ON THE HITCHCOCK DISTRIBUTION PROBLEM

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1. Introduction. Frank L. Hitchcock [1] has offered a mathematical formulation of the problem of determining the most economical manner of distribution of a product from several sources of supply to numerous localities of use, and has suggested a computational procedure for obtaining a solution of his system in any particular case. L. Kantorovitch [2], Tjalling C. Koopmans [3], George B. Dantzig [4b], C. B. Tompkins [5], Julia Robinson [7; 8], Alex Orden [6], and others [4] have also discussed the computational aspects of this problem; paper [5] illustrates the use of the "projection method," due to C. B. Tompkins, as a computational process applicable to either of the Fundamental Problems of the present paper.

We shall be concerned only with the mathematical justification of computational procedure, and shall limit our attention to one specific method of solution of general validity. No attempt will be made to compare the various methods already proposed, either as to their mathematical similarity or as to their relative efficiency in any particular case.

2. The problem. The problem is to find a set of values of the mn variables x_{ij}, subject to the following conditions:

$$\sum_{i=1}^{m} x_{ij} = c_j, \quad \sum_{j=1}^{n} x_{ij} = r_i, \tag{2.1}$$

$$x_{ij} \geq 0, \tag{2.2}$$

Received January 25, 1952. The author's interest in the problem was aroused by papers on transportation theory presented by Koopmans [4a] and Dantzig [4b] at a conference on linear programming in Chicago during June, 1949, under the auspices of the Cowles Commission for Research in Economics of the University of Chicago. Several other papers presented at this conference are of closely related interest. Professor Koopmans, in his Introduction to the Conference Proceedings [4], also discussed the background and interrelationship of the conference papers—including the bearing of some of these on the Hitchcock distribution problem. The results of the present paper have been presented in three seminar lectures: once in December, 1949, at The RAND Corporation in Santa Monica, once in July, 1950, at the Institute for Numerical Analysis of the National Bureau of Standards in Los Angeles, and once in June, 1951, at the National Bureau of Standards in Washington, D.C. The author is especially indebted to Dr. D.R. Fulkerson, who has given real assistance in simplifying notation and proofs of theorems, for a careful reading of the manuscript.

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$$(2.3)\qquad \sum_{i,j} x_{ij}\, d_{ij} = \text{minimum}.$$

The numbers m, n, r_i, c_j, and d_{ij} are given positive integers with $\sum c_j = \sum r_i$. The indices i and j are understood always to range over these same integers m and n, respectively; it is also assumed, for convenience, that $m \geq n$. Any set of values x_{ij} that satisfies all these conditions is called a *solution* of the problem.

There is no loss of generality in assuming that the d_{ij} are positive integers, rather than rational numbers, since the problem is essentially unchanged if d_{ij} is replaced by $ad_{ij} + b$, where a and b are any positive rational numbers. We have not examined the case in which some of the quantities r_i, c_j, and d_{ij} are irrational. The only effect of irrationality on the results of the present paper is a possible lack of convergence of the iterative process of solution. These considerations are not of importance in the usual applications.

It will sometimes be more convenient to use an alternative statement of the problem, in matrix notation, as follows:

$$(2.4)\qquad M'y \geq b,$$

$$(2.5)\qquad y \geq 0,$$

$$(2.6)\qquad a'y = \text{minimum}.$$

It is easily seen that the two formulations are equivalent if y, a, b, and M' are defined as follows:

$$y_{n(i-1)+j} = x_{ij},$$

$$a_{n(i-1)+j} = d_{ij},$$

$$M' = \left\| \begin{matrix} I_n & I_n & \cdots & I_n \\ -I_n & -I_n & \cdots & -I_n \\ J_1 & J_2 & \cdots & J_m \\ -J_1 & -J_2 & \cdots & -J_m \end{matrix} \right\|, \qquad b = \left\| \begin{matrix} c \\ -c \\ r \\ -r \end{matrix} \right\|,$$

where I_n is the identity matrix of order n, and J_i is the $m \times n$ matrix with all elements zero except for the ith row in which each element is unity. Of course, y, a, c, and r are column matrices (or vectors) with components $y_{n(i-1)+j}$, $a_{n(i-1)+j}$, c_j, and r_i, respectively, and a prime denotes the transpose of a matrix (or vector).

