & d_{nn} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ 0 & x_1 & \cdots & x_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & x_1^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} .$$

Thus,

$$(-1)^{n(n+1)/2}\, a_0^n\, C_n\, V_n\, I = \frac{-1}{a_0} \begin{vmatrix} \beta_1 & C_{12} & \cdots & C_{1n} \\ \beta_2 & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_n & C_{n2} & \cdots & C_{nn} \end{vmatrix} \cdot (-1)^{n(n+1)/2}\, a_0^n\, V_n\,.$$

The relation $\beta_l = (-1)^n\, g_l/2$ gives, finally, the desired formula:

$$(6) \qquad I \equiv \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{g(x)\,dx}{h(x)\,h(-x)} = \frac{(-1)^{n+1}}{2\,a_0} \cdot \frac{G_n}{C_n}\,,$$

where

$$G_n = |g_{ij}|,\; C_n = |c_{ij}| \qquad (1 \le i, j, n),$$

$$c_{ij} = a_{2i-j},\; g_{ij} = \begin{cases} g_i & \text{if } j = 1 \\ c_{ij} & \text{if } j > 1\,, \end{cases}$$

Since I is a continuous function of the coefficients of $h(x)$, and hence of the zeros, equation (6) remains true when two zeros coincide.

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