ON A THEOREM OF JORDAN

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1. Introduction. In 1872 Jordan [4] showed that a finite quadruply transitive group in which only the identity fixes four letters must be one of the following groups: the symmetric group on four or five letters, the alternating group on six letters, or the Mathieu group on eleven letters.

In this paper Jordan's theorem on quadruply transitive groups is generalized in two ways. The number of letters is not assumed to be finite; and instead of assuming that the subgroup fixing four letters consists of the identity alone, we only assume it to be a finite group of odd order. The conclusion is essentially the same as that of Jordan's theorem, the only other group satisfying the hypotheses being the alternating group on seven letters.

2. Proof of the main theorem. The theorem is the following:

THEOREM 2.1. *A group G quadruply transitive on a set of letters, finite or infinite, in which a subgroup H fixing four letters is of finite odd order, must be one of the following groups: S_4, S_5, A_6, A_7 or the Mathieu group on* 11 *letters.*

Case 1. *G on not more than seven letters.* A quadruply transitive group on 4 or 5 letters must be the symmetric group. On six letters its order must be at least $6 \cdot 5 \cdot 4 \cdot 3$, and hence it is A_6 or S_6. On seven letters, its index is at most 6 in S_7. As S_7 does not have a subgroup of index 3 or 6, the only possibilities are A_7 and S_7. In both S_6 and S_7 there are elements of order two fixing at least four letters, and so these groups do not satisfy our hypothesis.

To treat the case in which G is on more than seven letters, we begin with two simple lemmas.

LEMMA 2.1. *Elements a, b in a group, satisfying the relations*

$$a^2 = 1,\ b^2 = 1,\ (ab)^s = 1,$$

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generate the dihedral group of order $2s$. *If* $s = 2t - 1$ *is odd, then a power of* $y = ab$ *transforms* a *into* b. *If* $s = 2r$ *is even, then* a *and* b *permute with* y^r.

Proof. With $y = ab$, we have

$$a^2 = 1,\ y^s = 1,\ b = ay = y^{-1}a.$$

If $s = 2t - 1$, then

$$y^{-t}a\,y^t = ay^{2t} = b.$$

If $s = 2r$, then

$$ay^r = y^{-r}a = y^r a.$$

LEMMA 2.2. *If* G *is a* k*-ply transitive permutation group and* P *is a Sylow subgroup of the finite subgroup* H_k *fixing* k *letters, then* $N_G(P)$, *the normalizer in* G *of* P *is* k*-ply transitive on the letters fixed by* P.

Proof. (Compare [**1**, p. 212; **6**, p. 259].) Let $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ be letters fixed by P. Then in G there is an element x taking $a_1, \cdots, a_k$ into $b_1, \cdots, b_k$. Here $x^{-1}Px = P'$ is a Sylow subgroup of the group H'_k fixing $b_1, \cdots, b_k$. But P also fixes $b_1, \cdots, b_k$. Thus P and P' are conjugate in H'_k, and there will be a y fixing $b_1, \cdots, b_k$ with

$$y^{-1}P'y = P.$$

Hence $z = xy$ is an element normalizing P and taking $a_1, \cdots, a_k$ into $b_1, \cdots, b_k$. Trivially $N_G(P)$ takes the letters fixed by P into themselves. We note that P need not be a Sylow subgroup of G.

From here on G will denote, as in Theorem 2.1, a group quadruply transitive on more than seven letters, and H a subgroup of odd order m fixing four letters.

LEMMA 2.3. *The group* G *contains elements of order* 2, *and all elements of order* 2 *are conjugate. Either* 1) *every element of order* 2 *fixes two letters or* 2) *every element of order* 2 *fixes three letters.*

Proof. By quadruple transitivity G contains an element

$$g = (12)(34)\cdots.$$

Here g^2 fixes 1, 2, 3, 4, and so belongs to H and will be of finite odd order m_1.

