COMPLETELY CONTINUOUS NORMAL OPERATORS WITH PROPERTY L

IRVING KAPLANSKY

1. Introduction. Two matrices A and B are said to have property L if it is possible to arrange their characteristic roots

$$A: \lambda_1, \lambda_2, \cdots, \lambda_n$$

$$B: \mu_1, \mu_2, \cdots, \mu_n$$

in such a way that for every α, the characteristic roots of $\alpha A + B$ are given by $\alpha \lambda_i + \mu_i$. In [1] this property is investigated, and among other things a conjecture of Kac is confirmed by showing that if A and B are hermitian, then they commute. In [2] this is generalized by replacing "hermitian" by "normal".

In this note we launch the project of generalizing such results to (complex) Hilbert space. However, since it is not clear how to formulate the problem for general operators (especially in the presence of a continuous spectrum), we shall content ourselves with the completely continuous case. For self-adjoint operators we obtain a fully satisfactory generalization (Theorem 1). For the more general case of normal operators we find ourselves obliged to make an extra assumption roughly to the effect that nonzero characteristic roots are paired only to nonzero roots. In the finite-dimensional case such an assumption would be harmless; indeed, by adding suitable constants to A and B, we could even arrange to have all the characteristic roots of A and B nonzero. It would nevertheless be of interest to determine whether this blemish can be removed from Theorem 2.

2. Remarks. Before we state the results, some remarks are in order. The number λ is a characteristic root of A if $A - \lambda I$ has a nonzero null space. If A is a completely continuous normal operator, its characteristic roots are either finite in number or form a sequence approaching zero. We have an orthogonal decomposition of the Hilbert space:

Received August 23, 1952. The preparation of this paper was sponsored (in part) by the Office of Naval Research.

Pacific J. Math. 3 (1953), 721-724

$$H = H(0) \oplus H(\lambda_1) \oplus H(\lambda_2) \oplus \cdots,$$

where A acts on $H(\lambda_i)$ as a multiplication by λ_i. The dimension of $H(\lambda_i)$ is called the multiplicity of the characteristic root λ_i; it is finite except possibly for the characteristic root 0.

Now even though A and B are both to be normal, $\alpha A + B = C$ need not (a priori) be normal. We must accordingly give further attention to the meaning of the multiplicity of a characteristic root ν of C. For our purposes virtually any reasonable definition would do; we select the following one. We note that the null spaces of the operators $C - \nu I$, $(C - \nu I)^2$, $\cdots$ form an ascending chain, and we form their union; the dimension of this union is the multiplicity of ν. Note that this agrees with customary usage in the finite-dimensional case.

We shall need the (easily proved) additivity of the multiplicity. In detail: suppose H is an orthogonal direct sum of two closed subspaces both invariant under C; then the multiplicity of ν in the whole space is the sum of its multiplicities in the two subspaces.

3. Results. We are now ready to define property L. We do this in a way that is adequate for the proof, although it does not treat A and B symmetrically.

Let A and B be completely continuous normal operators. Let there be given two sequences λ_i, μ_i of complex numbers. We say that A and B have *property L* (relative to the two sequences) provided:

(1) The λ's constitute precisely the nonzero characteristic roots of A, each counted as often as its multiplicity.

(2) If, for a certain α and ν, there are k values of i such that $\nu = \alpha\lambda_i + \mu_i$, then $\alpha A + B$ has ν as a characteristic root at least of multiplicity k.

THEOREM 1. *Let A and B be completely continuous self-adjoint operators with property L. Then A and B commute.*

THEOREM 2. *Let A and B be completely continuous normal operators with property L, relative to the sequences λ_i and μ_i. Suppose further that the μ's are all nonzero. Then A and B commute.*

4. Proof. The two theorems can conveniently be proved simultaneously. We can suppose that λ_1 is a characteristic root of maximum absolute value, that is, $|\lambda_1| = ||A||$. For brevity write $\lambda = \lambda_1$, $\mu = \mu_1$. By an application of the definition of property L, with $\alpha = 0$, we see that μ is a characteristic root of B. We are going to prove that there exists a nonzero vector x with

$$Ax = \lambda x, \ Bx = \mu x.$$

If $\mu \neq 0$, we are ready to proceed. If $\mu = 0$, then by hypothesis both A and B are self-adjoint. We replace B by $A + B$ which is again self-adjoint; this replaces μ by

$$\lambda + \mu = \lambda \neq 0.$$

So in any event we are entitled to assume that μ is nonzero.

