THE TWO NONCHARACTERISTIC PROBLEM WITH DATA PARTLY ON THE PARABOLIC LINE

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1. Introduction. We consider the equation

$$K(y)u_{xx} + u_{yy} = 0, \tag{1}$$

where $K(y)$ is a monotone increasing, twice differentiable function of y with $K(0) = 0$. The equation is elliptic for $y > 0$, hyperbolic for $y < 0$, and $y = 0$ is a parabolic line. Equations of this type have been of interest recently because of certain problems arising in transonic flow. The equations for the compressible flow of an ideal fluid when transformed to the hodograph plane lead, in the transonic case, to an elliptic-hyperbolic equation of the above type.

In this paper the existence and uniqueness of the solution of a certain boundary value problem are discussed. It will be clear from the methods employed that estimates can be obtained for the solution in terms of the boundary values, although these estimates are not stated explicitly.

Equation (1) has real characteristics in the lower half-plane given by the equations

$$\frac{dy}{dx} = + \frac{1}{\sqrt{-K}}, \tag{2a}$$

$$\frac{dy}{dx} = - \frac{1}{\sqrt{-K}}. \tag{2b}$$

Let γ_1 be the characteristic of (2b) passing through (0, 0), and γ_2 the member of (2a) passing through (2, 0). Then the segment $0 \leq x \leq 2$, along with γ_1 and γ_2, will enclose a domain which we denote by D'. Let Γ, given by $y = h(x)$, be a curve lying in D' and emanating from the point (2, 0). It will be assumed that $h(x)$ intersects each characteristic of (1) at most once, and that there are two positive constants m and M such that $0 < m \leq h'(x) \leq M$. We call

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$P_0(x_0, y_0)$ the point where Γ intersects γ_1.

The following problem is treated. Let

$$F_0(x), \ (0 \le x \le 2), \qquad G_0(x), (x_0 \le x \le 2),$$

be two given functions possessing continuous derivatives of the fifth order. A solution $u(x, y)$ of (1) is sought in D' which satisfies the conditions $u(x, y) = F_0(x)$ $(0 \le x \le 2)$ and $u[x, h(x)] = G_0(x)$ $(x_0 \le x \le 2)$. We denote by D the domain bounded by γ_1, Γ, and the segment $0 \le x \le 2$. Then clearly all considerations may be confined to D instead of D'. For, once $u(x, y)$ is determined in D, the Cauchy problem may be solved with the function u and its first derivatives prescribed along $h(x)$ and this will yield u in the remainder of D'. The solution of this problem is well known for the case of purely hyperbolic equations [2]. The case where Γ coincides with one of the characteristics has been treated earlier [3], and under those circumstances certain simplifications take place and some of the hypotheses can be weakened.

2. The step-function case. Suppose $K^*(y)$ is a nondecreasing step-function with m steps:

$$K^*(y) = -\lambda_i^2, \quad y_i \le y \le y_{i-1} \qquad (i = 1, 2, \cdots, m).$$

We will take

$$\lambda_1^2 > 0, y_0 = 0, \text{ and } y_m = c < 0.$$

The boundary value problem proposed in § 1 will first be solved for the equation

$$K^*(y) u_{xx} + u_{yy} = 0. \tag{3}$$

The characteristic curves of equation (2) are polygonal arcs, and the domain D' will be divided into strips in each one of which the solution $u(x, y)$ will satisfy the wave equation with the appropriate constant λ_i^2.

