EIGENVALUES OF CIRCULANT MATRICES

RICHARD S. VARGA

1. Introduction. The integral equations

$$(1) \qquad u(z_j) = \lambda \oint_C A(z, z_j)\, u(z)\, dq + \Phi(z_j),$$

where C is a smooth closed curve, and

$$A(z, z_j) = d \arg (z - z_j)/dq,$$

has many important applications. Thus [6], iteration of (1) gives a solution for the conformal mapping problem for the interior and exterior of C.

In numerical work, the rate of convergence of such iterations depends on the eigenvalues of the integral operator $A(z, z_j)$. It is known that the absolute values of the nontrivial[1] eigenvalues of the integral operator $A(z, z_j)$ are less than one. A recent paper [1] gives a sharper bound to the eigenvalues.

However, in numerical computation, equation (1) must be replaced [6] by a discrete equation of the form

$$(2) \qquad u_{r+1}(z_j) = \lambda \sum_{k=1}^{N} A_{jk}\, u_r(z_j) + \Phi(z_j).$$

This makes it important to know the relation between the eigenvalues of $A(z, z_j)$ and those of the matrix A_{jk}.

We determine this relation below in the special case that C is an ellipse. In particular, we show that the eigenvalues of A_{jk} approach $N/2$ times those of $A(z, z_j)$ with exponential convergence. Since trapezoidal integration based on trigonometric interpolation gives exponential accuracy, this fact is probably

[1] It is easy to verify that for the eigenfunction $u(z) \equiv 1$, we have the simple eigenvalue unity. By the nontrivial eigenvalues of $A(z, z_j)$, we mean all other eigenvalues.

Received November 26, 1952. This work was done at Harvard University under Project N5ori-07634 with the Office of Naval Research. The author wishes to express his appreciation to Professor Garrett Birkhoff for helpful suggestions.

Pacific J. Math. 4 (1954), 151-160

true for any analytic curve. However, it seemed most interesting to get quantitative bounds in the special case of ellipses.

2. Circulant matrices. For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

it is known [1] that

$$A(z, z_j) = \pi^{-1} \frac{ab}{(a^2+b^2) - (a^2-b^2)\cos(q+q_j)}.$$

It follows that the associated matrix

$$A_{j,k} = \left\| \pi^{-1} \frac{ab}{(a^2+b^2) - (a^2-b^2)\cos(q_k+q_j)} \right\| = \|a_{j,k}\|$$

is a *circulant* matrix, in the usual sense that

$$(3) \qquad a_{i+h,j-h} = a_{i,j}$$

for all integers h, where subscripts are taken mod N. We first show how to compute the eigenvalues of a circulant matrix in a way which seems somewhat more simple and perspicuous than that given in the literature [7].

Following the notation of [5], let $\vec{\epsilon}_1, \cdots, \vec{\epsilon}_n$ denote the unit vectors in $V_n(C)$, and let

$$\vec{\epsilon}_i \longrightarrow \sum a_{ij} \vec{\epsilon}_j$$

denote the linear transformation associated with the matrix A_{jk}. It is convenient to introduce the new basis

$$\vec{\alpha}_1, \cdots, \vec{\alpha}_n \text{ defined by } \vec{\alpha}_l = \sum \omega^{lk} \vec{\epsilon}_k,$$

where $\omega = e^{i2\pi/n}$ is a primitive nth root of unity. The matrix

$$\Omega = \|\omega^{ik}\|$$

is closely related to that used in Lagrangian resolvents; it is symmetric, and $n^{-1/2}\Omega$ is *unitary*.

Relative to the basis $\vec{\alpha}_1, \cdots, \vec{\alpha}_n$, cyclic matrices [2, p. 124] are diagonalized, while circulant matrices (whose squares are cyclic matrices) reduce to *monomial* matrices which are reducible into 2×2 components. Specifically, easy computations show that the basic transposition

$$R_m : \vec{\epsilon}_k \longrightarrow \vec{\epsilon}_{m-k} \qquad (m = 0, 1, \cdots, n-1),$$

corresponding to a circulant matrix with ones on a reversed diagonal: $i + j \equiv m \pmod{n}$, carries $\vec{\alpha}_i$ into $\omega^{im}\, \vec{\alpha}_{n-i}$. Hence, a general circulant matrix $\sum c_m R_m$ carries $\vec{\alpha}_i$ into

$$(\sum c_m \omega^{im})\vec{\alpha}_{n-i}\,.$$

Thus, in general, a pair of eigenvalues is associated with each subspace spanned by $\vec{\alpha}_i$ and $\vec{\alpha}_{n-i}$ (we have an exception when $i = n$, and, if n is even, when $i = n/2$). On this subspace, A is similar to

$$\begin{pmatrix} 0 & c_m \omega^{im} \\ c_m \omega^{-im} & 0 \end{pmatrix}.$$

Hence, the eigenvalues λ_i, λ_{n-i} are the distinct roots of:

$$(4) \quad \lambda^2 = (\sum c_m \omega^{im})(\sum c_m \omega^{-im}) = \left(\sum c_m \cos \frac{2\pi im}{n}\right)^2 + \left(\sum c_m \sin \frac{2\pi im}{n}\right)^2.$$

