1. Introduction. In this paper we evaluate certain determinants whose elements are the Bernoulli, Euler, and related numbers of higher order. In the notation of Nörlund [1, Chapter 6] these numbers may be defined as follows: the Bernoulli numbers of order \( n \) by

\[
\left( \frac{t}{e^t - 1} \right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} B_v^{(n)},
\]

the related "D" numbers by

\[
\left( \frac{t}{\sin t} \right)^n \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} J_v^{(n)} (2v) = 0,
\]

the Euler numbers of order \( n \) by

\[
(\sec t)^n = \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} E_v^{(n)} (2v) = 0,
\]

and the "C" numbers by

\[
\left( \frac{2}{e^t + 1} \right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} C_v^{(n)}.
\]

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proofs of these results follow from the evaluation of a determinant of a more
general nature; see (3.4), below. Finally, a number of applications are given.

2. Preliminaries and notation. The numbers \( B_v^{(n)} \), \( D_{2v}^{(n)} \), \( E_{2v}^{(n)} \), and \( C_v^{(n)} \) may be expressed as polynomials in \( n \) of degree \( v \) [1, Chapter 6]; in particular,

\[
B_0^{(n)} = D_0^{(n)} = E_0^{(n)} = C_0^{(n)} = 1.
\]

Although little is known about these polynomials, it will suffice for our purposes
to give explicitly the values of the coefficients of \( n^v \) in each of the four cases.

Considering first the Bernoulli numbers, we use the recursion formula [1, p. 146]

(2.1) \[
B_v^{(n)} = -\frac{n}{v} \sum_{s=1}^{v} (-1)^s \binom{v}{s} B_s B_{v-s}^{(n)}.
\]

Let

\[
B_v^{(n)} = b_v n^v + b_{v-1} n^{v-1} + \cdots + b_0,
\]

\[
B_{v-1}^{(n)} = b_{v-1} n^{v-1} + b_{v-2} n^{v-2} + \cdots + b_0,
\]

and compare coefficients of \( n^v \) on both sides of (2.1). We find that

\[
b_v = -\frac{1}{v} (-1)^v \binom{v}{1} B_1 b_{v-1}.
\]

But \( B_1 = -1/2 \) and therefore \( b_v = -b_{v-1}/2 \). Since \( B_0^{(n)} = 1 \), the preceding leads us recursively to

(2.2) \[
B_v^{(n)} = \left(-\frac{1}{2}\right)^v n^v + b_{v-1} n^{v-1} + \cdots + b_0.
\]

In a similar fashion the formula [1, p. 146]

(2.3) \[
C_{v+1}^{(n)} = -n \sum_{s=0}^{v} (-1)^s \binom{v}{s} C_s C_{v-s}^{(n)},
\]

coupled with \( C_0^{(n)} = 1 \), permits us to write
(2.4) \[ C^{(n)}_v = (-1)^v n^v + c_{v-1} n^{v-1} + \cdots + c_0. \]

As for the Euler numbers, we consider the symbolic formula [1, p. 124]

(2.5) \[ (E^{(n)} + 1)^{2v} + (E^{(n)} - 1)^{2v} = 2E^{(n-1)}_{2v} \]

in which, after expansion, exponents on the left side are degraded to subscripts. Hence we have

(2.6) \[ E^{(n)}_{2v} + \frac{(2v)(2v-1)}{1 \cdot 2} E^{(n)}_{2v-2} + \cdots = E^{(n-1)}_{2v}. \]

Writing

\[ E^{(n)}_{2v} = e_v n^v + e_{v-1} n^{v-1} + \cdots + e_0, \]

and

\[ E^{(n)}_{2v-2} = e'_{v-1} n^{v-1} + e'_{v-2} n^{v-2} + \cdots + e'_0, \]

we see first that

\[ E^{(n)}_{2v} - E^{(n-1)}_{2v} = ve_v n^{v-1} + \text{terms of lower degree}. \]