Thus

$$x = g^{m_1} = (12)(34)\cdots,$$

with $x^2 = 1$. As H is of odd order, any element u of order two will fix at most three letters, and hence displace at least four letters. With

$$u = (ab)(cd)\cdots$$

there is a conjugate of u,

$$v = w^{-1}uw = (12)(34)\cdots.$$

Either $v = x$, or vx fixes four letters and is of odd order, whence, by Lemma 2.1, v and x are conjugate. Thus all elements of order two are conjugate. On the other hand, there is in G an element $z = (1)(2)(34)\cdots$, and either z or an odd power of z is an element of order two fixing at least two letters. Hence every element of order two fixes either two or three letters, since they fix at least two and not as many as four.

Case 2. *G on more than seven letters.* Let

$$a_1 = (1)(2)(34)\cdots$$

be an element of order two and

$$b = (12)(34)\cdots$$

another element of order two. Then $f = a_1 b = (12)(3)(4)\cdots$ will be of even order, and f^2 will be of odd order m_1. Hence $f^{m_1} = a_3$ is of order two and by Lemma 2.1 will permute with a_1. Hence in G we have permuting elements of order two, with $a_2 = a_1 a_3$:

$$\begin{aligned} a_1 &= (1)(2)(3\,4)\cdots \\ a_2 &= (1\,2)(3\,4)\cdots \qquad (2.1) \\ a_3 &= (1\,2)(3)(4)\cdots \end{aligned}$$

Now a_2 as an element of order 2 fixes either two letters 5 and 6, or three letters 5, 6, and 7. As a_1 permutes with the element a_2, it takes these letters into themselves. But a_1 fixes 1 and 2 and at most one other letter. Hence we have

$$(2.2)\quad \begin{aligned} a_1 &= (1)(2)(3\,4)(5\,6)\cdots \\ a_2 &= (1\,2)(3\,4)(5)(6)\cdots \\ a_3 &= (1\,2)(3)(4)(5\,6)\cdots \end{aligned} \quad \text{or} \quad \begin{aligned} a_1 &= (1)(2)(3\,4)(5\,6)(7)\cdots \\ a_2 &= (1\,2)(3\,4)(5)(6)(7)\cdots \\ a_3 &= (1\,2)(3)(4)(5\,6)(7)\cdots \end{aligned}$$

the first case arising if elements of order 2 all fix two letters, the second if all fix three letters. The elements a_1, a_2, a_3 of (2.2) and the identity form a four group V. Further letters will occur in sets of four which will be sets of transitivity for V:

$$(2.3)\quad \begin{aligned} a_1 &= (1)(2)(3\,4)(5\,6)(7)(hi)(jk)\cdots \\ a_2 &= (1\,2)(3\,4)(5)(6)(7)(hj)(ik)\cdots \\ a_3 &= (1\,2)(3)(4)(5\,6)(7)(hk)(ij)\cdots \end{aligned}$$

Here it is understood that the 7 may not be present.

The order of the subgroup K taking h, i, j, k into themselves will be $24m$, and $H = H(h, i, j, k)$ of order m will be normal in K. There will be a subgroup U, $K \supset U \supset H$, in which h, i, j, k are permuted in the following way:

$$(2.4)\quad \begin{aligned} &(h) \\ &(hi)(jk) \\ &(hj)(ik) \\ &(hk)(ij) \\ &(hjik) \\ &(hkij) \\ &(hi)(j)(k) \\ &(h)(i)(jk). \end{aligned}$$

