EXTENSION OF A RENEWAL THEOREM

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1. Introduction. A chance variable x will be called a *d-lattice variable* if

$$\sum_{n=-\infty}^{\infty} \Pr\{x = nd\} = 1 \tag{1}$$

and

$$d \text{ is the largest number for which (1) holds.} \tag{2}$$

If x is not a d-lattice variable for any d, x will be called a *nonlattice variable.* The main purpose of this paper is to give a proof of:

THEOREM 1. *Let $x_1, x_2, \cdots$ be independent identically distributed chance variables with $E(x_1) = m > 0$ (the case $m = +\infty$ is not excluded); let $S_n = x_1 + \cdots + x_n$; and, for any $h > 0$, let $U(a, h)$ be the expected number of integers $n \geq 0$ for which $a \leq S_n < a + h$. If the x_n are nonlattice variables, then*

$$U(a, h) \longrightarrow \frac{h}{m},\ 0 \qquad as\ a \longrightarrow +\infty,\ -\infty.$$

If the x_n are d-lattice variables, then

$$U(a, d) \longrightarrow \frac{d}{m},\ 0 \qquad as\ a \longrightarrow +\infty,\ -\infty.$$

(*If $m = +\infty$, h/m and d/m are interpreted as zero.*)

This theorem has been proved (A) for nonnegative d-lattice variables by Kolmogorov [5] and by Erdös, Feller, and Pollard [4]; (B) for nonnegative nonlattice variables by the writer [1], using the methods of [4]; (C) for d-lattice variables by Chung and Wolfowitz [3]; (D) for nonlattice variables such that the distribution of some S_n has an absolutely continuous part and $m < \infty$ by Chung

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and Pollard [2], using a purely analytical method; and (E) in the form given here by Harris (unpublished). Harris' proof does not essentially use the results of the special cases (A), (B), (C), (D); the proof given here obtains Theorem 1 almost directly from the special cases (A) and (B) by way of an integral identity and an equation of Wald.

2. **An integral identity.** Let N_1 be the smallest n for which $S_n > 0$, and write $z_1 = S_{N_1}$; let N_2 be the smallest $n > 0$, for which $S_{N_1+n} - S_{N_1} > 0$, and write $z_2 = S_{N_1+N_2} - S_{N_1}$, and so on. Continuing in this way, we obtain sequences N_1, $N_2, \cdots; z_1, z_2, \cdots$ of independent, positive, identically distributed chance variables such that

$$S_{N_1+\cdots+N_K} = z_1 + \cdots + z_K.$$

Let $V(t)$, $R(t)$ denote the expected number of integers $n \geq 0$ for which

$$T_n = z_1 + \cdots + z_n \leq t \text{ and } -t \leq S_n \leq 0,$$

$n < N_1$, respectively. That $V(t) < \infty$ follows from a theorem of Stein [6], and that $R(t) < \infty$ follows from $E(N_1) < \infty$, which we show in the next section. The integral identity is:

THEOREM 2. $U(a, h) = \int_0^\infty [R(t-a) - R(t-a-h)]\, dV(t).$

Proof. If n_K is the number of integers n with

$$N_1 + \cdots + N_K \leq n < N_1 + \cdots + N_{K+1} \text{ and } a \leq S_n < a + h,$$

we have

$$E(n_K \mid T_K = t) = R(t-a) - R(t-a-h),$$

so that

$$E(n_K) = \int_0^\infty [R(t-a) - R(t-a-h)]\, dF_K(t),$$

where $F_K(t) = \Pr\{T_K \leq t\}$. Summing over $K = 0, 1, 2, \cdots$, and using the fact that

$$V(t) = \sum_{K=0}^{\infty} F_K(t),$$

we obtain the theorem.

3. Wald's equation. The main purpose of this section is to note that $E(N_1)$ is finite, so that an equation of Wald [7, p. 142] holds.

THEOREM 3. *$E(N_1) < \infty$ and $mE(N_1) = E(z_1)$, so that m, $E(z_1)$ are both finite or both infinite.*

Proof. In showing $E(N_1)$ finite, we may suppose $\{x_n\}$ bounded above; for defining $x_n^* = \min\{s_n, M\}$ yields an $N_1^* \geq N_1$; choosing M sufficiently large makes $E(x_n^*) > 0$, and $E(N_1^*) < \infty$ implies $E(N_1) < \infty$. Since

$$\frac{T_K}{K} = \frac{S_{N_1 + \cdots + N_K}}{N_1 + \cdots + N_K} \cdot \frac{N_1 + \cdots + N_K}{K},$$

we obtain, letting $K \longrightarrow \infty$ and using the strong law of large numbers, first that $E(z_1) = mE(N_1)$ and next since if $\{x_n\}$ is bounded above and $\{z_n\}$ is bounded, that $E(N_1)$ is finite in this case and consequently in general.