Thus by a solution of (3) we mean a function $u(x, y)$ which satisfies the equation

$$\lambda_i^2 u_{xx} - u_{yy} = 0$$

in the ith strip, and in addition u, u_x, and u_y are continuous throughout D'. In the ith strip a solution of (3) will have the form

$$f_i(x+\lambda_i y)+g_i(x-\lambda_i y),$$

and this is valid for $y_i \leq y \leq y_{i-1}$. To preserve continuity of u, u_x, u_y at the junction of two strips we have

$$f_i(x+\lambda_i y_i)+g_i(x-\lambda_i y_i)=f_{i+1}(x+\lambda_{i+1}y_i)+g_{i+1}(x-\lambda_{i+1}y_i)$$

$$\lambda_i f_i'(x+\lambda_i y_i)-\lambda_i g_i'(x-\lambda_i y_i)=\lambda_{i+1}f_{i+1}'(x+\lambda_{i+1}y_i)-\lambda_{i+1}g_{i+1}'(x-\lambda_{i+1}y_i).$$

With a proper adjustment of constants this yields the relations

$$\begin{aligned} f_{i+1}(x+\lambda_{i+1}y_i) &= \frac{\lambda_{i+1}+\lambda_i}{2\lambda_{i+1}} f_i(x+\lambda_i y_i)+\frac{\lambda_{i+1}-\lambda_i}{2\lambda_{i+1}} g_i(x-\lambda_i y_i) \\ g_{i+1}(x-\lambda_{i+1}y_i) &= \frac{\lambda_{i+1}-\lambda_i}{2\lambda_{i+1}} f_i(x+\lambda_i y_i)+\frac{\lambda_{i+1}+\lambda_i}{2\lambda_{i+1}} g_i(x-\lambda_i y_i). \end{aligned} \tag{4}$$

Without loss of generality we may suppose $F_0(2)=G_0(2)=0$. We denote by $\gamma_1^{(m)}$ and $\gamma_2^{(m)}$ the characteristics of (3) which pass through (0, 0) and (2, 0), respectively, and which intersect. Then D_m will designate the domain bounded by $\gamma_1^{(m)}$, Γ, and the segment of the x-axis, $0 \leq x \leq 2$. Let $P_0^{(m)}(x_0^{(m)}, y_0^{(m)})$ be the point where Γ and $\gamma_1^{(m)}$ intersect. Since our ultimate purpose is to select a sequence of step-functions $K_n(y)$ converging to $K(y)$ it is no restriction to select $K^*(y)$ so that D_m lies entirely in the domain D_m' consisting of $\gamma_1^{(m)}$, $\gamma_2^{(m)}$ and $0 \leq x \leq 2$.

LEMMA. *Let $F_0(x)$ $(0 \leq x \leq 2)$ and $G_0(x)$ $(x_0 \leq x \leq 2)$ be given functions with continuous first derivatives with $F_0(2)=G_0(2)=0$. Then there exists a unique solution $u(x, y)$ of (3) in D_m' satisfying the conditions*

$$u(x,0)=F_0(x) \quad (0 \leq x \leq 2) \quad \textit{and} \quad u[x,h(x)]=G_0(x) \quad (x_0^{(m)} \leq x \leq 2).$$

