ON THE DIMENSION THEORY OF RINGS (II)

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1. Introduction. As in [3], we shall say that an integral domain O is n-dimensional if in O there is a proper chain

$$(0) \subset P_1 \subset \cdots \subset P_n \subset (1)$$

of prime ideals, but no such chain

$$(0) \subset P_1' \subset \cdots \subset P_{n+1}' \subset (1).$$

In Theorem 2 of [3] it was shown that if O is n-dimensional, then $O[x]$ is at least $(n+1)$-dimensional and at most $(2n+1)$-dimensional: here, as throughout, x is an indeterminate. After preparatory constructions in Theorems 1 and 2 below, this theorem is completed in Theorem 3 by showing that for any integers m and n with $n+1 \leq m \leq 2n+1$, there exist n-dimensional rings O such that $O[x]$ is m-dimensional. The other theorems mainly concern 1-dimensional rings. Such rings O can be divided into those for which $O[x]$ is 2-dimensional and those for which this condition fails, the so-called F-rings. The paper [3] was concerned with the existence of F-rings and showed [3, Theorem 8] that the 1-dimensional ring O is not an F-ring if and only if every quotient ring of the integral closure of O is a valuation ring. Below, in Theorem 5, we show more generally that if O is 1-dimensional but not an F-ring, then $O[x_1, \cdots, x_n]$ is $(n+1)$-dimensional, where the x_i are indeterminates: this theorem depends on the essentially more general Theorem 4, which says that if O is an m-dimensional multiplication-ring, then $O[x_1, \cdots, x_n]$ is $(m+n)$-dimensional. In the case that the x_i are not indeterminates, one can still say (Theorem 10) that

$$\dim O[x_1, \cdots, x_n] = 1 + \text{degree of transcendency of } O[x_1, \cdots, x_n]/O,$$

provided that the intersection of the prime ideals ($\neq (0)$) in O is $= (0)$, where O is a 1-dimensional ring such that $O[x]$ is 2-dimensional. For F-rings O, Theorem 6 shows that

$$n+2 \leq \dim O[x_1, \cdots, x_n] \leq 2n+1,$$

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where the x_i are indeterminates, while Theorem 7 constructs for any N and n with

$$n+2 \leq N \leq 2n+1$$

an F-ring O such that $O[x_1, \cdots, x_n]$ is N-dimensional. Similar results for rings of dimension greater than 1 would be interesting if one could get them.

2. Simple extensions. Let us call the integral domain O *of type* (n, m) if

$$\dim O = n \text{ and } \dim O[x] = m.$$

THEOREM 1. *Let O be integrally closed and of type (n, m), let K be its quotient field, and let K' be a proper extension of K in which K is algebraically closed. Let Σ be any field having a discrete rank 1 valuation with K' as residue field. Let O^* be the set of elements whose residues are finite and in O. Then O^* is integrally closed and of type $(n+1, m+2)$.*

Proof. Let $\alpha \in \Sigma$, with α integral over O^*,

$$\alpha^s + a_1 \alpha^{s-1} + \cdots + a_s = 0 \qquad (a_i \in O^*),$$

an equation of integral dependence. Dividing this equation by α^s and supposing $1/\alpha$ to have residue 0, we get the contradiction $1 = 0$. So α has finite residue, and

$$\bar{\alpha}^s + \bar{a}_1 \bar{\alpha}^{s-1} + \cdots + \bar{a}_s = 0,$$

where the bars indicate residues. Since K is algebraically closed in K', we have $\bar{\alpha} \in K$; and $\bar{\alpha} \in O$, since O is integrally closed. Hence O^* is integrally closed.

Let P be the set of $\alpha \in O^*$ having residue 0. Then P is a prime ideal. From the definitions one obtains

$$O^*/P \simeq O,$$

whence O^* is at least $(n+1)$-dimensional. If P' is a prime ideal in O^*, $P' \neq (0)$, then $P' \supseteq P$. In fact, let $g \in P'$; since g is O^*, we have $v(g) = s \geq 0$, where v is the given valuation (and the group of integers is the valuation group). Then the $(s+1)$th power of any element in P is divisible by g, whence $P \subseteq P'$. From this it follows that O^* is at most $(n+1)$-dimensional.