4. The d-lattice case. For d-lattice variables, Theorem 2 yields

$$U(nd, d) = \sum_{s=0}^{\infty} r(s-n)\, v(s) = \sum_{s=0}^{\infty} r(s)\, v(s+n), \tag{3}$$

where $r(s) = R(sd) - R([s-1]\,d)$ and $v(s) = V(sd) - V([s-1]\,d)$. Now

$$\sum_{s=0}^{\infty} r(s) = \lim_{t\to\infty} R(t) = E(N_1) < \infty .$$

Theorem (A) asserts that

$$v(n) \longrightarrow \frac{d}{E(z_1)},\ 0 \qquad \text{as } n \longrightarrow \infty,\ -\infty;$$

applying this to (1) yields

$$U(nd, d) \longrightarrow \frac{dE(N_1)}{E(z_1)},\ 0 \qquad \text{as } n \longrightarrow \infty,\ -\infty,$$

and Wald's equation yields Theorem 1 for d-lattice variables.

5. The nonlattice case. For nonlattice variables we have, rewriting Theorem

2 with a change of variable,

$$U(a, h) = \int_M^\infty [R(t) - R(t-h)]\, dV(t+a).$$

For any $M > 0$, write

$$U(a, h) = I_1(M, a, h) + I_2(M, a, h),$$

where

$$I_1 = \int_0^M [R(t) - R(t-h)]\, dV(t+a)$$

and

$$I_2 = \int_0^\infty [R(t) - R(t-h)]\, dV(t+a).$$

Theorem B applied to $\{z_n\}$ yields

$$V(t+h) - V(t) \longrightarrow \frac{h}{E(z_1)}$$

for all $h > 0$ as $t \longrightarrow \infty$, so that, since $R(t)$ is monotone,

$$I_1 = \int_0^M R(t)\, dV(t+a) - \int_0^{M-h} R(t)\, dV(t+a+h)$$

$$\longrightarrow \frac{1}{E(z_1)} \cdot \int_{M-h}^M R(t)\, dt,\ 0 \qquad \text{as } a \longrightarrow \infty,\ -\infty$$

for fixed M, h. We now show that, for fixed h, $I_2(M, a, h) \longrightarrow 0$ as $M \longrightarrow \infty$ uniformly in a. We have

$$I_2 = \sum_{n=0}^\infty \int_{M+nh}^{M+(n+1)h} [R(t) - R(t-h)]\, dV(t+a)$$

$$\leq \sum_{n=0}^\infty R_1(M, n)\, [V(a+M+(n+1)h) - V(a+M+nh)],$$

where

$$R_1(M, n) = \sup\, [R(t) - R(t-h)]$$

as t varies over the interval $(M + nh, M + (n+1)h)$. Since, by Theorem (B),

$$V(b+h) - V(b) \longrightarrow \frac{h}{E(z_1)} \qquad \text{as } b \longrightarrow \infty,$$

there is a constant c (for the given h) such that

$$I_2(M, a, h) \leq c \sum_{n=0}^{\infty} R_1(M, n) \qquad \text{for all } M \text{ and } a.$$

Now

$$\sum_{n=0}^{\infty} R_1(M, 2n) \leq E(N_1) - R(M) \text{ and } \sum_{n=0}^{\infty} R_1(M, 2n+1) \leq E(N_1) - R(M),$$

and $R(M) \longrightarrow E(N_1)$ as $M \longrightarrow \infty$. Thus

$$|U(a, h) - I_1(M, a, h)| < \epsilon(M, h)$$

for all a, where $\epsilon(M, h) \longrightarrow 0$ *as* $M \longrightarrow \infty$ for fixed h. Then

$$\left| U(a, h - \frac{hE(N_1)}{E(z_1)} \right| \leq \epsilon(m, h) + \left| I_1(M, a, h) - \frac{1}{E(z_1)} \int_{M-h}^{M} R(t)\, dt \right|$$
$$+ \left| \frac{1}{E(z_1)} \int_{M-h}^{M} R(t)\, dt - hE(N_1) \right|,$$

so that

$$\limsup_{a \to \infty} \left| U(a, h) - \frac{hE(N_1)}{E(z_1)} \right|$$
$$\leq \epsilon(M, h) + \frac{1}{E(z_1)} \left| \int_{M-h}^{M} R(t)\, dt - hE(N_1) \right|.$$

Letting $M \longrightarrow \infty$ yields

$$U(a, h) \longrightarrow \frac{hE(N_1)}{E(z_1)} \qquad \text{as } a \longrightarrow \infty,$$

and Wald's equation yields Theorem 1 for $a \longrightarrow \infty$. Similarly,

$$U(a, h) \leq \epsilon(M, h) + |I_1(M, a, h)|$$

for all a, so that

$$\limsup_{a \to -\infty} U(a, h) \leq \epsilon(M, h)$$

and $U(a, h) \longrightarrow 0$ as $a \longrightarrow -\infty$. This completes the proof.

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