For $i = n$, and $i = n/2$ for n even, we have, similarly, the respective eigenvalues:

$$\lambda_n = \sum c_m\,; \quad \lambda_{n/2} = \sum_{m=0}^{n-1} (-1)^m c_m\,.$$

If the coefficients c_m are real, then it follows from (4) that all the eigenvalues are real. Furthermore, if we have an evenness-property for c_m's, that is, $c_k = c_{n-k}$, then

$$\sum c_r \sin \frac{2\pi kr}{n} = 0,$$

which implies

$$\lambda_k = + \sum c_r \cos \frac{2\pi kr}{n} \; ; \; \lambda_{n-k} = - \sum c_r \cos \frac{2\pi kr}{n} .$$

If $c_k = - c_{n-k}$, then

$$\sum c_r \cos \frac{2\pi kr}{n} = 0,$$

which implies

$$\lambda_k = + \sum_{r=0}^{n-1} c_r \sin \frac{2\pi kr}{n} \; ; \; \lambda_{n-k} = - \sum_{k=0}^{n-1} c_r \sin \frac{2\pi kr}{n} .$$

The eigenvalues in the real or complex case can be conveniently calculated by the formulas

$$b_k = \sum_{j=0}^{n-1} c_{j+k}\, c_j ; \; \nu_j = \sum_{k=0}^{n-1} b_k \cos \frac{2\pi kj}{n}, \tag{5}$$

where λ_i, λ_{n-i} are the distinct roots of

$$\lambda^2 = \nu_i ; \; \lambda_0 = + \sqrt{\nu_0} \; ; \; \lambda_{n/2} = + \sqrt{\nu_{n/2}} .$$

This involves about fifty per cent fewer steps than that usually given.

3. Discrete approximation to eigenvalues. For the circulant matrix A_{jk}, associated with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$a > b > 0$, we have the real coefficients

$$c_j = \frac{ab}{(a^2 + b^2) - (a^2 - b^2) \cos (2\pi j/N)} \qquad (j = 0, 1, \cdots, N-1).$$

Since $c_j = c_{N-j}$, we have then as the positive eigenvalues:

$$(6) \qquad \lambda_k(N) = + \sum_{r=0}^{N-1} c_r \cos \frac{2\pi kr}{N} = ab \sum_{r=0}^{N-1} \frac{\cos(2\pi kr/N)}{(a^2+b^2)-(a^2-b^2)\cos(2\pi r/N)}$$

$$(k = 0, 1, \cdots, [N/2]).$$

Now

$$\lim_{N\to\infty} \frac{2\pi}{N} \lambda_k(N) = ab \int_{-\pi}^{+\pi} \frac{\cos k\theta \, d\theta}{(a^2+b^2)-(a^2-b^2)\cos\theta} = G(k).$$

But $G(k)$ is tabulated [**4**, Table 65, no. 3]:

$$G(k) = \pi\left(\frac{a-b}{a+b}\right)^k.$$

Hence, from (6), it follows that

$$(7) \qquad \lambda_k(N) \sim \frac{N}{2}\left(\frac{a-b}{a+b}\right)^k \qquad (k = 0, 1, 2, \cdots, [N/2]),$$

which gives us an asymptotic approximation to the eigenvalues of the matrix A_{jk}. The eigenvalues of $A(z, z_j)$ can be shown, by means of [**3**], to be:

$$\left(\frac{a-b}{a+b}\right)^k \qquad (k = 0, 1, 2, \cdots).$$

4. Error estimates. We define $E(m, N)$, the error, by

$$(8) \qquad \int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2+b^2)-(a^2-b^2)\cos\theta} + E(m, N)$$

$$= \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos(2\pi km/N)}{(a^2+b^2)-(a^2-b^2)\cos(2\pi k/N)}.$$