3. Fundamental theorems. There are several fundamental theorems concerning systems of linear inequalities that are useful for this paper. I reproduce their statements here in a form given by A. W. Tucker in an unpublished note dated December, 1949. The interested reader can find proofs of these theorems, and of others of similar type, in a paper by Gale, Kuhn, and Tucker [4c].

FUNDAMENTAL PROBLEMS. (Here lower case roman letters denote one-column vectors, while capitals denote rectangular matrices; M, a, and b are given, but d is to be determined.)

PROBLEM I. To satisfy the constraints $Mx \leq a$, $x \geq 0$, and make $b'x = d$ for d *maximal* in the sense that no x satisfying the constraints makes $b'x > d$.

PROBLEM II. To satisfy the constraints $M'y \geq b$, $y \geq 0$, and make $a'y = d$ for d *minimal* in the sense that no y satisfying the constraints makes $a'y < d$.

Problems I and II are said to be *dual*.

FUNDAMENTAL FEASIBILITY THEOREM. *The constraints in a problem are feasible* (*that is, satisfied by some* x *or* y) *if and only if the dual problem in homogeneous form* (*that is, with* $b = 0$ *or* $a = 0$) *has a null solution.*

FUNDAMENTAL EXISTENCE THEOREM. i. *The vectors* x *and* y *are solutions of Problems* I *and* II *if and only if they satisfy their constraints in the two problems and make* $a'y = b'x$. *Such* x *and* y *exist if the constraints in both problems are feasible.*

ii. *A problem has a solution if and only if its constraints are feasible and its homogeneous form has a null solution.*

FUNDAMENTAL DUALITY THEOREM. *A problem has a solution* (*for a unique* d) *if and only if the dual problem has a solution* (*for the same* d).

4. The dual and combined problems. We note that the problem, as stated in relations (2.4)-(2.6), is a Fundamental Problem of form II. The dual problem is:

$$Mx \leq a, \tag{4.1}$$

$$x \geq 0, \tag{4.2}$$

$$b'x = \text{maximum}. \tag{4.3}$$

This can be rewritten in a more convenient form, for our present purposes, as follows:

$$v_j - u_i \leq d_{ij}, \tag{4.4}$$

$$\sum c_j v_j - \sum r_i u_i = \text{maximum}, \tag{4.5}$$

where $v_j = x_j - x_{n+j}$, and $u_i = - x_{2n+i} + x_{2n+m+i}$; we omit the condition (4.2), that $x \geq 0$, since this imposes no limitation on u_i and v_j.

THEOREM 1. *The problem has a solution.*

Proof. By the Fundamental Existence Theorem, there is a solution if and only if the constraints are feasible and $y = 0$ is a solution of the problem when $b = 0$. Now

$$\sum c_j = \sum r_i ,$$

so

$$x_{ij} = \frac{r_i\, c_j}{\sum r_i}$$

satisfies the constraints. When $b = 0$, obviously the only values that satisfy the constraints are $x_{ij} = 0$, and so the theorem is proved.

By the Fundamental Duality Theorem, we see:

COROLLARY 1A. *The dual problem has a solution.*

THEOREM 2. *The numbers x_{ij}, and u_i, v_j, are solutions of the problem and the dual, respectively, if and only if they satisfy*:

$$\sum_j x_{ij} = r_i, \quad \sum_i x_{ij} = c_j, \quad x_{ij} \geq 0, \tag{4.6}$$

$$d_{ij} + u_i - v_j \geq 0, \tag{4.7}$$

$$x_{ij}(d_{ij} + u_i - v_j) = 0. \tag{4.8}$$

Proof. Since (4.6) and (4.7) are simply the constraints for the problem and the dual, respectively, it remains only to show that (4.8) is equivalent to the condition $a'y - b'x = 0$. Now

$$a'y - b'x = \sum_{i,j} x_{ij}\, d_{ij} - \sum_{j} c_j\, v_j + \sum_{i} r_i\, u_i$$

$$= \sum_{i,j} x_{ij}\, d_{ij} - \sum_{i,j} x_{ij}\, v_j + \sum_{i,j} x_{ij}\, u_i = \sum_{i,j} x_{ij}(d_{ij} + u_i - v_j).$$

Since each term in this sum is nonnegative,

$$a'y - b'x = 0$$

if and only if

$$x_{ij}(d_{ij} + u_i - v_j) = 0.$$

We refer to the problem of finding values for x_{ij}, u_i, and v_j that satisfy (4.6)-(4.8) as the "combined problem", and note that the combined problem always has a solution.