Further, for $y_1 \leq y \leq 0$, $u(x, y)$ may be represented in the form

$$u(x,y)=f_1(x+\lambda_1 y)+g_1(x-\lambda_1 y),$$

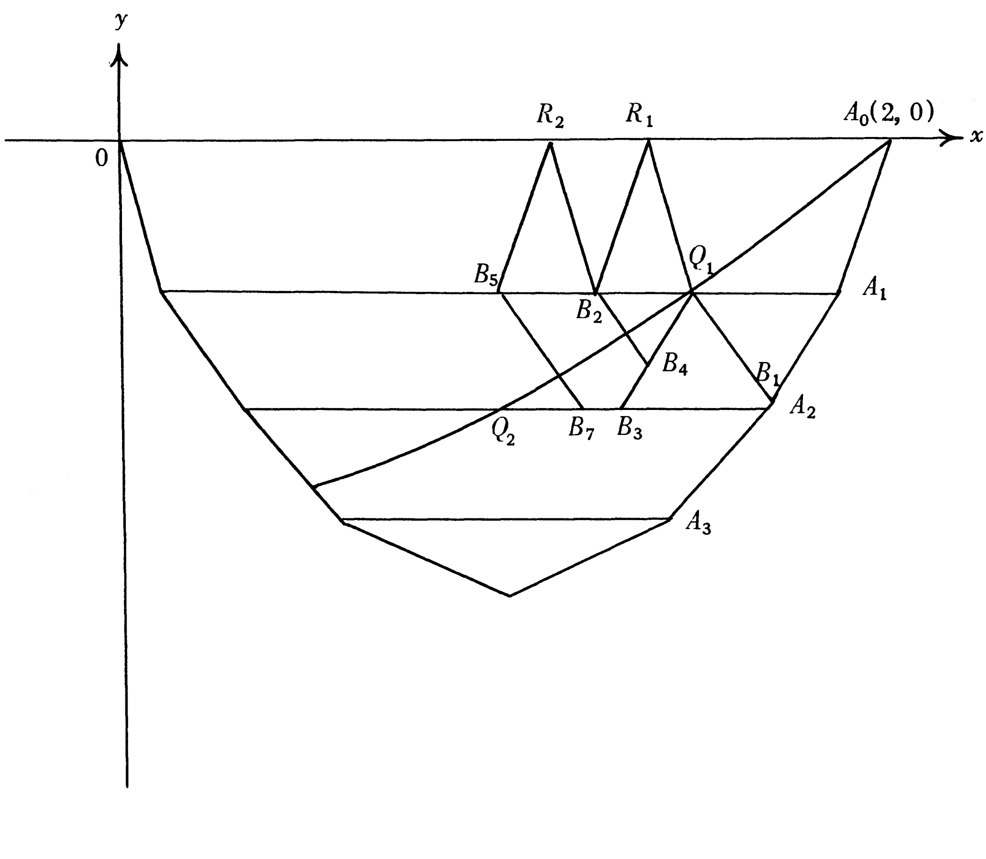
and the functions f_1, g_1 satisfy the inequalities

$$|f_1| \leq \frac{M}{\lambda_1}, \quad |g_1| \leq \frac{M}{\lambda_1},$$

where M is a constant depending on the slope of $h(x)$ and the maximum of

$$|F_0(x)|,\ |F_0'(x)|,\ |G_0(x)|,\ |G_0'(x)|.$$

Proof. The existence and uniqueness will be established simultaneously by constructing the solution. The solution itself will be obtained in a step-by-step process, and the method for constructing the first few steps will be shown in detail. From this it will be clear how to continue until the complete solution is obtained in a finite number of steps. Let $Q_1, Q_2, \cdots, Q_k$ $(k < m)$ be the points of intersection of $y = h(x)$ with the lines $y = y_1$, $y = y_2, \cdots, y = y_k$, respectively. Draw the characteristic $Q_1 R_1$ (see figure). The determination of the solution of (3) in the trapezoid $A_0 A_1 Q_1 R_1$ with data given along two noncharacteristics is a classical problem for the wave equation. However, since certain estimates are needed for the functions f_1 and g_1, this solution will be obtained explicitly. Let $P(x, y)$ be a point in the trapezoid $A_0 A_1 Q_1 R_1$ lying above Γ. Then $f_1(x + \lambda_1 y)$ is constant along the characteristic through P parallel to $R_1 Q_1$. This characteristic intersects Γ at, say, S_1, and we have



$$f_1(P) = f_1(S_1) = G_0(S_1) - g_1(S_1).$$

From S_1 draw the characteristic parallel to $A_0 A_1$ intersecting the x-axis at T_1. Then

$$g_1(S_1) = g_1(T_1) = F_0(T_1) - f_1(T_1),$$

and consequently

$$f_1(P) = G_0(S_1) - F_0(T_1) + f_1(T_1).$$

Through T_1 draw the characteristic parallel to $R_1 Q_1$ intersecting Γ at S_2. Through S_2 draw the characteristic parallel to $A_0 A_1$ intersecting the x-axis at T_2. Continuing in this way we find

$$f_1(P) = \sum_{n=1}^{\infty} G_0(S_n) - \sum_{n=1}^{\infty} F_0(T_n),$$

or

$$f_1(P) = \sum_{n=1}^{\infty} n[G_0(S_n) - G_0(S_{n+1})] - \sum_{n=1}^{\infty} n[F_0(T_n) - F_0(T_{n+1})].$$

