ON GROUPS OF ORTHONORMAL FUNCTIONS (II)

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An orthonormal group on a measure-space Ω is defined as an orthonormal system of functions which is simultaneously a group with respect to multiplication. In an earlier note [1] we showed that, essentially, all such systems are derived from character groups of compact abelian groups.[1] It may be of interest to know, for a given Ω, what orthonormal groups it can support. The answer to this question for $\Omega = I$, the unit interval, is given by the following theorem.

THEOREM. *Let G be any countable abelian group. Then there exists on the unit interval I an orthonormal group G' isomorphic to G. If G is infinite, then G' is complete in $L^2(I)$.*

Proof. Assign the discrete topology to G, and let H be its character group. If G is of finite order n, then H and G are isomorphic. To each $h_k \in H$, $k = 1, 2, \cdots, n$, we associate the interval $I_k = [(k-1)/n, k/n)$, and define the n functions f_j by

$$f_j(x) = h_k(g_j) \qquad (x \in I_k) ,$$

where g_j are the elements of G. Then $\{f_j\}$ is the required orthonormal group.

If G is infinite, H is uncountable. The measure-algebra of H (with respect to normalized Haar measure) is non-atomic, separable, and normalized. Hence [2, p. 173] it is isomorphic to the measure-algebra of I. Now H is a complete separable metric space, and the outer Haar measure is a regular Caratheodory outer measure; the same is true of Lebesgue measure on I. Therefore we can apply a theorem of von Neumann [3, Th. 1] to obtain a measure-preserving transformation from H to I. The characters of H, transferred to I, then form the required orthonormal group, complete in $L^2(I)$.

[1] Only the case of a countable orthonormal group was considered in [1], but the proofs carry over to the uncountable case with slight modification.

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We shall now examine a special case of this theorem, in which we can construct G' explicitly. Let G denote the additive group of rationals mod 1. G is the (weak) direct sum of the primary groups G_p consisting of all the rationals mod 1 with denominators powers of the prime p. The character group of G_p is H_p, the group of p-adic integers

$$\xi_p = c_{p0} + c_{p1}p + c_{p2}p^2 + \cdots \qquad (c_{pm} = 0, 1, 2, \cdots, p-1), \tag{1}$$

and H is the (strong) direct sum of all the H_p. The value of the character ξ_p at $1/p^m$, $m > 0$, is given by

$$\chi_{1/p^m}(\xi_p) = \exp 2\pi i\, A_m(\xi_p), \tag{2}$$

where

$$A_m(\xi_p) = \frac{1}{p^m}(c_{p0} + c_{p1}p + \cdots + c_{p,m-1}p^{m-1}). \tag{3}$$

For $n \le m$, $p^n A_m(\xi_p) \equiv A_{m-n}(\xi_p) \pmod 1$, so the definition

$$\chi_{k/p^m}(\xi_p) = \chi^k_{1/p^m}(\xi_p) \tag{4}$$

is unambiguous. For $h = (\xi_2, \xi_3, \xi_5, \cdots) \in H$, and for $r \in G$, we decompose r into

$$r = \frac{k_1}{p_1^{m_1}} + \frac{k_2}{p_2^{m_2}} + \cdots + \frac{k_t}{p_t^{m_t}} \tag{5}$$

and have

$$\chi_r(h) = \prod_{s=1}^{t} \chi_{k_s/p_s^{m_s}}(\xi_{p_s}). \tag{6}$$

Next we construct a mapping λ of H onto I. For this purpose we remark that, given any sequence of integers $n_k > 1$, $k = 1, 2, \cdots$, it is possible to get a representation of $x \in I$ by sequences $c_k(x)$ which generalizes the representation to the fixed base b. We divide I into n_1 equal intervals, each of these into n_2 equal intervals, and so on. If x falls in the jth interval $(j = 0, 1, 2, \cdots, n_k - 1)$ at the kth stage, we define $c_k(x) = j$. We then have

$$(7) \qquad x = \frac{c_1}{n_1} + \frac{c_2}{n_1 n_2} + \frac{c_3}{n_1 n_2 n_3} + \cdots,$$

and the representation is unique except for a countable set. We can remove the ambiguity by choosing the finite expansion where two are possible. It is easy to see that the measure of the set of $x \in I$ for which $c_1, c_2, \cdots, c_k$ have given values is $(n_1 n_2 \cdots n_k)^{-1}$, so that the functions $c_k(x)$ are statistically independent.