Now U is of order $8m$, and so a Sylow subgroup of U will be of order 8. The elements taking h, i, j, k into themselves in a particular way will be a coset of H in U. As H is normal in U, a group of order 8 in U will have one element from each coset and be isomorphic to H/U and hence faithfully represented by the permutations on these letters. V will be contained in a Sylow subgroup of order 8 in U. This yields

$$(2.5)\quad \begin{aligned} a_1 &= (1)(2)(3\,4)(5\,6)(7)(h\,i)(j\,k)\cdots \\ a_2 &= (1\,2)(3\,4)(5)(6)(7)(h\,j)(i\,k)\cdots \end{aligned}$$

$$\begin{aligned}
a_3 &= (1\ 2)(3)(4)(5\ 6)(7)(h\ k)(i\ j)\cdots \\
u &= (1)(2)(3\ 5\ 4\ 6)\quad (7)(h\ j\ i\ k)\cdots \\
a_1 u &= (1)(2)(3\ 6\ 4\ 5)\quad (7)(h\ k\ i\ j)\cdots \\
a_2 u &= (1\ 2)(3\ 6)(4\ 5)\quad (7)(hi)(j)(k)\cdots \\
a_3 u &= (1\ 2)(3\ 5)(4\ 6)\quad (7)(h)(i)(jk)\cdots
\end{aligned}$$

or the same permutations with 5 and 6 interchanged. The way in which the last four elements permute the letters $1, \cdots, 7$ is determined by the relations

$$u^2 = a_1, \quad u^{-1} a_2 u = a_3, \quad (a_2 u)^2 = 1.$$

Here u normalizes V and so fixes the only letter, 7, fixed by V, if the 7 occurs. Also u must take the fixed letters of a_3 into those of a_2, whence

$$u = \begin{pmatrix} 3 & 4 & \cdots \\ 5 & 6 & \cdots \end{pmatrix} \quad \text{or} \quad u = \begin{pmatrix} 3 & 4 & \cdots \\ 6 & 5 & \cdots \end{pmatrix};$$

but also $u^2 = a_1$, whence

$$u = (3546)\cdots \quad \text{or} \quad u = (3645)\cdots.$$

Finally, u must fix 1 and 2 or interchange them. But if u interchanges 1 and 2, then $a_2 u$ is of order 2 and fixes the letters 1, 2, j, k. Thus

$$u = (1)(2)(3546)\cdots \quad \text{or} \quad u = (1)(2)(3645)\cdots,$$

and the rest follows.

Each further transitive constituent of V such as h, i, j, k yields a group S such as that in (2.5). The elements

$$(12)(36)(45)\cdots \quad \text{and} \quad (12)(35)(46)\cdots$$

in each of these groups fix two letters of the constituent. Since an element of order two cannot fix four letters, each constituent yields a different element permuting the first six letters in the way (12) (36) (45). But there are at most m elements with this effect on the first six letters. Thus if there are t such constituents, $t \leq m$ is finite and G is a group on $n = 4t + 6$ or $4t + 7$ letters. If G is on 10 or 11 letters we have $t = 1$.

There is no quadruply transitive group on 10 letters (except of course A_{10} and S_{10}). For the normalizer of a cycle of length 7 by Lemma 2.2 is S_3 on the remaining three letters; and so this normalizer, which is the subdirect product of S_3 and the normalizer on the letters of the 7-cycle, will pair a 3-cycle with the identity. Hence G contains a 3-cycle, and, being quadruply transitive, all 3-cycles and so contains A_{10}.

On 11 letters G is of order $11 \cdot 10 \cdot 9 \cdot 8m$, and even without assuming m odd, consideration of normalizers of Sylow subgroups fixing four letters shows that we must have $m = 1$. The group of order 8 fixing three letters contains a single element of order 2 [3, p. 311], and so is the cyclic or quaternion group. The cyclic group, having only 4 automorphisms, could not have a normalizer triply transitive on the remaining three letters, for then G would contain a 3-cycle. Hence the subgroup fixing three letters must be the quaternion group Q. Then G will be a transitive extension of Q, and the methods of T. C. Holyoke [3] readily enable us to construct from Q not only the quadruply transitive Mathieu group on 11 letters, but the quintuply transitive group on 12 letters.