We shall assume that $N > 2m$, and that N is even. We have:

$$\int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2+b^2)-(a^2-b^2)\cos\theta} = \frac{1}{a^2+b^2} \int_0^{2\pi} \frac{\cos m\theta \, d\theta}{1-\gamma\cos\theta},$$

where

$$\gamma = \frac{a^2 - b^2}{a^2 + b^2} < 1.$$

Since $\gamma \cos \theta < 1$ for all values of θ, we can write:

$$\int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} = \frac{1}{a^2 + b^2} \int_0^{2\pi} \cos m\theta \left(\sum_{k=0}^{\infty} \gamma^k \cos^k \theta \right) d\theta$$

$$= \frac{1}{a^2 + b^2} \sum_{k=0}^{\infty} \gamma^k \int_0^{2\pi} \cos^k \theta \cos m\theta \, d\theta,$$

since the series converges uniformly and absolutely. Now

$$\cos^k \theta = \frac{1}{2} \beta_0^k + \sum_{p=1}^{k} \beta_p^k \cos p\theta ,$$

where the Fourier coefficients are given by

$$\beta_p^k = \frac{1}{\pi} \int_0^{2\pi} \cos^k \theta \cos p\theta \, d\theta. \tag{9}$$

Rewriting, we have

$$\int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta}$$

$$= \frac{1}{a^2 + b^2} \sum_{k=0}^{\infty} \gamma^k \left\{ \frac{1}{2} \beta_0^k \int_0^{2\pi} \cos m\theta \, d\theta + \sum_{p=1}^{k} \beta_p^k \int_0^{2\pi} \cos m\theta \cos p\theta \, d\theta \right\}.$$

Using the orthogonality of the cosines in the interval $[0, 2\pi]$, we obtain:

$$\int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} = \frac{\pi}{a^2 + b^2} \sum_{k=m}^{\infty} \gamma^k \beta_m^k . \tag{10}$$

We shall now obtain a similar expression for the sum in (8):

$$\frac{2\pi}{N}\sum_{k=0}^{N-1}\frac{\cos(2\pi km/N)}{(a^2+b^2)-(a^2-b^2)\cos(2\pi k/N)}$$

$$=\frac{2\pi}{N(a^2+b^2)}\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}\sum_{j=0}^{\infty}\gamma^j\cos^j\frac{2\pi k}{N}\ .$$

Since $\gamma < 1$, the sum is absolutely convergent, and we have

$$\frac{2\pi}{N}\sum_{k=0}^{N-1}\frac{\cos(2\pi km/N)}{(a^2+b^2)-(a^2-b^2)\cos(2\pi k/N)}$$

$$=\frac{2\pi}{N(a^2+b^2)}\sum_{j=0}^{\infty}\gamma^j\left\{\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}\cos^j\frac{2\pi k}{N}\right\}.$$

Now,

$$\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}\cos^j\frac{2\pi k}{N}=\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}\left\{\frac{1}{2}\beta_0^j+\sum_{p=1}^{j}\beta_p^j\cos\frac{2\pi kp}{N}\right\}.$$

Since this is a finite sum, then

$$\frac{2\pi}{N}\sum_{k=0}^{N-1}\frac{\cos(2\pi km/N)}{(a^2+b^2)-(a^2-b^2)\cos(2\pi k/N)} \tag{11}$$

$$=\frac{2\pi}{N(a^2+b^2)}\sum_{j=0}^{\infty}\gamma^j\left\{\frac{1}{2}\beta_0^j\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}+\sum_{p=1}^{j}\beta_p^j\sum_{k=0}^{N-1}\cos\frac{2\pi km}{N}\cos\frac{2\pi kp}{N}\right\}.$$

From [8, p. 212], we have the result that

$$\sum_{j=0}^{N-1}\cos\frac{2\pi kj}{N}\cos\frac{2\pi lj}{N}=\begin{cases} N \text{ for } k=0,\ N,\ 2N,\cdots, \text{ if } l=0; \text{ zero otherwise}\\ \dfrac{N}{2} \text{ for } k=l,\ N-l,\ N+l,\ 2N-l,\cdots, \text{ if } l\neq 0; \text{ zero}\\ \text{otherwise}.\end{cases}$$

Thus, in the case that $m \neq 0$, we have, for example

$$\sum_{p=0}^{j} \beta_p^j \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \cos \frac{2\pi pk}{N} = \frac{N}{2} \{ \beta_m^j + \beta_{N-m}^j + \cdots$$

$$+ \beta_{(r_j-1)N+m}^j + \beta_{r_j N-m}^j \},$$

where

$$r_j = \left[\frac{j+m}{N} \right].$$

Thus, we obtain, for $m \neq 0$

$$(12) \qquad \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos (2\pi km/N)}{(a^2+b^2) - (a^2-b^2) \cos (2\pi k/N)}$$

$$= \frac{\pi}{a^2+b^2} \cdot \sum_{j=m}^{\infty} \gamma^j \{ \beta_m^j + \cdots + \beta_{r_j N-m}^j \}.$$