The quotient ring O^*_P is integrally closed and has only one prime ideal

($\neq (0)$). Moreover it is not a valuation ring. In fact, let $\alpha \in \Sigma$ be an element having residue in K' but not in K. Since α can clearly be written as a quotient of two elements of positive value, we have that α is in the quotient field of O_P^*; but neither α nor $1/\alpha$ has residue in K, so neither α nor $1/\alpha$ is in O_P^*. Thus O_P^* is not a valuation ring, and hence is an F-ring, by [3, Theorem 8]. It follows at once that $O^*[x] \cdot P$ is not minimal in $O^*[x]$. Now

$$O^*[x]/O^*[x] \cdot P \simeq O^*/P[x] \simeq O[x],$$

so $O^*[x]$ is at least $(m+2)$-dimensional.

Finally, let $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_s \subset (1)$ be a chain of prime ideals in $O^*[x]$. Let P_1 be minimal; then $P_1 \cap O^* = (0)$, as otherwise

$$P_1 \cap O \supseteq P \quad \text{and} \quad P_1 \supseteq O^*[x] \cdot P.$$

Similarly one concludes that if no chain of prime ideals $P_1' \subset P_1''$ can be inserted between (0) and P_2, then

$$P_2 \cap O^* = P \quad \text{and} \quad P_2 = O^*[x] \cdot P$$

(by [3, Theorem 1], P_2 cannot contract in O to (0)). From this it follows at once that $O^*[x]$ is at most $(m+2)$-dimensional, and the proof is complete.

REMARK. The above construction stems from an example of Krull showing that an integrally closed integral domain with only one proper prime ideal need not be a valuation ring; see [2, p. 670f].

THEOREM 2. *Let O, K, K', Σ, O^* be as in Theorem 1 except that we assume $K = K'$. Then O^* is integrally closed and of type $(n+1, m+1)$.*

Proof. The proof follows exactly the lines of the proof of Theorem 1, except that here O_P^* *is* a valuation ring, as one easily sees.

THEOREM 3. *For every n and m such that $n+1 \leq m \leq 2n+1$ there exist integrally closed rings of type (n, m).*

Proof. Any field is of type $(0,1)$. Theorem 1 gives us an integrally closed ring of type $(1,3)$, and Theorem 2 gives us one of type $(1,2)$—the required valuations obviously exist. Suppose now by induction that for some n and each m, $n+1 \leq m \leq 2n+1$, we have an integrally closed ring of type (n,m). If $n+3 \leq m \leq 2n+3$, then $n+1 \leq m-2 \leq 2n+1$, and from an integrally closed ring of type $(n, m-2)$ we get, by Theorem 1, an integrally closed ring of type

$(n+1, m)$. If $m = n+2$, we apply Theorem 2 similarly to get an integrally closed ring of type $(n+1, m)$.

As for simple algebraic extensions $O[\alpha]$ of an n-dimensional ring O, it is clear that $\dim O[\alpha] \le 2n$. On the other hand, let O be an integrally closed ring of type (n, m) and let O^* be a ring constructed as in Theorem 1; also let Σ and P be as in Theorem 1. Let

$$\alpha \in \Sigma, \ \alpha \notin O_P^*, \ 1/\alpha \notin O_P^*.$$

Then

$$O^*[\alpha]/O^*[\alpha] \cdot P \simeq O^*/P[x] \simeq O[x],$$

by [3, Theorem 7], so $O^*[\alpha]$ is at least $(m+1)$-dimensional; it is also at most $(m+1)$-dimensional, since $O^*[x]$ is $(m+2)$-dimensional. Hence

$$(n+1)+1 \le \dim O^*[\alpha] \le 2(n+1).$$

It is thus clear that for any $n' > 0$ and m' with $n'+1 \le m' \le 2n'$, there exists an n'-dimensional ring O^* such that for some α in the quotient-field of O^* we have $\dim O^*[\alpha] = m'$. — Also

$$\dim O[\alpha] < \dim O$$

is possible. In fact, let O be a valuation ring of rank n, $(0) \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n \subset (1)$, the chain of prime ideals in O. Let $c \in \mathfrak{p}_{i+1}$, $c \notin \mathfrak{p}_i$; then $\dim O[1/c] = 1$. *In short,* $\dim O[\alpha]$ *covers precisely the range from* 0 *to* $2n$ *as* O *varies over the* n*-dimensional rings* O.

3. Multiple transcendental extensions. We recall that a *multiplication-ring* may be defined as an integral domain O such that $O_\mathfrak{p}$ is a valuation ring for each prime ideal $\mathfrak{p}$ in O (see [2, p. 554]).