The convergence of these series under the hypotheses of the lemma follows easily. Let

$$M_1 = \max(|F_0'(x)|, |G_0'(x)|),$$

and denote the length of the line segment from T_n to T_{n+1} by $|T_n - T_{n+1}|$. Then an application of the theorem of the mean yields

$$|f_1| \leq M_1 \sum_{n=1}^{\infty} n\,|T_n - T_{n+1}| + M_1 \sum_{n=1}^{\infty} n\,|S_n - S_{n+1}|.$$

To obtain an estimate for f_1 we first note that the lengths $|T_n - T_{n+1}|$ and $|S_n - S_{n+1}|$ form geometric progressions. Let L denote the length of that part of Γ between A_0 and Q_1. For simplicity we may replace the arc $A_0 Q_1$ by the chord and let k be the tangent of the angle this chord makes with the horizontal. In the actual case k is replaced by a variable for which we have upper and lower bounds. Construct the perpendiculars from the points S_n to the x-axis and denote these lengths by b_n. It is easily seen that these lengths are given by

$$b_n = \frac{Lk}{(1+\lambda_1 k)^n},$$

and hence

$$| T_{n-1} - T_n | = \frac{2\lambda_1 Lk}{(1+\lambda_1 k)^n}.$$

Since a similar estimate holds for the lengths $| S_n - S_{n+1} |$, we find

$$|f_1| \leq \frac{M_1 CL}{\lambda_1},$$

where C is a constant depending only on the slope of $h(x)$. To determine $g_1(x - \lambda_1 y)$ we proceed in a similar way. From the point P in the trapezoid $A_0 A_1 Q_1 R_1$ lying above Γ we draw the characteristic parallel to $A_0 A_1$, and denote by t_1 the point where this characteristic meets the x-axis. Through t_1 we construct the characteristic parallel to $R_1 Q_1$ intersecting Γ at s_1. The sequences $\{ t_n \}$ and $\{ s_n \}$ are constructed as before, and we obtain

$$g_1(P) = g_1(t_1) = F_0(t_1) - f_1(t_1) = F_0(t_1) - f_1(s_1) = F_0(t_1) - G_0(s_1) + g_1(t_2).$$

Hence

$$g_1(P) = \sum_{n=1}^{\infty} F_0(t_n) - \sum_{n=1}^{\infty} G_0(s_n).$$

A similar estimate to that made for f_1 yields

$$|g_1| \leq \frac{M_1 CL}{\lambda_1}.$$

The solution $u(x, y)$ is now completely determined in that part of the trapezoid $A_0 A_1 Q_1 R_1$ lying above Γ. However, from the fact that f_1 is constant along the characteristics $x + \lambda_1 y =$ const., and g_1 along the characteristics $x - \lambda_1 y =$ const., we see that u is completely determined in the remainder of $A_0 A_1 Q_1 R_1$. From the compatibility relations (4) this determines the solution in the triangle (or trapezoid) $A_1 B_1 Q_1$. (See figure.) Construct now the characteristics $Q_1 B_3$,