5. Linear graphs. It will be convenient, for some purposes, to associate linear graphs [9] with certain subsets of the elements of a matrix $S = \|s_{hk}\|$. If I is a given subset of the elements of S, we define the *I-graph* L of S as follows: the *vertices* of L are all the points (h, k) in the Cartesian plane for which $s_{hk} \in I$; the *arcs* of L are all line-segments joining pairs of *neighboring* vertices with either equal abscissas or equal ordinates, where two vertices with equal abscissas (ordinates) are neighboring if they are not separated by another vertex of L with the same abscissa (ordinate). For the moment, denote the vertices of L by symbols $a, b, c, \cdots, f$, and the arcs by symbols such as $ab, bc, \cdots, cf$ (no distinction is made between the arcs ab and ba). Then a *chain* is a set of one or more distinct arcs that can be arranged as $ab, bc, \cdots, de, ef$, where vertices denoted by different symbols are distinct. A *cycle* is a set of distinct arcs (at least four are necessary) that can be ordered as $ab, bc, \cdots, ef, fa$, the vertices being distinct as in the case of a chain. A graph is *connected* if each pair of vertices is joined by a chain. A *forest* is a graph containing no cycles, and a *tree* is a connected forest.

If L contains v vertices, a arcs, and p connected pieces, the number $\mu = a - v + p$ is known as the *cyclomatic number* (or first Betti number) of L. It follows from a well-known theorem [9] concerning linear graphs in general that: (i) L is a forest if and only if $\mu = 0$, and (ii) L contains just one cycle if and only if $\mu = 1$.

Note that L contains a cycle if and only if there is a subset of I that can be arranged as a sequence

$$s_{h_1 k_1},\ s_{h_1 k_2},\ s_{h_2 k_2},\ s_{h_2 k_3},\ \cdots,\ s_{h_\sigma k_\sigma},\ s_{h_\sigma k_1},$$

where the h's and k's are distinct among themselves; and L contains a single cycle if and only if I contains just one subset that can be arranged in the displayed form. We call such a subset of I an *I-circuit* on S, and denote it by $[S_\sigma]$. For a particular arrangement of $[S_\sigma]$, we also refer to the terms $s_{h_\alpha k_\alpha}$ as *odd-terms*, the others as *even-terms*.

In case I consists of all $s_{hk} > 0$, as it frequently will, we speak of the *positive graph* of S, *positive circuits* on S, and abbreviate such statements as "the positive graph of S is a forest" to "S is a forest".

6. The method of solution. In the method of solution to be developed for the problem, we start with a special set of values $X^0 = ||x_{ij}^0||$ that satisfy the constraints (4.6). We then test to determine whether or not there exist u_i and v_j satisfying the relations (4.7) and (4.8) for the given X^0. If so, then X^0 is a solution, otherwise not. The method next yields a new trial matrix $X^1 = ||x_{ij}^1||$, if X^0 is not a solution, such that

$$\sum_{i,j} (x_{ij}^0 - x_{ij}^1) d_{ij} \geq 1.$$

After a finite number of steps this process necessarily must terminate, and it leads to an exact integral solution of the problem.

The first trial matrix X^0 is a forest of t trees, and has $m + n - t$ nonzero elements. According as $t = 1$ or $t > 1$, two essentially different cases may be met at each stage of the solution process.[1]

At each stage when $X = ||x_{ij}||$ is a tree, the equations (4.8) have a general solution for u_i and v_j with one free parameter, say u_1. However, the quantities $d_{ij} + u_i - v_j$ are uniquely determined in this case, so it is sufficient to calculate them and note whether or not they are all nonnegative in order to decide whether or not X is a solution. If some

[1] These are the nondegenerate and degenerate cases in the work of Dantzig [4b]. We shall use these terms also. The method of solution developed by Dantzig [4b] for the nondegenerate case is essentially the same as the one in the present paper, although the derivations of the results are quite different. Orden [6] has subsequently given an elegant method for reducing the degenerate case to the nondegenerate one, as an extension of the ϵ-method proposed by Dantzig [4b]. The author believes that the treatment of the degenerate case provides the only results in the present paper that are new, or at least fresh for the Hitchcock problem, and also of some mathematical interest. It also seems likely that the method given here will often be more efficient computationally, in the degenerate case, than the Dantzig-Orden ϵ-method.