THEOREM 4. *If O is an m-dimensional multiplication-ring, then $O[x_1, \cdots, x_n]$ is $(m+n)$-dimensional, where the x_i are indeterminates.*

Proof. To facilitate the proof, we define the *dimension of a prime ideal* P in an extension $O' = O[\alpha_1, \cdots, \alpha_n]$ of a finite-dimensional ring O (relative to O) as follows:

$$\dim P = \text{d.t.}\,(O'/P)/(O/P) + \dim O/\mathfrak{p},$$

where $\mathfrak{p} = P \cap O$ (and "d.t." abbreviates "degree of transcendence"). The following points (a), (b) do not assume O to be a multiplication-ring.

(a) Let $\overline{O}'$, $\overline{O}$, $\overline{P}$, $\overline{\mathfrak{p}}$ be the images of O', O, P, $\mathfrak{p}$, respectively, under a homomorphism with kernel contained in P. Then $\dim \overline{P} = \dim P$.

In fact, $\overline{O}'/\overline{P} = O'/P$ and $\overline{O}/\overline{\mathfrak{p}} = O/\mathfrak{p}$; also $\overline{P} \cap \overline{O} = \overline{\mathfrak{p}}$.

(b) Let M be a nonempty multiplicatively closed system in O not meeting $\mathfrak{p}$,

$$O_M = \{\alpha \mid \alpha = a/b,\ a \in O,\ b \in M\},$$

$$O'_M = \{\alpha \mid \alpha = a/b,\ a \in O',\ b \in M\}.$$

Then

$$\dim P - \dim O/\mathfrak{p} = \dim O'_M \cdot P - \dim O_M/O_M \cdot \mathfrak{p}.$$

In fact, the rings O'/P and $O'_M/O'_M \cdot P$ have the same quotient field, as do the rings $O/\mathfrak{p}$ and $O_M/O_M \cdot \mathfrak{p}$. Note also that $O'_M \cdot P \cap O_M = O_M \cdot \mathfrak{p}$, whence the required equality follows.

Let P_1, P_2 be two prime ideals in O', $P_1 \subset P_2$, $\mathfrak{p}_i = P_i \cap O$, $i = 1, 2$. We want to compare $\dim P_1$ with $\dim P_2$. If $\mathfrak{p}_1 = \mathfrak{p}_2$, then, passing to a residue class ring, we may assume $\mathfrak{p}_1 = \mathfrak{p}_2 = (0)$. Taking $M = O - (0)$, we pass to the quotient-ring O'_M, which is a finite integral domain. Thus $\dim P_1 > \dim P_2$ if $\mathfrak{p}_1 = \mathfrak{p}_2$. This conclusion holds also if $\mathfrak{p}_1 \subset \mathfrak{p}_2$ provided O is a multiplication-ring.

(c) If P_1 and P_2 are prime ideals in $O[x_1, \cdots, x_n]$ and $P_1 \subset P_2$, then

$$\dim P_1 > \dim P_2;$$

also

$$\dim P_1 - \dim P_2 \geq \dim O/\mathfrak{p}_1 - \dim O/\mathfrak{p}_2,$$

provided that O is a multiplication-ring.

In fact, we may suppose $\mathfrak{p}_1 \subset \mathfrak{p}_2$, and have only to prove the second point. Also, by (b), we may pass to any quotient-ring O_M, where M does not meet $\mathfrak{p}_2$. Taking $M = O - \mathfrak{p}_2$, we may assume that O is a valuation-ring and that $\mathfrak{p}_2$ is its ideal of non-units. Let $z_1, \cdots, z_r$ be elements of O' which are algebraically dependent mod P_1 over O. Then they are also dependent mod P_2. In fact, let

$$f(z_1, \cdots, z_r) \equiv 0 (P_1),$$

where the coefficients of the polynomial f are in O but not all in p_1. Dividing by a coefficient of least value, we may suppose f to have a coefficient equal to unity. But then we have a relation mod P_2. This proves that

$$\text{d.t. } (O'/P_2)/(O/p_2) \leq \text{d.t. } (O'/P_1)/(O/p_1),$$

that is, (c) is proved.

The theorem now follows from (c) since $\dim (0) = m + n$.