Hence comparing coefficients of \( n^{v-1} \) in (2.6) we have

\[ e_v = - \frac{(2v)(2v-1)}{2v} e'_{v-1}. \]

Since \( E^{(n)}_0 = 1, \) we obtain recursively

(2.7) \[ E^{(n)}_{2v} = \frac{(2v)!}{(-2^v v!)} n^v + e_{v-1} n^{v-1} + \cdots + e_0. \]

Next, from [1, p. 129]

(2.8) \[ (D^{(n)} + 1)^{2v+1} - (D^{(n)} - 1)^{2v+1} = 2(2v + 1)D^{(n-1)}_{2v}, \]

we find that
We shall employ the difference operator \( \Delta_d = \Delta \) for which
\[
\Delta f(x) = f(x + d) - f(x) \quad \text{and} \quad \Delta^v = \Delta \cdot \Delta^{v-1}.
\]

We recall that if
\[
f(x) = a_v x^v + a_{v-1} x^{v-1} + \cdots + a_0,
\]
then
\[
(2.10) \quad \Delta^v f(x) = a_v d^v v!
\]

3. Main results. Let
\[
f_n(x) = a_{n,n} x^n + a_{n,n-1} x^{n-1} + \cdots + a_{n,0} \quad (a_{n,n} \neq 0),
\]
and consider the determinant
\[
(3.2) \quad |f_i(x_j)| \quad (i, j = 0, 1, \ldots, m).
\]

This may be written as the product of the two determinants
\[
(3.3) \quad \begin{vmatrix}
  a_{0,0} & 0 & \cdots & 0 \\
  a_{1,0} & a_{1,1} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m,0} & a_{m,1} & \cdots & a_{m,m}
\end{vmatrix} \cdot \begin{vmatrix}
  1 & 1 & \cdots & 1 \\
  x_0 & x_1 & \cdots & x_m \\
  \vdots & \vdots & \ddots & \vdots \\
  x_0^m & x_1^m & \cdots & x_m^m
\end{vmatrix}.
\]

The first determinant in (3.3) reduces simply to the product of the elements on the main diagonal, and the second is the familiar Vandermond determinant. Hence
\[
(3.4) \quad |f_i(x_j)| = \prod_{k=0}^{m} a_{k,k} \prod_{r > s} (x_r - x_s) \quad (r, s = 0, 1, \ldots, m).
\]

If we let
\[
f_i(x_j) = B_i^{(x_j)},
\]
then it follows from (3.4) and (2.2) that

$$|B^{(x_j)}_i| = \prod_{k=0}^{m} \left( -\frac{1}{2} \right)^k \prod_{r>s} (x_r - x_s) \quad (i, j, r, s = 0, 1, \ldots, m).$$

Application of (3.4) to (2.4), (2.7), and (2.9) yields results of a similar nature for the $C$, $D$, and $E$ numbers. Consequently we have:

**Theorem 1.** For $i, j = 0, 1, \ldots, m$,

(i) $$|B^{(x_j)}_i| = \prod_{k=0}^{m} \left( -\frac{1}{2} \right)^k \prod_{r>s} (x_r - x_s),$$

(ii) $$|C^{(x_j)}_i| = \prod_{k=0}^{m} (-1)^k \prod_{r>s} (x_r - x_s),$$

(iii) $$|D^{(x_j)}_{2i}| = \prod_{k=0}^{m} \left( -\frac{1}{6} \right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s),$$

(iv) $$|E^{(x_j)}_{2i}| = \prod_{k=0}^{m} \left( -\frac{1}{2} \right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s).$$

If we take $x_j = a + jd$ then we obtain:

**Corollary 1.** For $i, j = 0, 1, \ldots, m, a$ and $d$ constants,

(i) $$|B^{(a+jd)}_i| = \prod_{k=0}^{m} \left( -\frac{d}{2} \right)^k k!,$$

(ii) $$|C^{(a+jd)}_i| = \prod_{k=0}^{m} (-d)^k k!,$$

(iii) $$|D^{(a+jd)}_{2i}| = \prod_{k=0}^{m} \left( -\frac{d}{6} \right)^k (2k)!,$$