$R_1 B_2$, and $B_2 B_4$. Since g_1 is a function of $x - \lambda_1 y$, the determination of g_1 in $A_0 A_1 Q_1 R_1$ defines it also in the triangle $R_1 Q_1 B_2$ and in particular along the segment $Q_1 B_2$. This together with the fact that u is prescribed along the arc $Q_1 Q_2$ enables us to determine u throughout the triangle $Q_1 B_2 B_4$. Let $P(x, y)$ be a point on the segment $Q_1 B_2$. From (4) we have

$$f_1(P) = \frac{2\lambda_2}{\lambda_2 + \lambda_1} f_2(P) - ag_1(P),$$

where we have set

$$\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = a.$$

Through P construct the characteristic parallel to $B_2 B_4$ and intersecting $Q_1 Q_2$ at the point r_1. From r_1 we next draw the characteristic parallel to $Q_1 B_3$ and meeting $Q_1 B_2$ at the point v_1. Continuing this process we obtain the sequences $\{r_n\}$ along $Q_1 Q_2$ converging to Q_1, and $\{v_n\}$ along $Q_1 B_2$ converging to Q_1. Then, by the same argument employed above,

$$f_1(P) = -ag_1(P) + \frac{2\lambda_2}{\lambda_2 + \lambda_1} f_2(r_1) = -ag_1(P) + \frac{2\lambda_2}{\lambda_2 + \lambda_1} [G_0(r_1) - g_2(r_1)]$$

and

$$f_1(P) = -ag_1(P) + \frac{2\lambda_2}{\lambda_2 + \lambda_1} G_0(r_1) - g_1(v_1) - af_1(v_1).$$

Continuing, we obtain

$$f_1(P) = -ag_1(P) + \frac{2\lambda_2}{\lambda_2 + \lambda_1} \sum_{n=1}^{\infty} (-a)^{n-1} G_0(r_n)$$

$$- \frac{4\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} \sum_{n=1}^{\infty} (-a)^{n-1} g_1(v_n).$$

This yields not only the complete determination of f_1 on the segment $Q_1 B_2$ but also the estimate

$$|f_1| \leq \frac{3\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_2}{\lambda_1} M_2 ,$$

where M_2 depends only on the given data and Γ. Knowledge of the function f_1 along $Q_1 B_2$ together with relations (4) yields the solution u in the triangle $Q_1 B_2 B_4$. Draw now the characteristic $B_2 R_2$ as shown. Since f_1 is a function of $x + \lambda_1 y$, we now know f_1 in the parallelogram $R_1 Q_1 B_2 R_2$. Along $R_1 R_2$, $g_1 = F_0 - f_1$, and thus g_1 and therefore u is determined in this parallelogram. The transition from the second to the third step is completely analogous and may be carried out in the same way. The estimates for f_1 and g_1 are easily obtained by an induction argument that parallels that given in [3] and need not be repeated. The bounds show that the solution obtained is unique.

We note that $u_x(x, y)$ also satisfies equation (3). Therefore we may consider again the same problem treated in the lemma with the following data:

$$F_0'(x) \qquad (0 \leq x \leq 2, \; y = 0)$$

and

$$f_i'(x + \lambda_i h(x)) + g_i'(x - \lambda_i h(x)) \qquad (x_0^{(m)} \leq x \leq 2, \text{ along } y = h(x)).$$

To do this we assume that $F_0(x)$, $G_0(x)$ possess continuous second derivatives. If the argument employed in the lemma is repeated and the relation

$$G_0'(x) = f_i'[x + \lambda_i h(x)] [1 + \lambda_i h'(x)] + g_i'[x - \lambda_i h(x)] [1 - \lambda_i h'(x)]$$

is employed then estimates of the form

$$|f_1'|, \; |g_1'| \leq \frac{M_3}{\lambda_1}$$

may be obtained. Here M_3 depends only on $\max |K^*(y)|$, the given data, and $h(x)$. In the first strip, that is, the strip bounding the x-axis, we have

$$u_y(x, y) = \lambda_1 f_1'(x + \lambda_1 y) - \lambda_1 g_1(x - \lambda_1 y).$$

Hence the estimates for f_1' and g_1' show that $u_y(x, y)$ is *uniformly bounded* with the bound depending only on the given data, the domain, and $\max |K^*(y)|$.

