SOME THEOREMS ON THE SCHUR DERIVATIVE

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1. Introduction. Given the sequence $\{a_m\}$ and $p \neq 0$, Schur [5] defined the derivative a'_m by

$$(1.1) \qquad a'_m = \Delta a_m = (a_{m+1} - a_m)/p^{m+1};$$

higher derivatives are defined by means of

$$a_m^{(r)} = \Delta^r a_m = \Delta(a_m^{(r-1)}), \quad a_m^{(0)} = a_m.$$

In particular if p is a prime, a an integer and $a_m = a^{p^m}$, then by Fermat's theorem

$$a'_m = (a^{p^{m+1}} - a^{p^m})/p^{m+1}$$

is integral. Schur proved that if $p \nmid a$, then also the derivatives

$$\Delta^2 a^{p^m}, \ \Delta^3 a^{p^m}, \cdots, \Delta^{p-1} a^{p^m}$$

are all integral. Moreover if $a'_0 \equiv 0 \pmod p$ then all the derivatives $\Delta^r a^{p^m}$ are integral, while if $a'_0 \not\equiv 0 \pmod p$ then every number of $\Delta^p a^{p^m}$ has the denominator p.

A. Brauer [1] gave another proof of Schur's results. About the same time Zorn [6] proved these results by p-adic methods and indeed proved the following stronger theorem. For $x \equiv 1 \pmod p$, define

$$X_m = (x^{p^m} - 1)/p^{m+1},$$

and as above let $\Delta^r X_m$ denote the r-th derivative of X_m; then

$$(1.2) \qquad \Delta^r X_m \equiv \frac{(p-1)(p^2-1)\cdots(p^r-1)}{(r+1)!} X_m^{r+1} \pmod{p^m}$$

provided $r < p$; for $r < p-2$, the congruence (1.2) holds $\pmod{p^{m+1}}$. It is also shown that Schur's theorem is an easy consequence of Zorn's results.

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In the present paper we shall give a simple elementary proof of Zorn's congruences. In addition we prove, for example, that for $r \le p$,

$$\Delta^r a^{p^m} \equiv \frac{1}{r!} a^{p^m} q_m^r \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m}, \tag{1.3}$$

where

$$a^{(p-1)p^m} = 1 + p^{m+1} q_m;$$

for $r < p - 1$, (1.3) holds (mod p^{m+1}).

We next (§4) extend Schur's and Zorn's theorems to algebraic numbers. In §5 we consider a generalization of another kind suggested by the arithmetic function (see for example [2, p. 84-86])

$$F(a, m) = \sum_{de=m} \mu(d) a^e. \tag{1.4}$$

Finally (§6), we give some applications of Schur's theorem to the Euler and Bernoulli polynomials and numbers; the results are analogous to Kummer's congruences [3, Ch. 12]. In particular $\Delta^r E_{k+p^m}$ is integral (mod p) for $p > 2, r < p$, $r \le m$; also $\Delta^r(B_{k+p^m}/(k+p^m))$ is integral (mod p) for $p - 1 \nmid k + 1$, $r < p$, $r \le m$. Here E_k and B_k denote the Euler and Bernoulli numbers in the notation of Nörlund [3].

2. Formulas for $\Delta^r a_m$. We shall require some preliminary results.

LEMMA 1. *The following identity holds*:

$$\prod_{i=0}^{r=1} (x - p^i) = \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} x^{r-i}, \tag{2.1}$$

where

$$\left[{r \atop i}\right] = \frac{(p^r - 1)(p^{r-1} - 1)\cdots(p^{r-i+1} - 1)}{(p-1)(p^2-1)\cdots(p^i - 1)} = \left[{r \atop r-i}\right], \left[{r \atop 0}\right] = 1. \tag{2.2}$$

LEMMA 2. *Put*

$$W_{k,r} = \sum_{i=0}^{k} (-1)^i \left[{k \atop i}\right] \binom{p^{k-i}}{r} p^{i(i-1)/2},$$

where $\binom{m}{r}$ *denotes a binomial coefficient. Then*

$$
(2.3)\qquad W_{k,r} = \begin{cases} 0 & (r<k) \\[2mm] \dfrac{1}{r!}\displaystyle\prod_{i=0}^{r-1}(p^r-p^i) & (r=k) \\[2mm] \dfrac{1}{r!}\, p^{k(k-1)/2}\, U_{k,r} & (r>k), \end{cases}
$$

where $U_{k,r}$ *is an integer.*

Lemma 1 is will known. To prove Lemma 2, we note first that the binomial coefficient $\binom{x}{r}$ is a polynomial in x of degree r. Since by (2.1)

$$
\sum_{i=0}^{k}(-1)^i \left[{k \atop i}\right] p^{i(i-1)/2}\, p^{r(k-i)} = \prod_{i=0}^{k-1}(p^r-p^i),
$$

the several parts of (2.3) follow without much difficulty.

