# THE SPACE $H^p$, $0 < p < 1$, IS NOT NORMABLE

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**1. Introduction.** For $p > 0$, the space $H^p$ is defined to be the class of functions $x(z)$ of the complex variable $z$, which are analytic in the interior of the unit circle, and satisfy

$$\sup_{0 \le r < 1} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta < \infty.$$

Set

$$A_p(r; x) = \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

and

$$||x|| = \sup_{0 \le r < 1} A_p(r; x).$$

S. S. Walters has shown [2] that $H^p$, $0 < p < 1$, is a linear topological space under the topology: $U \subset H^p$ is open if $x_0 \in U$ implies the existence of a "sphere" $S$: $||x - x_0|| < r$ such that $S \subset U$. He conjectured in [3] that $H^p$, $0 < p < 1$, does not have an equivalent normed topology, and it is shown here that this conjecture is correct. Since the conjugate space $(H^p)^*$ has sufficiently many members to distinguish elements of $H^p$, the space $H^p$, $0 < p < 1$, affords an interesting nontrivial example of a locally bounded linear topological space which is not locally convex.

2. *Proof.* For $x \in H^p$, $p > 0$, it is known [4, 160] that $A_p(r; x)$ is a nondecreasing function of $r$. Consequently, if $P(z)$ is a polynomial, then $P \in H^p$ and $||P|| = A_p(1; P)$. This observation will be used below.

According to a theorem of Kolmogoroff [1], a linear topological space has an equivalent normed topology if and only if the space contains a bounded open convex set. It will be shown here that the "sphere" $K_1$: $||x|| < 1$ of $H^p$,

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$0 < p < 1$, contains no convex neighborhood of the origin; this is clearly sufficient to show that $H^p$, $0 < p < 1$, contains no bounded open convex set, and hence is not normable.

To accomplish this, the contrary is assumed. Thus, it is assumed that $K_1$ contains a convex neighborhood $V$ of the origin. Since $V$ is open, $V$ contains a "sphere" $K_\epsilon$: $||x|| < \epsilon$. There will be exhibited $x_1, \cdots, x_N \in K_\epsilon$, and $a_1 > 0, \cdots, a_N > 0$, with $\sum a_k = 1$, such that $\sum a_k x_k \notin K_1$ and, *a fortiori*, $\sum a_k x_k \notin V$, in contradiction to the assumed convexity of $V$.

If $x(\theta)$ is a complex function of the real variable $\theta \in I$: $0 \leq \theta \leq 2\pi$, define

$$A(x) = \left(\frac{1}{2\pi}\int_0^{2\pi} |x(\theta)|^p \, d\theta\right)^{1/p}.$$

Once and for all, $k$ is any integer in the range $1, \cdots, N$. Let $I_k$ denote the interval

$$\frac{2\pi(k-1)}{N} < \theta < \frac{2k\pi}{N},$$

and let $i_k$ denote the degenerate interval consisting of the point $(2\pi/N)(k-1/2)$. Define the continuous function $c_k(\theta)$ to be zero on $I - I_k$, to be equal to $\epsilon N^{1/p}$ on $i_k$, and to be linear on each of the two intervals in $I_k - i_k$. Let

$$a_k = B_N k^{-1/p}, \; B_N = \left(\sum_1^N k^{-1/p}\right)^{-1},$$

so that $a_k > 0$ and $\sum a_k = 1$. It is easily verified that

$$A(c_k) = \epsilon(p+1)^{-1/p} < \epsilon$$

and

$$A(\sum a_k c_k) = \epsilon B_N (p+1)^{-1/p}\left(\sum_1^N k^{-1}\right)^{1/p}.$$

Since $B_N$ is bounded away from zero below, $N$ can be chosen such that

$$A(\sum a_k c_k) > 1.$$

Each $c_k(\theta)$ is absolutely continuous on $I$. Given $\alpha > 0$, it follows that there is a trigonometrical polynomial

$$T_k(\theta) = \sum_{n=-m_k}^{m_k} a_{nk}\, e^{in\theta}$$

such that

$$|T_k(\theta) - c_k(\theta)| < \alpha$$

uniformly in $\theta$. Setting

$$p_k(\theta) = e^{im_k\theta}\, T_k(\theta)$$

gives

$$|p_k(\theta) - e^{im_k\theta}\, c_k(\theta)| < \alpha$$

uniformly in $\theta$. Set

$$C_k(\theta) = e^{im_k\theta}\, c_k(\theta).$$

It is clear that $A(C_k) = A(c_k)$ and $A(\sum a_k C_k) = A(\sum a_k c_k)$. Since $A(x)$ is a continuous function of $x$, it follows, if $\alpha$ is small enough, that $A(p_k) < \epsilon$ and $A(\sum a_k p_k) > 1$.

Let

$$P_k(z) = \sum_{n=-m_k}^{m_k} a_{nk}\, z^{n+m_k},$$

so that $P_k(e^{i\theta}) = p_k(\theta)$. As previously remarked,

$$||P_k|| = A_p(1; P_k) = A(p_k)$$

and

$$||\sum a_k P_k|| = A_p(1; \sum a_k P_k) = A(\sum a_k p_k).$$

Since $P_1, \cdots, P_N \in K_\epsilon \subset V \subset K_1 \subset H^p$, $a_1 > 0, \cdots, a_N > 0$, $\sum a_k = 1$, and $\sum a_k P_k \notin K_1$, we have obtained the required contradiction.

## REFERENCES

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