$$d_{i_1 j_1} + u_{i_1} - v_{j_1} < 0,$$

then there is a unique I-circuit $[X_s]$ on X, where I consists of $x_{i_1 j_1}$ and all positive x_{ij}, that may be arranged with $x_{i_1 j_1}$ as the second term, say. Let g denote the smallest odd-term of $[X_s]$. Then the new trial matrix X^* is obtained from X by adding g to the even terms of $[X_s]$, subtracting g from the odd-terms, and leaving the other elements of X unchanged.

At each stage when X is a forest of $t > 1$ trees, the equations (4.8) have a general solution for u_i and v_j with t independent parameters, and the quantities $d_{ij} + u_i - v_j$ involve $t - 1$ independent parameters. The rows and columns of the matrix X are rearranged so that it can be represented as a square matrix of order t whose t^2 elements are submatrices X_{ab} such that $X_{ab} = 0$ if $a \neq b$, and X_{aa} is a tree with $m_a + n_a - 1$ nonzero elements and is of order $m_a \times n_a$. It may also be assumed that each X_{aa} is a solution of its subproblem. We can select

$$u_1,\ u_{m_1+1},\ \cdots,\ u_{m_1+\cdots+m_{t-1}+1}$$

to be the t parameters. If we assign these the value zero and denote this particular solution of (4.8) by $\bar{u}_i$ and $\bar{v}_j$, then we may define numbers

$$\bar{p}_{ij} = d_{ij} + \bar{u}_i - \bar{v}_j.$$

We partition the matrix $\bar{P} = ||\bar{p}_{ij}||$ into submatrices corresponding to the X_{ab} and denote them $\bar{P}_{ab}$. Let p_{ab} be the smallest element in $\bar{P}_{ab}$ and define the square matrix P of order t by $P = ||p_{ab}||$. To designate the position of p_{ab} in the matrix $\bar{P} = ||\bar{p}_{ij}||$, we may write p_{ab} alternatively as

$$\bar{p}_{a\,b}^{i_a j_b},$$

the subscripts referring to the submatrix and the superscripts to the rows and columns in the submatrix. When it introduces no ambiguity, the subscripts on the superscripts will be omitted in order to simplify the notation.

The test as to whether or not X is a solution consists of forming all sums

$$p_{a_1 a_2 \cdots a_h} = p_{a_1 a_2} + p_{a_2 a_3} + \cdots + p_{a_h a_1}$$

for $h = 2, 3, \cdots, t$, where $(a_1 a_2 \cdots a_h)$ is any permutation of h different positive integers, none greater than t; X is a solution if and only if all such sums are nonnegative.

If any

$$p_{a_1 a_2 \cdots a_h} < 0,$$

then there is a unique I-circuit $[X_s]$ on X, where I consists of all positive x_{ij} together with all x_{ij} that correspond to the terms

$$\overline{p}^{i}_{a_k}\,{}^{j}_{a_{k+1}} \quad \text{of } p_{a_1 a_2 \cdots a_h},$$

which can be arranged to involve all

$$x^{i}_{a_k}\,{}^{j}_{a_{k+1}}$$

as even-terms. If g is the smallest odd-term in $[X_s]$, then (as in the nondegenerate case) the new trial matrix X^* is obtained by adding g to the even-terms of $[X_s]$, subtracting g from the odd-terms, and leaving the other elements of X unchanged.

7. The initial trial solution. An X that satisfies (4.6) will be called a *trial solution.* It would be all right to take the positive values

$$\frac{r_i c_j}{\sum r_i}$$

for the initial trial solution $X^0 = ||x^0_{ij}||$. An alternative is to construct an initial trial solution that is a forest. It is always possible to do this in integral values. The following theorem certifies the existence of such an integral trial solution. The method of proof shows how to construct one.

THEOREM 3. *There is a matrix* $X^0 = ||x^0_{ij}||$ *with integral elements that satisfies* (4.6) *and is a forest.*

Proof. The theorem is trivial for $m = 1$. Assume the theorem is true for m and consider the case $m + 1$.