LEMMA 3. *For an arbitrary sequence* $\{a_m\}$,

$$
(2.4)\qquad \Delta^r a_m = p^{-rm-r(r+1)/2}\sum_{i=0}^{r}(-1)^i\left[{r \atop i}\right] p^{i(i-1)/2}\, a_{m+r-i}\,.
$$

This formula, which is given by Schur, is easily proved. In view of (2.1) it can be put in the following symbolic form:

$$
(2.5)\qquad \Delta^r a_m = p^{-rm-r(r+1)/2}\, a^m \prod_{i=0}^{r-1}(a-p^i),
$$

where it is understood that after expansion of the right member a^k is to be replaced by a_k.

Suppose now that $p \nmid a$ and put

$$
(2.6)\qquad a^{(p-1)p^m} = 1 + p^{m+1}q_m,
$$

so that q_m is integral. Then by the binomial theorem we have

$$
a^{(p-1)p^{m+s}} = \sum_{i=0}^{p^r}\binom{p^s}{i} p^{(m+1)i}\, q_m^i \qquad (r \ge s),
$$

and by (2.4) this implies

$$p^{rm+r(r+1)/2}\ \Delta^r a^{(p-1)p^m}$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{p^r} \binom{p^s}{i} p^{(m+1)i} q_m^i$$

$$= \sum_{i=0}^{p^r} p^{(m+1)i} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2}$$

$$= \sum_{i=0}^{p^r} p^{(m+1)i} q_m^i\, W_{r,i}$$

$$= \frac{1}{r!} p^{rm+r(r+1)/2} q_m^r \prod_{i=1}^{r} (p^i - 1)$$

$$+ \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)i+r(r-1)/2} q_m^i\, U_{r,i},$$

by (2.3); $W_{r,i}$ and $U_{r,i}$ have the same meaning as in Lemma 2. We thus get

$$(2.7)\quad \Delta^r a^{(p-1)p^m} = \frac{1}{r!} q_m^r \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+1}^{p^r} \frac{1}{i!} p^{(m+1)(i-r)} q_m^i\, U_{r,i}\,.$$

We next set up a similar formula for $\Delta^r q_m$, where q_m is defined by (2.6). Indeed substitution in (2.4) gives

$$p^{rm+r(r+1)/2}\ \Delta^r q_m = \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2-(m+s+1)} \left(a^{(p-1)p^{m+s}} - 1\right)$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2-(m+s+1)} \sum_{i=1}^{p^r} \binom{p^s}{i} p^{(m+1)i} q_m^i$$

$$= \sum_{i=1}^{p^r} p^{(m+1)(i-1)} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2-s}$$

$$= \frac{1}{(r+1)!} p^{rm+r(r+1)/2} q_m^{r+1} \prod_{i=1}^{r} (p^i - 1)$$

$$+ \sum_{i=r+2}^{p^r} \frac{1}{i!} p^{(m+1)(i-1)+r(r-1)/2} q_m^i\, U'_{r,i},$$

by a slight modification of Lemma 2; the coefficient $U'_{r,i}$ is integral and is defined by

$$\frac{1}{i!}\, p^{r(r-1)/2}\, U'_{r,i} = \sum_{s=0}^{r} (-1)^s \left[{r \atop s}\right] \binom{p^{r-s}}{i} p^{s(s-1)/2-(r-s)}.$$

Hence

$$(2.8)\quad \Delta^r q_m = \frac{1}{(r+1)!}\, q_m^{r+1} \prod_{i=1}^{r} (p^i - 1) + \sum_{i=r+2}^{p^r} \frac{1}{i!}\, p^{(m+1)(i-r-1)}\, q_m^i\, U'_{r,i}.$$