COROLLARY. *If O is an m-dimensional multiplication-ring then*

$$\dim O[\alpha_1, \cdots, \alpha_n] \leq m + r,$$

where

$$r = \text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O.$$

Proof. The foregoing proof shows that

$$\dim O[x_1, \cdots, x_n] \leq \dim (0) = m + \text{d.t. } O[x_1, \cdots, x_n]/O,$$

and in doing so makes no use of the fact that the x_i are indeterminates; this fact is used only to get that

$$\dim O[x_1, \cdots, x_n] \geq m + n.$$

THEOREM 5. *If O is a 1-dimensional ring such that $O[x]$ is 2-dimensional, then*

$$\dim O[\alpha_1, \cdots, \alpha_n] \leq 1 + \text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O;$$

if the α_i are indeterminates, then

$$\dim O[\alpha_1, \cdots, \alpha_n] = 1 + n.$$

Proof. We may suppose O to be integrally closed. In that event, O is a multiplication-ring, by [3, Theorem 8]. The present theorem now follows immediately from the preceding corollary.

THEOREM 6. *If O is 1-dimensional, then $O[x_1, \cdots, x_n]$ is at most $(2n+1)$-dimensional, where the x_i are indeterminates.*

Proof. Let $(0) \subset p_1 \subset p_2 \subset \cdots \subset p_s \subset (1)$ be a chain of prime ideals in

$O[x_1, \cdots, x_n]$. Let K = quotient field of O. If $p_s \cap O = (0)$, then the above chain extends to a chain of s prime ideals in $K[x_1, \cdots, x_n]$, so $s \leq n$. Suppose, then, that

$$p_i \cap O = (0), \quad p_{i+1} \cap O = p \neq (0),$$

whence also $p_{i+k} \cap O = p$, since O is 1-dimensional. Passing to $K[x_1, \cdots, x_n]$, we see that $i \leq n$; and passing to

$$O[x_1, \cdots, x_n] / p_{i+1} = O/p\, [\bar{x}_1, \cdots, \bar{x}_n],$$

we have $s - (i + 1) \leq n$, since O/p is a field. Hence $s \leq 2n + 1$.

THEOREM 7. *If O is an F-ring, then $O[x_1, \cdots, x_n]$ is at least $(n + 2)$-dimensional and at most $(2n + 1)$-dimensional. For any N, $n+2 \leq N \leq 2n+1$, there is an F-ring O such that $O[x_1, \cdots, x_n]$ is N-dimensional, where the x_i are indeterminates.*

Proof. Let K be a field, x, $y_1, \cdots, y_m$ indeterminates. Let

$$K' = K(y_1, \cdots, y_m), \quad \Sigma = K'(x),$$

and let v be the discrete rank 1 valuation of Σ obtained by placing

$$v(a_i x^i + a_{i+1} x^{i+1} + \cdots + a_s x^s) = i,$$

where $a_j \in K'$, $a_i \neq 0$. Let O^* be the set of elements whose residues are finite and in K. The ring O^* consists of the elements in $K(x, y_1, \cdots, y_m)$ which can be written in the form

$$\alpha(x, y_1, \cdots, y_m) / \beta(x, y_1, \cdots, y_m),$$

where

$$\alpha, \beta \in K[x, y_1, \cdots, y_m], \; \beta(0, y_1, \cdots, y_m) \neq 0,$$

$$\alpha(0, y_1, \cdots, y_m) / \beta(0, y_1, \cdots, y_m) \in K.$$

By Theorem 1, O^* is an F-ring; and O^* contains only one proper prime ideal, namely the ideal P consisting of the elements α/β with

$$\alpha(0, x_1, \cdots, x_m) / \beta(0, x_1, \cdots, x_n) = 0.$$

We shall prove that for $m \leq n$, $O^*[x_1, \cdots, x_n]$ is $(m+n+1)$-dimensional. In $O^*[x_1, \cdots, x_n]$ let P_m be the ideal of polynomials which vanish for $x_i = y_i$ $(i = 1, \cdots, m)$. We claim this ideal is in

$$O^*[x_1, \cdots, x_n] \cdot P = P'.$$

In fact, let

$$\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \in O^*[x_1, \cdots, x_n]$$

be in P_m, and write

$$\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} = \sum c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} + \sum d_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $c_{i_1 \cdots i_n} \in K$, $d_{i_1 \cdots i_n} \in P$. This polynomial vanishes for $x_i = y_i$, $i = 1, \cdots, m$; hence also for $x_i = y_i$, $i = 1, \cdots, m$, $x = 0$. Hence

$$\sum c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

vanishes for $x_i = y_i$, $i = 1, \cdots, m$, whence

$$\sum c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} = 0,$$

and

$$\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} = \sum d_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \in O^*[x_1, \cdots, x_n] \cdot P = P'.$$