Let the notation be chosen so that

$$r_1 \geq r_2 \geq \cdots \geq r_{m+1} > 0, \text{ and } c_1 \geq c_2 \geq \cdots \geq c_n > 0.$$

If $n < m + 1$, then $c_1 > r_{m+1}$. If $n = m + 1$ then $c_1 > r_{m+1}$ unless $c_i = r_j = \lambda$

(for all i and j); in this latter case $X^0 = \lambda$ satisfies the conditions of the theorem. Hence, by the induction hypothesis, there is a set of nonnegative integers x^*_{ij} $(i = 1, \cdots, m)$ such that

$$\sum_i x^*_{ij} = c_j - \delta_{1j} r_{m+1}, \quad \sum_j x^*_{ij} = r_i, \quad \text{and} \quad X^* = \| x^*_{ij} \|$$

is a forest. Then X^0, defined by

$$x^0_{ij} = x^*_{ij}, \quad x^0_{m+1\, j} = \delta_{1\,j}\, r_{m+1},$$

satisfies (4.6). Now since the $(m + 1)$st row, with only one positive element, clearly cannot contribute terms to a positive circuit, X^0 is also a forest; the theorem is proved.

To apply this method, in the construction of a trial solution, search for the smallest r_{i_1} and the largest c_{j_1}, and then set

$$x^0_{i_1 j_1} = r_{i_1}.$$

In effect, this deletes the i_1st row, after c_{j_1} is replaced by $c_{j_1} - r_{i_1}$, and the process is repeated (with interchanged rows and columns as necessary) until all x^0_{ij} have been determined. For automatic machine calculation, the procedure is easily made unique, for any one starting order of rows and columns, by specifying that the search is first on row-totals when the number of rows is the same as the number of columns at any stage, and that the row-total or column-total with the smallest index is chosen whenever at any stage there are several equal values to choose from. This initial trial solution will be called "preferred" for identification.[2]

THEOREM 4. *A trial solution that is a forest of t trees has $m + n - t$ nonzero elements.*

Proof. Observe first that if the trial solution X is a forest of t trees, the rows and columns of X can be rearranged so that X has the form

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[2]Sometimes, as in this instance, we indicate how to make a unique choice among possible alternatives at each computational step, but usually do not. It is necessary to do this in order completely to routinize the computing steps, of course, but the matter presents no difficulty and we omit it here.

$$\begin{Vmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_{tt} \end{Vmatrix},$$

where each X_{aa} is a tree. Consequently, the theorem amounts to proving that an $m \times n$ matrix with no zero rows or columns, which is a tree, has $m + n - 1$ positive elements. If $m + n = 2$, this is obvious, so assume the statement to be true for all matrices for which $m + n = k$ and consider one for which $m + n = k + 1$. Since $m \geq n$, clearly some row has only one positive element, as otherwise there would be a positive circuit. Delete this row and apply the induction hypothesis.

In actual cases when m and n are relatively small, or when there is other reason to believe that an initial trial solution better than the preferred one can be found by trial and error, it may be better to construct the initial trial solution in some other way than the one given in the proof of Theorem 3, in order to reduce the number of steps required in the iterative process.

The methods developed in this paper apply directly for any trial solution that is a forest, and are readily extended for other cases. It is easy to see that there must be at least one solution which is a forest.

8. Nondegenerate case. We consider now the case of a trial solution X which is a tree. Let the positive elements of X be

$$x_{i_a j_a} \qquad (a = 1, \cdots, m + n - 1).$$

We shall need the following theorem.

THEOREM 5. *If X is a trial tree, the set of equations*

$$d_{ij} + u_i - v_j = 0 \text{ for } (i, j) = (i_a, j_a), \tag{8.1}$$

has the general solution

$$u_i = u_i^* + z, \quad v_j = v_j^* + z,$$

where (u_i^*, v_j^*) is a particular solution and z is arbitrary.

Proof. The theorem is apparent for $m = 1$, and we proceed by induction.