Using the same method we can also evaluate $\Delta^r a^{p^m}$. It follows from (2.6) that

$$(2.9)\qquad a^{p^{m+s}} = a^{p^m} (1 + p^{m+1} q_m)^{e_s} \qquad \left(e_s = \frac{p^s - 1}{p-1}\right),$$

and thus substitution in (2.4) yields

$$p^{rm+r(r+1)/2}\, \Delta^r a^{p^m} = a^{p^m} \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{e_r} \binom{e_s}{i} p^{(m+1)i} q_m^i$$

$$= a^{p^m} \sum_{i=0}^{e_r} p^{(m+1)i} q_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{e_s}{i} p^{(r-s)(r-s-1)/2}.$$

Since $\binom{e_s}{i}$ is a polynomial in p^s of degree i, the same reasoning as before applies and we get after a little manipulation

$$(2.10)\qquad \Delta^r a^{p^m} = \frac{1}{r!}\, a^{p^m}\, q_m^r\, \frac{\prod_{i=1}^{r} (p^i - 1)}{(p-1)^r}$$

$$+ a^{p^m} \sum_{i=r+1}^{e_r} \frac{1}{i!}\, p^{(m+1)(i-r)}\, q_m^i\, U''_{r,i},$$

where $U''_{r,i}$ is integral.

Comparison of (2.7) and (2.10) shows that (2.7) is included in (2.10). Indeed it is easy to set up the following formula which includes both (2.7) and (2.10):

$$(2.11)\qquad \Delta^r a^{kp^m} = \frac{1}{r!}\, a^{kp^m}\, q_m^r\, k^r\, \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} + a^{kp^m} \sum_{i=r+1}^{e_r} \frac{1}{i!}\, p^{(m+1)(i-r)}\, q_m^i\, V_{r,i},$$

where $V_{r,i} = V_{r,i}^{(k)}$ is integral and $k \geq 1$. The proof of (2.11) is exactly like the proof of (2.10); the first step is to raise both members of (2.9) to the k-th power.

3. The main results. In order to make use of (2.7) and (2.10) it is evidently necessary to examine $p^{(m+1)(i-r)}/i!$. We suppose $i > r$, $r \leq p$. Then in the first place it is easily seen [6, p. 462] that $p^{i-r}/i!$ is integral (mod p), and a simple discussion shows that $p^{i-r}/i!$ is divisible by p unless (i) $i = p$, $r = p - 1$, or (ii) $i = p + 1$, $r = p$. We now state:

THEOREM 1. *The derivative* $\Delta^r a^{(p-1)p^m}$ *is integral for* $1 \leq r \leq p - 1$, *while* $\Delta^p a^{(p-1)p^m}$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r a^{(p-1)p^m}$ *are integral.*

THEOREM 2. *For* $1 \leq r \leq p$, $m \geq 0$,

$$(3.1)\qquad \Delta^r a^{(p-1)p^m} \equiv \frac{1}{r!}\, q_m^r \prod_{i=1}^{r} (p^i - 1) \pmod{p^m};$$

if $r < p - 1$, *the congruence is valid* $\pmod{p^{m+1}}$.

THEOREM 3. *The derivative* $\Delta^r a^{p^m}$ *is integral for* $1 \leq r \leq p - 1$, *while* $\Delta^p a^{p^m}$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r a^{(p-1)p^m}$ *are integral.*

THEOREM 4. *For* $1 \leq r \leq p$, $m \geq 0$,

$$(3.2)\qquad \Delta^r a^{p^m} \equiv \frac{1}{r!}\, a^{p^m}\, q_m^r\, \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m};$$

if $r < p - 1$, *the congruence is valid* $\pmod{p^{m+1}}$.

If we make use of (2.11) rather than (2.7) or (2.10) we get the following more general result.

THEOREM 4′. *For* $1 \leq r \leq p$, $m \geq 0$

$$\Delta^r a^{kp^m} \equiv \frac{1}{r!} a^{kp^m} q_m^r k^r \frac{\prod_{i=1}^r (p^i - 1)}{(p-1)^r} \pmod{p^m};$$

if $r < p - 1$, *the congruence is valid* $\pmod{p^{m+1}}$.