We shall now show that $t > 1$ conflicts with the hypothesis that H is of odd order, and thus complete the proof of our theorem. If w, x, y, z is another transitive constituent of V, we have

$$a_2u = (12)(36)(45)(7)(hi)(j)(k)\cdots$$

from (2.5) and will have another element

$$a_2u' = (12)(36)(45)(7)(wx)(y)(z)\cdots.$$

Each of these elements permutes with a_1 and transforms a_2 into a_3 and a_3 into a_2. Their product is an element q fixing the first six (or seven) letters and so of odd order. Also q centralizes V. By Lemma 2.1 a power of q transforms a_2u into a_2u', and so takes the fixed letters j, k of a_2u into the fixed letters y, z of a_2u'. Centralizing V, this element must take the entire constituent $hijk$ into $wxyz$. Hence there is a group C in G which fixes the first six (or seven) letters, centralizes V, and is transitive on the t remaining constituents of V. An element of C taking a constituent of V into itself, being of odd order, must fix all four letters. Thus the transitive constituents of C are (1) (2) (3) (4) (5) (6) (7) T_h, T_i, T_j, T_k, the last four sets of t each, the letters h, i, j, k being in different constituents of C.

Let p be a prime dividing t. (Here we use the assumption $t > 1$.) Let P be the corresponding Sylow subgroup of C. Then P displaces all $4t$ letters which C displaces, since a subgroup of C fixing a letter is of index $t \equiv 0 \pmod{p}$ and cannot contain such a Sylow subgroup. Now let P_1 be a Sylow subgroup of H, the subgroup fixing 1, 2, 3, 4, which contains P. Then P_1 displaces the $4t$ letters of C and no others, unless possibly we have the case

$$p = 3,\ t = 3^w,\ n = 4t + 7,$$

where P_1 might be on $4t + 3$ letters. This possibility will be considered later. With P_1 on $4t$ letters, by Lemma 2.2, the group $N_G(P_1)$ is quadruply transitive on the first six or seven letters and so contains A_6 or A_7 on these letters. But the subgroup taking the first six (or seven) letters into themselves also contains the element u of (2.5) which is not in the alternating group on these letters. Thus in G we have the full symmetric group on the first six or seven letters and hence some element fixing the first four letters and interchanging the fifth and sixth. This conflicts with the hypothesis that H is of odd order. Finally, consider the possibility that

$$t = 3^w,\ n = 4t + 7,$$

and that P_1 displaces 5, 6, 7 as well as the $4t$ letters of P. If $w > 1$, then surely 5, 6, 7 are a transitive constituent of P_1 and there is an element

$$z = (1)(2)(3)(4)(567)\cdots$$

in G. If $w = 1$, then P is of order 3, and even if, in P_1, 5, 6, 7 are in a constituent with 8, 9, 10 and 11, 12, 13 of P, since there is an element $(5)(6)(7)(8, 9, 10)(11, 12, 13)$, there will also be one such as z fixing 8, 9, 10. But with

$$z = (1)(2)(3)(4)(567)\cdots$$

and u of (2.5) we have

$$(zu)^3 = (1)(2)(35)(4)(6)(7)\cdots,$$

contradicting the assumption that a subgroup H fixing 4 letters is of odd order.

References

1. W. Burnside, *The theory of groups*, Cambridge University Press, second edition 1911.

2. R. D. Carmichael, *Theory of groups of finite order*, Ginn and Co., Boston, 1937.

3. T. C. Holyoke, *On the structure of multiply transitive permutation groups*, Amer. J. Math. 74 (1952), 787-796.

4. C. Jordan, *Recherches sur les substitutions*, J. Math. Pures Appl. (2) **17** (1872), 351-363.

5. G. A. Miller, *Limits of the degree of transitivity of substitution groups*, Bull. Amer. Math. Soc. 22 (1915), 68-71.

6. E. Witt, *Die 5-fach transitiven Gruppen von Mathieu*, Abh. Math. Sem. Univ. Hamburg **12** (1938), 256-264.

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