Let P_j be the ideal of elements in $O^*[x_1, \cdots, x_n]$ which vanish for $x_i = y_i$, $i = 1, \cdots, j$. Then P_j is prime and $(0) \subset P_1 \subset \cdots \subset P_m \subset P'$. Since any chain of n prime ideals in $O^*/P[x_1, \cdots, x_n]$ gives rise to such a chain in $O^*[x_1, \cdots, x_n]$ containing P', we see that $O^*[x_1, \cdots, x_n]$ is at least $(m+n+1)$-dimensional. On the other hand, $O^*[x_1, \cdots, x_n]$ is of degree of transcendency $m + n + 1$ over K, and so $O^*[x_1, \cdots, x_n]$ is at most $(m+n+1)$-dimensional. This last point follows from the following lemma, the proof of which is exactly as in the well-known case that O is a valuation ring.

LEMMA. *Let O be an arbitrary integral domain containing a field K, and let O be of degree of transcendency r over K. Then O is at most r-dimensional.*

Proof. This follows at once if we can show that the degree of transcendency of O/P over K is less than r for any proper prime ideal P in O. If $\theta_1, \theta_2, \cdots, \theta_s \in O$ map into (given) algebraically independent elements in O/P, and $\theta \in P$, $\theta \neq 0$, then $\theta, \theta_1, \cdots, \theta_s$ are algebraically independent over K. Hence

$$\text{d.t. } O/K > \text{d.t. } (O/P)/K.$$

4. **Arbitrary finite extensions.** Let O be an arbitrary integral domain which is not a field. It is certainly possible, for appropriate O, that some simple ring extension $O[\alpha]$ of O will be a field. In fact, let O be such that the intersection of all its prime ideals $(\neq (0))$ is not the ideal (0); for example, any integral domain with a finite, positive number of prime ideals $(\neq (0))$ will do. If $c\,(\neq 0)$ is an element in all the prime ideals, then $O[1/c]$ is a field; for if P is a prime ideal in $O[1/c]$, $P \neq (0)$, then

$$P \cap O = \mathfrak{p} \neq (0)$$

and

$$1 = (1/c) \cdot c \in O[1/c] \cdot \mathfrak{p} \subseteq P.$$

We also have the converse.

THEOREM 8. *Given an integral domain O, there exists a field F which is a simple ring extension of O if and only if the intersection of all the prime ideals $(\neq (0))$ in O is $\neq (0)$.*

Proof. Let $F = O[\alpha]$. Here α must be algebraic over O, say

$$c_0\alpha^m + c_1\alpha^{m-1} + \cdots + c_m = 0, \quad c_i \in O, \quad c_0 \neq 0.$$

Then $c_0\alpha$ is integral over O, as is the ring $O_1 = O[c_0\alpha]$. Let $F_1 = O_1[\alpha]$; then F_1 is a field [**1**, p. 253]. Over every prime ideal in O there lies a prime ideal in O_1; since O_1 is algebraic over O, if the intersection of the prime ideals $(\neq (0))$ in O_1 is $\neq (0)$, then the like is true in O. Hence we may assume that $O = O_1$, that is, that α is in the quotient field of O. By a similar reasoning we may suppose O is integrally closed. From the fact that $1/\alpha \in O[\alpha]$, one finds that $1/\alpha$ is integral over O, hence in O. Thus $\alpha = 1/b$, $b \in O$. The element b must be in every prime ideal $\mathfrak{p}(\neq (0))$; in fact, if $b \notin \mathfrak{p}$, then $O[1/b] \subseteq O_{\mathfrak{p}}$, whence $O[1/b] = O[\alpha]$ is not a field. This completes the proof. —This theorem has been previously proved in [**4**, p. 76].

A study of algebraic extensions of O must therefore separate the cases that the intersection of the prime ideals ($\neq (0)$) is $= (0)$ or is $\neq (0)$.

THEOREM 9. *If O is a* 1-*dimensional ring such that $O[x]$ is* 2-*dimensional, and the intersection of the prime ideals* ($\neq (0)$) *in O is* $= (0)$, *then the like is true of any simple algebraic ring extension $O[\alpha]$ of O* (*where it is assumed, of course, that $O[\alpha]$ is an integral domain*).