From our original definition, we have

$$(13) \quad E(m,N) = \frac{\pi}{a^2+b^2} \sum_{j=N-m}^{\infty} \gamma^j \{ \beta_{N-m}^j + \cdots + \beta_{r_j N-m}^j \}, \; m \neq 0$$

$$E(0,N) = \frac{\pi}{a^2+b^2} \sum_{j=N}^{\infty} \gamma^j \{ \beta_N^j + \cdots + \beta_{r_j N}^j \}.$$

We establish the following:

LEMMA.

$$\beta_j^l = \begin{cases} 0; \; l-j \not\equiv 0 \pmod 2 \\ \dfrac{1}{2^l} \left(C_{(l-j)/2}^l + C_{(l+j)/2}^l \right); \; l-j \equiv 0 \pmod 2. \end{cases}$$

Proof. From (9), we have

$$\beta_j^l = \frac{1}{\pi}\int_0^{2\pi} \cos^l \theta \cos j\theta \, d\theta = \frac{1}{\pi}\oint \frac{(z+1/z)^l}{2^l}\left(\frac{z^j + z^{-j}}{2}\right)\frac{dz}{zi},$$

where the path of integration is the circumference of the unit circle. This reduces to

$$\beta_j^l = \frac{1}{2\pi i}\cdot\frac{1}{2^l}\sum_{p=0}^{l} C_p^l\left\{\oint \frac{z^{j+2p}}{z^{l+1}}\,dz + \oint \frac{z^{-j+2p}}{z^{l+1}}\,dz\right\}.$$

Applying Cauchy's residue theorem, we have the desired result.

COROLLARY.

$$\frac{1}{2}\beta_0^l + \sum_{j=1}^{l}\beta_j^l = 1.$$

Proof. This is an immediate consequence of the Lemma. From the Lemma, we see that $E(m,N)$ is nonnegative, since the terms in the sum in (6) are nonnegative. Furthermore, by the Corollary, it is clear that

$$E(m,N) < \frac{\pi}{a^2+b^2}\sum_{j=N-m}^{\infty}\gamma^j \tag{14}$$

$$= \frac{\pi}{a^2+b^2}\,\frac{\gamma^{N-m}}{1-\gamma} = \frac{\pi}{2b^2}\left(\frac{a^2-b^2}{a^2+b^2}\right)^{N-m}.$$

In the particular case $a = 3$, $b = 2$, this reduces to

$$E(m,N) < \frac{\pi}{8}\left(\frac{5}{13}\right)^{N-m},$$

which is in good agreement with the numerical results in § 5.

5. Numerical results. For $N = 16$, $a = 3$, $b = 2$, the following numerical results were obtained:

Table 1

		Calculated	Approximated by (7) of § 3
1.	$\sqrt{\nu_0}$	8.00000	8.00000
2.	$\sqrt{\nu_1}$	1.60000	1.60000
3.	$\sqrt{\nu_2}$	0.32000	0.32000
4.	$\sqrt{\nu_3}$	0.06400	0.06400
5.	$\sqrt{\nu_4}$	0.01279	0.01280
6.	$\sqrt{\nu_5}$	0.00256	0.00256
7.	$\sqrt{\nu_6}$	0.00051	0.00051
8.	$\sqrt{\nu_7}$	0.00011	0.00010
9.	$\sqrt{\nu_8}$	0.00003	0.00002

References

1. L. V. Ahlfors, *Remarks on the Neumann-Poincaré integral equation*, Pacific J. Math. 2 (1952), 271-280.

2. A. C. Aitken, *Determinants and matrices*, Oliver and Boyd, Edinburgh, 1939.

3. S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, Composito Math. 8 (1951), 205-250.

4. D. Bierens De Haan, *Nouvelles tables d'integralés definies*, Steckert, New York, 1939.

5. G. Birkhoff and S. MacLane, *A survey of modern algebra*, Macmillan, New York, 1949.

6. G. Birkhoff, D. M. Young, and E. H. Zarantonello, *Numerical methods in conformal mapping*, Proceedings of the fourth symposium on applied mathematics, American Mathematical Society, McGraw-Hill, New York, 1953.

7. T. Muir and W. Metzler, *A treatise on the theory of determinants*, Longmans Green, New York, 1933.

8. C. Runge and H. König, *Vorlesungen über numerisches Rechnen*, Springer, Berlin, 1924.

HARVARD UNIVERSITY