Suppose the theorem is true for all trial trees of m rows, and let X be an $(m+1)\times n$ trial tree. Obviously, there must be at least one row of X that has exactly one nonzero element; we may suppose it to be $x_{m+1\,n}$ without loss of generality—also that

$$i_{m+n-1} = m+1 \text{ and } j_{m+n-1} = n.$$

Since X is a trial tree, the matrix obtained from X by deleting the last row (or, if $m+1=n$, its transpose) is also. The induction hypothesis implies that the general solution of (8.1), with the final equation omitted, is of the form

$$u_i = u_i^* + z,\ v_j = v_j^* + z.$$

We note next that this final equation becomes

$$u_{m+1} = (v_n^* - d_{m+1\,n}) + z = u_{m+1}^* + z.$$

The theorem follows easily.

It will be convenient to call the particular solution $\overline{u}_i, \overline{v}_j$ of (8.1) obtained by setting $u_1 = 0$ the *preferred* trial solution of the dual problem corresponding to the trial tree X. As an obvious consequence of Theorem 5, we state:

COROLLARY 5A. *If X is a trial tree, then it is a solution of the problem if and only if the corresponding preferred trial solution $(\overline{u}_i, \overline{v}_j)$ of the dual problem satisfies*

$$d_{ij} + \overline{u}_i - \overline{v}_j \geq 0$$

for all i and j.

All that is needed now in order to establish the method for the nondegenerate case is to show how to construct a new trial matrix X^*, if X is not a solution, such that

$$\sum_{i,j} (x_{ij} - x_{ij}^*) d_{ij} \geq 1.$$

In this case, it follows by Corollary 5A that

$$d_{kl} + \overline{u}_k - \overline{v}_l < 0$$

for at least one pair (k, l) and, of course, $x_{kl} = 0$.

THEOREM 6. *If the trial solution X is a tree, and $x_{kl} = 0$, then there is a unique I-circuit on X, where I consists of all positive x_{ij} together with x_{kl}.*

Proof. It suffices to show that the I-graph of X has cyclomatic number $\mu = 1$. By assumption, the positive graph of X has cyclomatic number zero; and since X must have positive elements x_{al} and x_{kb} for some a and b, the I-graph of X has two more arcs, one more vertex, and the same number (one) of connected pieces. Hence $\mu = 1$, and the proof is complete.

Now arrange this unique I-circuit $[X_s]$ with x_{kl} as the second term, and let g be the minimum of the odd-terms of $[X_s]$ in this arrangement. If we subtract g from the odd-terms, add g to the even terms, and leave the remaining elements of X unchanged, we get a matrix X^* that satisfies (4.6) and is a forest (since $[X_s]$ is unique).

THEOREM 7. *The following relation holds*:

$$\sum_{i,j} (x_{ij} - x^*_{ij}) d_{ij} \geq 1.$$

Proof. Let

$$[X_s] = [x_{i_1 j_1}, x_{i_1 j_2}, x_{i_2 j_2}, x_{i_2 j_3}, \cdots, x_{i_s j_s}, x_{i_s j_1}],$$

where

$$x_{i_1 j_2} = x_{kl}.$$

Then

$$\sum_{i,j} (x_{ij} - x^*_{ij}) d_{ij} = g(d_{i_1 j_1} - d_{i_1 j_2} + d_{i_2 j_2} - d_{i_2 j_3} + \cdots + d_{i_s j_s} - d_{i_s j_1})$$

$$= - g(d_{i_1 j_2} + u_{i_1} - v_{j_2}) \geq 1.$$

The theorem follows.

If X^* is a tree, then the whole process is repeated until at some stage a trial matrix is obtained that either (i) is a solution, or (ii) is not a solution and is a forest of $t > 1$ trees. We shall now discuss (ii).

9. The degenerate case. Let X be a trial matrix which is a forest of $t > 1$ trees. As we have seen, we may suppose that the rows and columns of X are ordered so that

$$X = \left\| \begin{matrix} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_{tt} \end{matrix} \right\|,$$

where each submatrix X_{aa} of order $m_a \times n_a$ is a tree. We can apply the methods of the nondegenerate case to the subproblems corresponding to the submatrices X_{aa}, and either obtain a solution to each subproblem or further decompose the matrix X; thus we may also assume that each X_{aa} is a solution to its subproblem.