Let $H(\mu)$ be the (finite-dimensional) characteristic subspace of B for the characteristic root μ, and K the orthogonal complement; let E and F be the projections on $H(\mu)$ and K. We note that $B - \mu I$ is nonsingular on K; let S be defined as its inverse on K and as 0 on $H(\mu)$. Thus we have

$$SF(B - \mu I) = F. \tag{1}$$

Next we consider $E(A - \lambda I)E$ as an operator on $H(\mu)$, and we are going to prove that it is singular. Suppose the contrary and define R to be its inverse on $H(\mu)$, 0 on K. Then R will satisfy

$$R(B - \mu I) = 0, \ R(A - \lambda I)E = E, \ RF = 0. \tag{2}$$

Choose $\alpha \neq 0$ so that

$$||\alpha SF(A - \lambda I)(F - RAF)|| < 1. \tag{3}$$

By hypothesis, the operator $\alpha A + B$ has $\alpha\lambda + \mu$ as a characteristic root, say with characteristic vector $y \neq 0$. We have

$$\alpha(A - \lambda I)y + (B - \mu I)y = 0. \tag{4}$$

Write $y = Ey + Fy$ in (4), apply R, and then use (2); we find that $Ey = -RAFy$, and so

$$y = Ey + Fy = (F - RAF)y. \tag{5}$$

Next apply SF to (4), and use (1) and (5):

$$Fy = -\alpha SF(A - \lambda I)(F - RAF)y. \tag{6}$$

On contemplating (6) in conjunction with (3) we see that Fy must be 0. But then $y = 0$ by (5). This contradiction shows that we were in error in supposing $E(A - \lambda I)E$ to be nonsingular on $H(\mu)$. Consequently we can find in $H(\mu)$

a nonzero vector x annihilated by $E(A - \lambda I)E$. Then since $Ex = x$, we have $EAx = \lambda x$. Form the orthogonal decomposition

$$Ax = EAx + FAx = \lambda x + FAx. \tag{7}$$

But

$$||Ax|| \leq |\lambda| \, ||x||,$$

since the norm of A is $|\lambda|$. Hence in (7) we must actually have $Ax = \lambda x$. Also $Bx = \mu x$ since x is in $H(\mu)$, and we have fulfilled our initial objective.

Let M be the orthogonal complement of x. It follows from the additivity of multiplicity (see above) that when the operator $\alpha A + B$ is confined to M, the multiplicity of its characteristic root $\alpha\lambda + \mu$ is diminished by precisely 1, while all other characteristic roots have unchanged multiplicity. Thus A and B, confined to M, satisfy property L relative to the sequences λ_i and μ_i for $i \geq 2$. The procedure may now be repeated to get within M another joint characteristic vector for A and B. In this way we proceed down the nonzero characteristic roots of A. Finally we are left with the null space of A, which of course commutes with whatever is left of B. Hence A and B commute.

5. Remark. As soon as we know that A and B commute (and hence can be simultaneously put in diagonal form), we can assert that they satisfy property L symmetrically, and indeed various stronger statements are obvious consequences of simultaneous diagonal form.

References

1. T. S. Motzkin and O. Taussky, *Pairs of matrices with property L*, Trans. Amer. Math. Soc. 73 (1952), 108-114.

2. N. A. Wiegmann, *A note on pairs of normal matrices with property L*, Proc. Amer. Math. Soc. 4 (1953), 35-36.

University of Chicago and
National Bureau of Standards, Los Angeles