To apply (2.8) we first examine $p^{i-r-1}/i!$ for $i > r+1$, $r+1 \le p$. We have:

THEOREM 5. *The derivative* $\Delta^r q_m$ *is integral for* $1 \le r \le p-2$, *while* $\Delta^{p-1} q_m$ *has the denominator* p *provided* $a^{p-1} \not\equiv 1 \pmod{p^2}$; *if* $a^{p-1} \equiv 1 \pmod{p^2}$ *then all* $\Delta^r q_m$ *are integral.*

THEOREM 6. *For* $1 \le r \le p-1$, $m \ge 0$,

$$\Delta^r q_m \equiv \frac{1}{(r+1)!} q_m^{r+1} \prod_{i=1}^{r} (p^i - 1) \pmod{p^m}; \tag{3.3}$$

if $r < p - 2$, *the congruence is valid* $\pmod{p^{m+1}}$.

Theorem 3 is of course Schur's theorem; Theorems 5 and 6 are due to Zorn. The remaining theorems are presumably new.

4. Generalization for algebraic numbers. Let k be an algebraic number field of degree n and let $\mathfrak{p}$ denote a prime ideal of k; also let

$$N\mathfrak{p} = p^f; \qquad \mathfrak{p}^e \mid p, \qquad \mathfrak{p}^{e+1} \nmid p; \tag{4.1}$$

for simplicity we assume $p > n$. If α k is integral (mod $\mathfrak{p}$) and $\mathfrak{p} \nmid \alpha$, then by Fermat's Theorem

$$\alpha^{p^f - 1} = 1 + \beta, \qquad \beta \equiv 0 \pmod{\mathfrak{p}}. \tag{4.2}$$

It follows from (4.2) that

$$\alpha^{(p^f-1)p^m} = 1 + \beta_m, \qquad \beta_m \equiv 0 \pmod{\mathfrak{p}^{me+1}}, \tag{4.3}$$

while (4.3) implies

$$\alpha^{(p^f-1)p^{m+s}} = \sum_{i=0}^{p^r} \binom{p^s}{i} \beta_m^i \qquad (r \ge s). \tag{4.4}$$

Then, exactly as in § 2,

$$p^{rm+r(r+1)/2}\ \Delta^r\ \alpha^{(p^f-1)p^m} = \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] p^{(r-s)(r-s-1)/2} \sum_{i=0}^{p^r} \binom{p^s}{i} \beta_m^i$$

$$= \sum_{i=0}^{p^r} \beta_m^i \sum_{s=0}^{r} (-1)^{r-s} \left[{r \atop s}\right] \binom{p^s}{i} p^{(r-s)(r-s-1)/2};$$

application of Lemma 2 now leads to

$$(4.5)\ \Delta^r\ \alpha^{(p^f-1)p^m} = \frac{1}{r!}\ p^{-r(m+1)}\ \beta_m^r \prod_{i=1}^{r}(p^i-1) + \sum_{i=r+1}^{p^r} \frac{1}{i!}\ p^{-r(m+1)}\ \beta_m^i \omega_{r,i},$$

where $\omega_{r,i}$ is integral. Note that for $e > 1$ the right member of (4.5) need not be integral. Accordingly we assume $e = 1$; the assumption $p > n$ is then no longer needed.

We now have:

THEOREM 7. *Let* $N\mathfrak{p} = p^f$, $\mathfrak{p}^2 \nmid p$, $\mathfrak{p} \nmid \alpha$; *then* $\Delta^r\ \alpha^{(p^f-1)p^m}$ *is integral for* $1 \le r \le p-1$, *while* $\Delta^p\ \alpha^{(p^f-1)p^m}$ *has the denominator* p *provided* $\alpha^{p^f-1} \not\equiv 1 \pmod{\mathfrak{p}^2}$; *if* $\alpha^{p^f-1} \equiv 1 \pmod{\mathfrak{p}^2}$ *then all* $\Delta^r\ \alpha^{(p^f-1)p^m}$ *are integral.*

THEOREM 8. *With the hypotheses of Theorem* 7,

$$(4.6)\qquad \Delta^r\ \alpha^{(p^f-1)p^m} \equiv \frac{1}{r!}\left(\frac{\beta_m}{p^{m+1}}\right)^r \prod_{i=1}^{r}(p^i-1) \pmod{\mathfrak{p}^m}$$

for $r \le p$; *if* $r < p-1$ *the congruence is valid* $(\text{mod } \mathfrak{p}^{m+1})$.

In order to extend Theorems 3 and 4′ it is convenient to suppose that $\mathfrak{p}$ is a prime ideal of the first degree. The following two theorems may be proved.