By Corollary 5A, we know that

$$(9.1) \qquad d_{a\,a}^{i\,j} + \bar{u}_a^i - \bar{v}_a^j \geq 0 \qquad (a = 1, \cdots, t;\ i_a = 1, \cdots, m_a;\ j_a = 1, \cdots, n_a),$$

where $\bar{u}_a$, $\bar{v}_a$ is the preferred trial solution of the dual subproblem corresponding to the solution X_{aa}, and that

$$(9.2) \qquad d_{aa}^{i\,j} + \bar{u}_a^i - \bar{v}_a^j = 0 \quad \text{if} \quad x_{aa}^{i\,j} > 0.$$

We recall also that the most general values for u_a^i and v_a^j are given by

$$u_a^i = \bar{u}_a^i + z_a, \qquad v_a^j = \bar{v}_a^j + z_a,$$

where the z_a are arbitrary parameters.

It follows from Theorem 2 that X is a solution if and only if there are values of z_a that satisfy inequalities corresponding to (4.7), or in our present notation:

$$(9.3) \qquad d_{ab}^{i\,j} + u_a^i - v_b^j \geq 0 \qquad \text{for all } a,\ b,\ i_a, \text{ and } j_b.$$

But (9.3) has a solution for z_a if and only if the following inequalities have a solution for z_a:

$$(9.4) \qquad p_{ab} + z_a - z_b \geq 0,$$

where

$$\overline{p}^{ij}_{ab} = d^{ij}_{ab} + \overline{u}^{i}_{a} - \overline{v}^{j}_{b}, \quad p_{ab} = \min_{(i,j)} \{\overline{p}^{ij}_{ab}\}.$$

We have proved:

LEMMA A. *The matrix X is a solution if and only if there are real numbers* z_a *such that*

$$p_{ab} + z_a - z_b \geq 0 \qquad (a, b = 1, \cdots, t) .$$

In order to establish a criterion for the solvability of (9.4), we consider a special case of the original problem, defined as follows:

$$d_{ab} = p_{ab}, \; r_a = c_b = 1, \; a, b = 1, \cdots, t.$$

We call this the *special problem*, the corresponding dual the *special dual*, and now consider the *special combined problem*:

$$\sum_b y_{ab} = \sum_a y_{ab} = 1 \qquad (y_{ab} \geq 0),$$

$$p_{ab} + z_a - w_b \geq 0, \quad y_{ab}(p_{ab} + z_a - w_b) = 0.$$

If we set $y_{ab} = \delta_{ab}$, then for this trial solution the conditions reduce to:

$$p_{ab} + z_a - w_b \geq 0 \qquad \text{for } a \neq b,$$

$$p_{aa} + z_a - w_a = 0.$$

Since $p_{aa} = 0$, it follows that $z_a = w_a$, and so these conditions are equivalent to (9.4). Hence, by Theorem 2, (9.4) has a solution if and only if $||\delta_{ab}||$ is a solution of the special problem. Using Lemma A, we now have:

LEMMA B. *The matrix X is a solution of the original problem if and only if the identity matrix is a solution of the special problem.*

THEOREM 8. *The matrix X is a solution of the problem if and only if*

$$p_{a_1 a_2 \cdots a_h} \geq 0 \qquad (h = 2, 3, \cdots, t),$$

where $(a_1, a_2, \cdots, a_h)$ *is any permutation of h different positive integers, none greater than t, and*

$$p_{a_1 a_2 \cdots a_h} = p_{a_1 a_2} + p_{a_2 a_3} + \cdots + p_{a_h a_1}.$$

Proof. By Lemma B, it suffices to show that the condition of the theorem is equivalent to the statement that $\|\delta_{ab}\|$ is a solution of the special problem.

First of all, it is easy to see that at least one solution $Y = \|y_{ab}\|$ of the special problem is a forest, and hence has less than $2t$ nonzero elements. That the elements of Y are all either zero or unity can be seen by induction as follows. The basis of the induction is obvious, and we consider the case $t + 1$, assuming the statement for t. There must be at least one element of Y that is unity, as otherwise Y would have at least $2t$ nonzero elements. We may suppose that this element is $y_{t+1\,t+1}$. But then the induction hypothesis implies that each element y_{ab} $(a, b = 1, \cdots, t)$ is zero or one. It follows that there are exactly t elements of Y that are unity, whence we can write

$$\sum_{a,b} y_{ab}\, p_{ab} = p_{a_1 b_1} + p_{a_2 b_2} + \cdots \quad p_{a_t b_t},$$

where $(a_1\ a_2 \cdots a_t)$ and $(b_1\ b_2 \cdots b_t)$ are permutations of the first t integers. Then $\|\delta_{ab}\|$ is a solution of the special problem if and only if always

$$p_{a_1 b_1} + p_{a_2 b_2} + \cdots + p_{a_t b_t} \geq p_{11} + p_{22} + \cdots + p_{tt} = 0.$$

The proof is completed by noting that this sum can be written as

$$p_{a_1 a_2} + p_{a_2 a_3} + \cdots + p_{a_h a_1},$$

with $(a_1\ a_2 \cdots a_h)$ as described in the theorem.