Proof. By Theorem 5, we know that $O[\alpha]$ is 0- or 1-dimensional, and the previous theorem excludes the first alternative. Also $O[\alpha, x]$ is 2-dimensional, for otherwise $O[y, x]$, y an indeterminate, would be of dimension more than 3, contradicting Theorem 5. Thus it remains to prove that the intersection of the prime ideals ($\neq (0)$) in $O[\alpha]$ is $= (0)$. Let

$$c_0\alpha^n + c_1\alpha^{n-1} + \cdots + c_n = 0, \quad c_i \in O, \quad c_0 \neq 0,$$

and let $S = \{p\}$ be the set of prime ideals ($\neq (0)$) in O which do not contain c_0; S is not empty. Then $\cap p = (0)$, for if $d \in \cap p$, $d \neq 0$, then $c_0 d$ is in every prime ideal ($\neq (0)$) of O. Over every prime ideal $p \in S$ there lies a prime ideal P in $O[\alpha]$. If $T = \{P\}$ is the set of prime ideals in $O[\alpha]$ contracting to prime ideals in S, then one concludes immediately that $\cap P = (0)$. A fortiori the intersection of all prime ideals ($\neq (0)$) in $O[\alpha]$ is $= (0)$. This completes the proof.

If O is an integral domain in which the intersection ($\cap p$) of the prime ideals ($\neq (0)$) is $\neq (0)$, then for every r it is possible to define a finite extension $O[\alpha_1, \cdots, \alpha_n]$ of O such that

$$\dim O[\alpha_1, \cdots, \alpha_n] = r$$

and

$$\text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O = r;$$

namely, we adjoin to O an element $1/c$, $c \in \cap p$, so that $O[1/c]$ is the quotient field of O, and thereupon adjoin r indeterminates. The situation is different for a 1-dimensional ring which is not an F-ring and in which the intersection of the prime ideals ($\neq (0)$) is $= (0)$.

THEOREM 10. *Let O be a* 1-*dimensional ring such that $O[x]$ is* 2-*dimensional, and let the intersection of the prime ideals* ($\neq (0)$) *in O be* $= (0)$. *Then for any integral domain $O[\alpha_1, \cdots, \alpha_n]$,*

$$\dim O[\alpha_1, \cdots, \alpha_n] = 1 + \text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O.$$

Proof. Let

$$K = \text{quotient field of } O, \quad r = \text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O.$$

Then $K[\alpha_1, \cdots, \alpha_n]$ is r-dimensional and a chain $(0) \subset P_1 \subset \cdots \subset P_r \subset (1)$ of prime ideals in $K[\alpha_1, \cdots, \alpha_n]$ contracts to a chain

$$(0) \subset p_1 \subset \cdots \subset p_r \subset (1), \text{ and } p_i \cap O = (0), \ i = 1, \cdots, r.$$

Moreover, p_r is not maximal, for if it were, then

$$O[\alpha_1, \cdots, \alpha_n]/p_r = O[\bar{\alpha}_1, \cdots, \bar{\alpha}_n]$$

would be a field; hence also $K[\bar{\alpha}_1, \cdots, \bar{\alpha}_n]$ would be a field, whence the $\bar{\alpha}_i$ would be algebraic over K, therefore also over O. This contradicts the previous theorem. Hence

$$\dim O[\alpha_1, \cdots, \alpha_n] \geq 1 + \text{d.t. } O[\alpha_1, \cdots, \alpha_n]/O,$$

and we have already seen the reverse inequality.

Since the theory of 1-dimensional rings must separate the cases that the intersections of prime ideals $(\neq (0))$ is $= (0)$ or $\neq (0)$, it may be of interest to have an example of a 1-dimensional ring, not an F-ring, *with infinitely many prime ideals* $(\neq (0))$ having intersection $\neq (0)$. We construct such a ring O as follows. Let K be a field containing all roots of unity, x an indeterminate, L the algebraic closure of $K(x)$, S the integral closure in L of $K[x]$, and O, the quotient-ring of S with respect to the multiplicatively closed system of polynomials in $K[x]$ which are not divisible by x. Infinitely many prime ideals in S lie over (x) in $K[x]$; to see this, let n be any integer not divisible by the characteristic of K, $a_1, \cdots, a_n$ the nth roots of unity, $y = \sqrt[n]{1+x}$. In $K[x, y]$ there lie n prime ideals over (x), namely $(x, y - a_i)$, since $(0, a_i)$ is a point of $y^n = 1 + x$. Going up to S, we see that there exist at least n prime ideals over (x). Every prime ideal in O which differs from (0) contains x, and there are infinitely many such ideals. We now verify immediately that O is a ring of the required type.

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