THEOREM 9. *Let* $N\mathfrak{p} = p$, $\mathfrak{p}^2 \nmid p$, $\mathfrak{p} \nmid \alpha$; *then* $\Delta^r\ \alpha^{p^m}$ *is integral for* $1 \le r \le p-1$, *while* $\Delta^p\ \alpha^{p^m}$ *has the denominator* p *provided* $\alpha^{p-1} \not\equiv 1 \pmod{\mathfrak{p}^2}$; *if* $\alpha^{p-1} \equiv 1 \pmod{\mathfrak{p}^2}$ *then all* $\Delta^r\ \alpha^{p^m}$ *are integral.*

THEOREM 10. *With the hypotheses of Theorem* 9,

$$(4.7)\qquad \Delta^r\ \alpha^{kp^m} \equiv \frac{1}{r!}\left(\frac{k\,\beta_m}{p^{m+1}}\right)^r \frac{\prod_{i=1}^{r}(p^i-1)}{(p-1)^r} \pmod{\mathfrak{p}^m}$$

for $r \le p$; *if* $r < p-1$ *the congruence is valid* $(\text{mod } \mathfrak{p}^{m+1})$.

For brevity we omit the extension of Theorems 5 and 6 for algebraic numbers.

5. Another generalization. Changing slightly the notation (1.1) we put

$$\Delta_p a_{mp^i} = (a_{mp^{i+1}} - a_{mp^i})/p^{i+1}, \tag{5.1}$$

and

$$\Delta_p^r a_{mp^i} = (\Delta_p^{r-1} a_{mp^{i+1}} - \Delta_p^{r-1} a_{mp^i})/p^{i+1}.$$

Then clearly $\Delta_p \Delta_q = \Delta_q \Delta_p$. If a and k are arbitrary integers then if follows from a well-known theorem concerning (1.4) that

$$\delta_k a^k = \Delta_{p_1} \cdots \Delta_{p_s} a^k \qquad (k = p_1^{e_1} \cdots p_s^{e_s}) \tag{5.2}$$

is integral. In view of Schur's theorem we can state the following generalization.

THEOREM 11. *Let $(a,k) = 1$ and let $r <$ the smallest prime dividing k; define*

$$\delta_k^r a^k = \delta_k \delta_k^{r-1} a^k. \tag{5.3}$$

Then $\delta_k^r a_k$ is integral for $k > 1$.

Indeed because of the commutativity of the operators Δ_{p_i} we need only observe that (5.2) and (5.3) imply

$$\delta_k^r a^k = \Delta_{p_1}^r \cdots \Delta_{p_s}^r a^k \tag{5.4}$$

and the theorem follows immediately.

The restriction $(a,k) = 1$ can be removed by taking k sufficiently large as we shall see below.

A slight extension of Theorem 11 is contained in:

THEOREM 12. *Let*

$$(a,k) = 1, \quad k = p_1^{e_1} \cdots p_s^{e_s},$$

and let $r_i < p_i$, $j = 1, \cdots, s$; then

$$\Delta_{p_1}^{r_1} \cdots \Delta_{p_s}^{r_s} a^k \tag{5.5}$$

is integral for all $k > 1$.

We remark that the function defined in (5.2) can also be expressed in the form

$$\delta_k a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d)\, a^{dk},$$

where $\mu(d)$ is the Möbius function and

$$k_1 = p_1^{e_1+1} \cdots p_s^{e_s+1};$$

similarly (5.3) becomes

$$\delta_k^r a^k = \frac{(-1)^s}{k_1} \sum_{d \mid k} \mu(d)\, \delta_k^{r-1} a^{dk}.$$

Formulas of a different kind can be obtained by applying (2.4) to (5.4) and (5.5); for example, (2.5) suggests the following symbolic formula:

$$\delta_k^r a^k = k^{-r} \prod_{j=1}^{s} p_j^{r(r+1)/2} \cdot \prod_{j=1}^{s} a_j^{e_j} \prod_{i=0}^{r=1} (a_j - p_j^i),$$

where after expansion $a_1^{f_1} \cdots a_s^{f_s}$ is to be replaced by a^m,

$$m = p_1^{f_1} \cdots p_s^{f_s}.$$

A similar but slightly more complicated formula can be stated for (5.5). We shall omit the generalization of Theorems 11 and 12 to algebraic numbers.