Returning to the mapping λ to be defined, let $h = (\xi_2, \xi_3, \xi_5, \cdots) \in H$ be given. We run through the integers c_{pm} by the diagonal method, taking $n_k = p$ whenever we reach c_{pm}. Thus $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$, $n_5 = 3$, $n_6 = 2$, $n_7 = 7$, and so on. This defines a number $x \in I$ which we designate as $\lambda(h)$. Thus

$$(8) \qquad \lambda(h) = \frac{c_{20}}{2} + \frac{c_{30}}{2\cdot 3} + \frac{c_{21}}{2^2\cdot 3} + \frac{c_{50}}{2^2\cdot 3\cdot 5} + \frac{c_{31}}{2^2\cdot 3^2\cdot 5} + \frac{c_{22}}{2^3\cdot 3^2\cdot 5}$$

$$+ \frac{c_{70}}{2^3\cdot 3^2\cdot 5\cdot 7} + \cdots = \sum_{\substack{p \text{ prime} \\ m \geq 0}} \frac{c_{pm}}{D_{pm}};$$

D_{pm} is defined, for $p = k$th prime p_k, by

$$(9) \qquad D_{pm} = \prod_{j=1}^{k-1} p_j^{m+k-j} \prod_{j=k}^{m+k} p_j^{m+k-j+1},$$

empty products being interpreted as 1. With the convention given above, λ is one-to-one, and the inverse μ is given by

$$(10) \qquad c_{pm}(x) \equiv [D_{pm}x] \pmod p \qquad 0 \leq c_{pm}(x) < p.$$

It is easily verified that λ is continuous. More important, however, is the fact that it is measure-preserving, since the $c_{pm}(h)$ are statistically independent with respect to Haar measure on H, and have the same distribution as the $c_{pm}(x)$ on I; the measures on both spaces are determined by the measures on the sets of constancy of finite collections of c_{pm}. It follows that the functions

$$(11) \qquad f_r(x) = \chi_r(\mu(x)) \qquad (r \in G;\ x \in I)$$

form a complete orthonormal group on I.

It may be of interest to note the following Fourier expansion of the function $((x)) = x - [x] - 1/2$:

$$(12) \qquad ((x)) = -\sum_{p,m,n} \left\{p^m D_{pm}\left(1 - e^{-2\pi i n/p^{m+1}}\right)\right\}^{-1} f_{n/p^{m+1}}(x),$$

where the summation is extended over all primes p, all $m \geq 0$, and $0 < n < p^{m+1}$, with $(n, p) = 1$. To derive (12), write

$$M = c_{p0}(x) + c_{p1}(x)p + \cdots + c_{pm}(x)p^m,$$

$$N = c_{p0}(x) + \cdots + c_{p,m-1}(x)p^{m-1},$$

so that

$$(13) \qquad c_{pm}(x) = \frac{M-N}{p^m}.$$

On the other hand, from (2), (3), and (11),

$$f_{k/p^m}(x) = \exp 2\pi i \frac{kN}{p^m}.$$

Thus

$$(14) \qquad \begin{aligned} N &= \sum_{r=0}^{p^m-1} r\delta(r, N) = \frac{1}{p^m}\sum_{r=0}^{p^m-1} r \sum_{k=0}^{p^m-1} e^{-2\pi i k r/p^m} f_{k/p^m}(x) \\ &= \sum_{k=0}^{p^m-1} f_{k/p^m}(x)\left\{\frac{1}{p^m}\sum_{r=0}^{p^m-1} r e^{-2\pi i k r/p^m}\right\}. \end{aligned}$$

There is a similar expression for M. Combined with (13), they yield the Fourier expansion for $c_{pm}(x)$. Substituting this in (8) and rearranging, we obtain (12). The series converges uniformly and absolutely for all x. This, incidentally, furnishes an alternate proof of the completeness of $\{f_r\}$, since every power of x has an expansion of the same kind, and the polynomials are dense in $L^2(I)$.

If we wish to exhibit the rationals R as a complete orthonormal group, this time on the unit square I_2, we need merely take

$$(15) \qquad g_r(x, y) = e^{2\pi iry} f_r(x) \qquad (r \in R;\ (x, y) \in I_2).$$

To see this, let $\hat{R}$ denote the character group of R. Every element $\sigma \in \hat{R}$ determines a number y, $0 \le y < 1$, by $\sigma(1) = \exp 2\pi i y$, and an element $h \in H$, defined by $\chi_r(h) = \sigma(r) \exp(-2\pi i r y)$. Conversely, every pair (h, y) determines a unique σ. If $h = (\xi_2, \xi_3, \cdots)$, $h' = (\xi_2', \xi_3', \cdots)$, then $\sigma(h, y) \cdot \sigma(h', y') = \sigma(h'', y'')$, where $y'' \equiv y + y'$ (mod 1) and $h'' = (\xi_2'', \xi_3'', \cdots)$ is given by $\xi_p'' = \xi_p + \xi_p' + [y + y']$. Using these relations, we can show that the mapping $\sigma(h, y) \longrightarrow (\lambda(h), y)$ is a continuous and measure-preserving transformation from $\hat{R}$ to I_2. Under this transformation R, realized as the character group of $\hat{R}$, goes over into the system (15). We could, of course, map I_2 onto I, preserving measure, and thus realize R as an orthonormal group on I.

REFERENCES

1. N. J. Fine, *On groups of orthonormal functions* (I), Pacific J. Math. **5** (1955), 51-59.

2. P. R. Halmos, *Measure theory*, New York, 1950.

3. J. von Neumann, *Einige Sätze über messbare Abbildungen*, Ann. of Math. **33** (1932), 574-586.

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