We now need to show how to construct an improved trial solution X^* in the event that X is not a solution. In this case, we know from Theorem 8 that there is a sum

$$\bar{p}^{i^0 j^0}_{a_1 a_2} + \bar{p}^{i^0 j^0}_{a_2 a_3} + \cdots + \bar{p}^{i^0 j^0}_{a_h a_1} < 0.$$

Let I consist of all positive elements x^{ij}_{aa} together with all

$$x_{a_k a_{k+1}}^{i^0 j^0}$$

of X. Then we assert:

THEOREM 9. *There is a unique l-circuit on X that can be arranged to involve as even-terms all the*

$$x_{a_k a_{k+1}}^{i^0 j^0}.$$

Proof. The positive graph of X has $m + n - t$ vertices, $m + n - 2t$ arcs, and t connected pieces. Also, for each

$$x_{a_k a_{k+1}}^{i^0 j^0}$$

there are nonzero elements

$$x_{a_k a_k}^{i^0 j^1}, \quad x_{a_{k+1} a_{k+1}}^{i^1 \quad j^0}.$$

Hence in passing from the positive graph to the l-graph, h vertices and $2h$ arcs are added, and the number of connected pieces is decreased from t to $t - h + 1$. Thus the cyclomatic number of the l-graph is

$$\mu = (2h + m + n - 2t) - (h + m + n - t) + (t - h + 1) = 1,$$

so there is a unique l-circuit $[X_s]$ on X. Since the graph obtained by omitting from l any

$$x_{a_k a_{k+1}}^{i^0 j^0}$$

clearly has no cycle, $[X_s]$ contains all of these.

Evidently $[X_s]$ can be arranged, for example, as

$$\left[x_{a_1 a_1}^{i^0 j^1}, \; x_{a_1 a_2}^{i^0 j^0}, \; x_{a_2 a_2}^{i^1 j^0}, \ldots, x_{a_2 a_2}^{i^0 j^1}, \; x_{a_2 a_3}^{i^0 j^0}, \ldots\right]$$

so that all

$$x_{a_k a_{k+1}}^{i^0 j^0}$$

appear as even-terms.

As in the nondegenerate case, let g be the smallest odd-term in $[X_s]$ (hence $g > 0$), and define a new trial matrix X^* by replacing the elements of X that appear in $[X_s]$ by new ones increased by g for even-terms and decreased by g for odd-terms; the other elements of X are left unchanged. Again X^* satisfies the conditions for a trial matrix. To complete the discussion of the degenerate case, it remains only to prove:

THEOREM 10. *The following relation holds*:

$$\sum_{i,j} (x_{ij} - x_{ij}^*) d_{ij} \geq 1.$$

Proof. Since X and X^* differ only on

$$[X_s] = [x_{i_1 j_1}, x_{i_1 j_2}, x_{i_2 j_2}, x_{i_2 j_3}, \cdots, x_{i_s j_s}, x_{i_s j_1}],$$

then

$$\sum_{i,j} (x_{ij} - x_{ij}^*) d_{ij} = -g(d_{i_1 j_1} - d_{i_1 j_2} + d_{i_2 j_2} - d_{i_2 j_3} + \cdots + d_{i_s j_s} - d_{i_s j_1}).$$

The proof is completed by noting that

$$d_{ij} = \overline{p}_{ij} + \overline{v}_j - \overline{u}_i \quad \text{and} \quad \overline{p}_{ij} = 0 \quad \text{if} \quad x_{ij} > 0,$$

so that

$$\sum_{i,j} (x_{ij} - x_{ij}^*) d_{ij} = -g(p_{a_1 a_2 \cdots a_h}) \geq 1.$$

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