6. Applications. In the theorems of §2 it is assumed that $p \nmid a$. However Theorem 3, for example, is easily extended to the case $p \mid a$. We can state that $\Delta^r a^{p^m}$ is integral for $r \le p-1$ and arbitrary a provided $m \ge r$. For let $p \mid a$; then, in view of (2.4), it is only necessary to verify that

$$p^{m+r-i} + \frac{1}{2}\, i(i-1) \ge rm + \frac{1}{2}\, r(r+1)$$

for $0 \le i \le r \le p-1$, $r \ge m$. This can be proved by induction with respect to m. In the next place since Theorem 11 is a direct consequence of Theorem 3 we infer that it also holds for all a provided $r \le \min(e_1, \cdots, e_s)$ in the notation of Theorem 11.

Now consider the number

$$C_k = \sum_{a=1}^{n} A_a a^k, \tag{6.1}$$

where A_a denote integers (mod p) and $n \geq 1$ is arbitrary. Then

$$(6.2) \qquad \Delta^r C_{k+p^m} = \sum_{a=1}^{n} A_a \Delta^r a^{k+p^m} \qquad (k \geq 0),$$

so that by the remark in the previous paragraph $\Delta^r C_{p^m}$ is certainly integral (mod p) provided $r \leq p-1$ and $r \leq m$. In the second place we may apply the operator δ_k^r defined in (5.2) and (5.3) and get

$$(6.3) \qquad \delta_k^r C_{h+k} = \sum_{a=1}^{n} A_a \delta_k^r a^{h+k};$$

we infer that $\delta_k^r C_k$ is integral provided $r <$ the smallest prime dividing k and $r \leq \min(i_1, \cdots, i_s)$, the notation being that of (5.2). Indeed a somewhat more general result can be obtained by applying Theorem 15, namely,

$$(6.4) \qquad \Delta_{p_1}^{r_1} \cdots \Delta_{p_s}^{r_s} C_{h+k} \qquad (h \geq 0)$$

is integral provided $r_t < p_t$, $r_t \leq e_t$, $t = 1, \cdots, s$.

As an instance of (6.1) we take the well-known formula for the Euler polynomial

$$(6.5) \qquad E_m(x) = \sum_{s=0}^{m} \frac{1}{2^s} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (x+i)^m.$$

(We use the notation of Nörlund [4] for the Euler and Bernoulli polynomials.) If $p > 2$ and x is integral (mod p) the preceding discussion applies. In particular using (2.4) we have:

THEOREM 13. *Let* $p > 2$ *and* x *be integral* (mod p). *Then*

$$\Delta^r E_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} E_{k+p^{m-i}}(x)$$

is integral (mod p) *provided* $r < p$, $r \leq m$.

For brevity we omit the generalizations corresponding to (6.3) and (6.4). The special case

$$(6.6) \qquad \sum_{de=m} \mu(d) E_{k+e}(x) \equiv 0 \qquad (\text{mod } m)$$

may be noted

As for the Bernoulli polynomials, it can be shown that if $p \nmid a$ and x is integral (mod p) then a formula of the type (6.1) holds for

$$\beta_k(x) = \frac{a^{k+1} - 1}{k+1} B_{k+1}(x). \tag{6.7}$$

(See for example Nielsen [3, Ch. 14].) Thus it follows that

$$\Delta^r \beta_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} \beta_{k+p^{m-i}}(x)$$

is integral for $r < p$, $r \le m$. If now we assume $p - 1 \nmid k$ and take a a primitive root (mod p) such that $a^{p-1} \equiv 1 \pmod{p^r}$ we get:

THEOREM 14. *Let $p > 2$ and x be integral* (mod p); *put* $H_k(x) = B_k(x)/k$. *Then if $p - 1 \nmid k + 1$,*

$$\Delta^r H_{k+p^m}(x) = p^{-rm - r(r+1)/2} \sum_{i=0}^{r} (-1)^i \left[{r \atop i}\right] p^{i(i-1)/2} H_{k+p^{m-i}}(x)$$

is integral for $r < p$, $r \le m$.

Finally corresponding to (6.6) we state

$$\sum_{de=m} \mu(d)\, \beta_{k+e}(x) \equiv 0 \pmod{m},$$

for $\beta_k(x)$ as defined in (6.7).

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