(iv) $$|E^{(a+jd)}_{2i}| = \prod_{k=0}^{m} \left( -\frac{d}{2} \right)^k (2k)!.$$
If we let
\[ f_i(a + x d_i) = g_i(x), \]
\( f_i(x) \) defined as in (3.1), then we can readily show by the above method that

\[ |f_i(a + j d_i)| = \prod_{k=0}^{m} a_{k,k} d_k^k k! \quad (i, j = 0, 1, \ldots, m). \]  

Hence (3.6) implies

\[ |B_i^{(a+j d_i)}| = \prod_{k=0}^{m} \left( -\frac{d_k^k}{2^k} \right) k!, \]

with like results for the other numbers.

We remark that the determinants of Corollary 1 may also be evaluated by a succession of column subtractions.

**4. Applications.** We consider first the determinant

\[ |B_i^{(a+j d)}(x)| \quad (i, j = 0, 1, \ldots, m; a, d \text{ constants}), \]

where \( B_i^{(n)}(x) \) is the Bernoulli polynomial of order \( n \) defined by [1, p. 145]

\[ \left( \frac{t}{e^t - 1} \right)^n e^{x t} = \sum_{r=0}^{\infty} \frac{t^v}{v!} B_v^{(n)}(x). \]

(For \( x = 0 \), \( B_v^{(n)}(0) = B_v^{(n)} \), the Bernoulli number of order \( n \)). Also, by [1, p. 143],

\[ B_v^{(n)}(x) = \sum_{s=0}^{v} \binom{v}{s} x^{v-s} B_s^{(n)}. \]

Consequently

\[ |B_i^{(a+j d)}(x)| = \left| \sum_{s=0}^{i} \binom{i}{s} x^{i-s} B_s^{(a+j d)} \right|. \]

If we define
\[
\binom{0}{0} = 1 \text{ and } \binom{i}{j} = 0 \text{ for } j > i,
\]
then the right member of (4.2) may be written as the product of the two determinants;
\[
\begin{vmatrix} i \\ j \end{vmatrix} x^{i-j} \cdot |B_{i}^{(a+jd)}|.
\]

The first determinant has value 1 and hence, by Corollary 1(i),
\[
|B_{i}^{(a+jd)}(x)| = \prod_{k=0}^{m} \left( -\frac{d}{2} \right)^{k} k!.
\]

The Bernoulli polynomials may also be expressed in terms of the \( D \) numbers by [1, p. 130]
\[
B_{v}^{(n)}(x) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left( x - \frac{n}{2} \right)^{v-2s} D_{2s}^{(n)} / 2^{2s}.
\]
If in (4.3) we let \( x = h n \), \( h \neq 1/2 \), then
\[
B_{v}^{(n)}(hn) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left( h - \frac{1}{2} \right)^{v-2s} n^{v-2s} D_{2s}^{(n)} / 2^{2s}.
\]
Since \( b_{2n}^{(n)} \) may be written as a polynomial in \( n \) of degree \( s \), and \( D_{0}^{(n)} = 1 \), it follows readily from (4.4) that, expressed as a polynomial in \( n \),
\[
B_{v}^{(n)}(hn) = \left( h - \frac{1}{2} \right)^{v} n^{v} + \text{terms of lower degree}.
\]
Consequently, using the same procedure that gave (3.4), we can show for \( a, d \) fixed constants, \( i, j = 0, 1, \ldots, m \), that
\[
|B_{i}^{(a+jd)}(h(a+jd))| = \prod_{k=0}^{m} \left( h - \frac{1}{2} \right)^{k} d^{k} k!.
\]
For \( h = 0 \), (4.7) reduces to the case of Corollary 1(i). If \( h = 1/2 \) and \( v \) is odd, then it follows from (4.4) that
Therefore for \( m \geq 1 \), the value of the determinant in (4.6) is zero. However, if \( v \) is even, then

\[
B_{\nu}^{(n)}(n/2) = D_{\nu}^{(n)}/2^{2\nu},
\]

and

\[
(4.7)' \quad \left| B_{2i}^{(a+jd)} \left( \frac{a+jd}{2} \right) \right| = |D_{2i}^{(a+jd)}/2^{2i}| = \prod_{k=0}^{m} \left( -\frac{d}{24} \right)^k (2k)!,
\]

where in evaluating the second determinant we have applied Corollary 1(iii).