3. The limiting process. We now consider a sequence $K_n(y)$ of nondecreasing step-functions, each with a finite number of steps, which converges uniformly to $K(y)$. The fact that $u(x, 0)$ and $u_y(x, 0)$ are uniformly bounded for all n enables us to employ the following theorem of Bers [1]:

THEOREM (Bers). *Let $\tau(x)$ and $\nu(x)$ be once continuously differentiable functions defined for $0 \leq x \leq 2$. Then there exists a unique solution $u(x, y)$ of equation* (1) *in D' satisfying the initial conditions*

$$u(x, 0) = \tau(x), \ u_y(x, 0) = \nu(x) \qquad (0 \leq x \leq 2).$$

In D', $u(x, y)$ satisfies the inequalities

$$|u| \leq T + |y| N, \qquad |u_y| \leq AT' + BN',$$

where

$$A = A(y) = \sqrt{-K(y)}, \ B = x + |y| A(y), \ T = \max |\tau(x)|, \ N = \max |\nu(x)|,$$

$$T' = \max |\tau'(x)|, \ N' = \max |\nu'(x)|.$$

The theorem of Bers applies equally well to equation (3). Employing this theorem together with the bounds we obtained for f_1' and g_1', we obtain uniform bounds for the solution $u(x, y)$ in D' in terms of F_0, G_0, and their first two derivatives.

Denote by $u^{(n)}(x, y)$ the solution of the boundary value problem corresponding to $K_n(y)$. Then

$$u^{(n)}(x, 0) = F_0(x)$$

for all n, and $\{u_y^{(n)}(x, 0)\}$ is a uniformly bounded sequence. The assumption that F_0 and G_0 possess continuous fourth derivatives gives us a uniform bound on $\{u_{yx}^{(n)}(x, 0)\}$; hence the sequence $\{u_y^{(n)}(x, 0)\}$ is equicontinuous, and there exists a convergent subsequence. Let $u_y(x, 0)$ be the limiting value. This fact together with the estimates obtained above allows us to apply a lemma of the author [3, p. 427] and conclude that a subsequence of $\{u_n(x, y)\}$ converges to a function $u(x, y)$ which satisfies (1). It is clear that $u(x, y)$ assumes the proper boundary values as each $u_n(x, y)$ does.

To determine the uniqueness of the solution, a method previously exploited [4] may be used. We assume that $u(x, y)$ is a solution which vanishes on the x-axis, $0 \leq x \leq 2$, and on Γ. We consider the integral

$$2 \iint_D (au_x + bu_y + cu_z)(Ku_{xx} + u_{yy})\,dxdy = 0,$$

where a, b, and c are functions yet to be determined. In this case we may take $a = 0$ and b and c constant. An application of Green's theorem yields

$$0 = \int_0^2 c(x, 0)u_y^2(x, 0)\,dx + \iint_D cK'u_x^2\,dx$$

$$- \int_{\gamma_1} (c\sqrt{-K} - b)\left(\sqrt{-K}\,u_x^2 - 2u_x u_y + \frac{1}{\sqrt{-K}}\,u_y^2\right) dx$$

$$- \int_\Gamma (c - h'(x)b)\left(K + \frac{1}{h'^2}\right) u_x^2\,dx.$$

An appropriate selection for b and c makes all these integrals have the same sign. This can only happen if u vanishes identically.

The preceding has proved the following:

THEOREM. *Let $F_0(x)$ $(0 \leq x \leq 2)$, $G_0(x)$ $(x_0 \leq x \leq 2)$ be functions with continuous fifth derivatives and $F_0(2) = G_0(2)$. Let $y = h(x)$ and D be defined as in* §1. *Then there exists a unique solution $u(x, y)$ of* (1) *in D satisfying the boundary conditions $u(x, 0) = F_0(x)$ $(0 \leq x \leq 2)$ and $u[x, h(x)] = G_0(x)$ $(x_0 \leq x \leq 2)$. Further, estimates for $u(x, y)$ may be obtained in terms of the given data.*

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