Finally, it is of interest to point out that \([1, p. 4]\)

\[
\Delta^{\nu} f(x) = \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} f(x+jd)
\]

together with (2.2), (2.4), (2.7), (2.9), and (2.10) yield the recursion formulas

\[
\sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} B_{\nu}^{(a+jd)} = \left( -\frac{d}{2} \right)^{\nu} \nu!,
\]

\[
\sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} C_{\nu}^{(a+jd)} = (-d)^{\nu} \nu!,
\]

\[
\sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} E_{2\nu}^{(a+jd)} = \left( -\frac{d}{2} \right)^{\nu} (2\nu)!
\]

and

\[
\sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} D_{2\nu}^{(a+jd)} = \left( -\frac{d}{6} \right)^{\nu} (2\nu)!.\]

5. Some additional results. The above methods may also be applied to the evaluation of determinants involving the classic orthogonal polynomials. We consider first the Laguerre polynomials defined by \([2, p. 97]\)
Setting $\alpha = a + jd$ and writing (5.1) as a polynomial in $j$ we have

$$L_n^{(a+jd)}(x) = j^n \frac{d^n}{n!} + \text{terms of lower degree}.$$ 

Consequently, as in § 3, we obtain

$$L_n^{(a+jd)}(x)^m = \prod_{k=0}^{m-1} d^k = d^{\frac{1}{2}m(m-1)}$$ 

For the Jacobi polynomials defined by [2, p. 67]

$$P_n^{(a, \beta)}(x) = \sum_{v=0}^{n} \binom{n + \alpha}{n - \nu} \binom{n + \beta}{v} \left( \frac{x - 1}{2} \right)^v \left( \frac{x + 1}{2} \right)^{n-v}$$

we set $\alpha = a + jd$ and hold $\beta$ fixed. Then, as a polynomial in $j$

$$P_n^{(a+jd, \beta)}(x) = j^n \frac{d^n}{2^n} \frac{(x + 1)^n}{n!} + \text{terms of lower degree}.$$ 

Hence, we find

$$|P_i^{(a+jd, \beta)}(x)| = \left( \frac{(x + 1)d}{2} \right)^{\frac{1}{2}m(m-1)}$$ 

Similarly

$$|P_i^{(a, b+je)}(x)| = \left( \frac{(x - 1)e}{2} \right)^{\frac{1}{2}m(m-1)}$$ 

We consider next, as a polynomial in $j$,

$$P_n^{(a+jd, b+je)}(x)$$

$$= j^n \sum_{v=0}^{n} \frac{\alpha^{n-v} e^v}{(n-v)! v!} \left( \frac{x - 1}{2} \right)^v \left( \frac{x + 1}{2} \right)^{n-v} + \text{terms of lower degree}$$
\[
\frac{j^n}{n!} \left( \frac{(d + e)x + d - e}{2} \right)^n + \text{terms of lower degree},
\]

which yields

\[\text{(5.6)} \quad |p_{i}^{(a+jd,b+je)}(x)| = \left[ \frac{(d + e)x + d - e}{2} \right]^{\frac{1}{2}m(m-1)} \]

\[(i, j = 0, 1, \ldots, m - 1).\]

Finally, for \(\alpha = \beta\), the Jacobi polynomials reduce to the ultraspherical polynomials \(P^{(\alpha)}(x)\). It follows from (5.6) that

\[\text{(5.7)} \quad |p_{i}^{(a+jd)}(x)| = (dx)^{\frac{1}{2}m(m-1)} \quad (i, j = 0, 1, \ldots, m - 1).\]

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