SOME DETERMINANTS INVOLVING BERNOULLI AND EULER NUMBERS OF HIGHER ORDER

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1. Introduction. In this paper we evaluate certain determinants whose elements are the Bernoulli, Euler, and related numbers of higher order. In the notation of Nörlund [1, Chapter 6] these numbers may be defined as follows: the Bernoulli numbers of order n by

$$(1.1)\qquad \left(\frac{t}{e^t-1}\right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} B_v^{(n)},$$

the related "D" numbers by

$$(1.2)\qquad \left(\frac{t}{\sin t}\right)^n \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} D_{2v}^{(n)} \qquad (D_{2v+1}^{(n)} = 0),$$

the Euler numbers of order n by

$$(1.3)\qquad (\sec t)^n = \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} E_{2v}^{(n)} \qquad (E_{2v+1}^{(n)} = 0),$$

and the "C" numbers by

$$(1.4)\qquad \left(\frac{2}{e^t+1}\right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} \frac{C_v^{(n)}}{2^v}.$$

(By n we denote an arbitrary complex number. When $n = 1$, we omit the upper index in writing the numbers; for example, $B_v^{(1)} = B_v$.)

We evaluate determinants such as

$$|B_i^{(x_j)}| \qquad (i, j = 0, 1, \cdots, m)$$

for the Bernoulli numbers, and similar determinants for the other numbers. The

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proofs of these results follow from the evaluation of a determinant of a more general nature; see (3.4), below. Finally, a number of applications are given.

2. Preliminaries and notation. The numbers $B_v^{(n)}$, $D_{2v}^{(n)}$, $E_{2v}^{(n)}$, and $C_v^{(n)}$ may be expressed as polynomials in n of degree v [**1**, Chapter 6]; in particular,

$$B_0^{(n)} = D_0^{(n)} = E_0^{(n)} = C_0^{(n)} = 1.$$

Although little is known about these polynomials, it will suffice for our purposes to give explicitly the values of the coefficients of n^v in each of the four cases.

Considering first the Bernoulli numbers, we use the recursion formula [**1**, p. 146]

$$(2.1) \qquad B_v^{(n)} = -\frac{n}{v} \sum_{s=1}^{v} (-1)^s \binom{v}{s} B_s B_{v-s}^{(n)}.$$

Let

$$B_v^{(n)} = b_v n^v + b_{v-1} n^{v-1} + \cdots + b_0,$$

$$B_{v-1}^{(n)} = b'_{v-1} n^{v-1} + b'_{v-2} n^{v-2} + \cdots + b'_0,$$

and compare coefficients of n^v on both sides of (2.1). We find that

$$b_v = -\frac{1}{v}(-1)\binom{v}{1} B_1 b'_{v-1}.$$

But $B_1 = -1/2$ and therefore $b_v = -b'_{v-1}/2$. Since $B_0^{(n)} = 1$, the preceding leads us recursively to

$$(2.2) \qquad B_v^{(n)} = \left(-\frac{1}{2}\right)^v n^v + b_{v-1} n^{v-1} + \cdots + b_0.$$

In a similar fashion the formula [**1**, p. 146]

$$(2.3) \qquad C_{v+1}^{(n)} = -n \sum_{s=0}^{v} (-1)^s \binom{v}{s} C_s C_{v-s}^{(n)},$$

coupled with $C_0^{(n)} = 1$, permits us to write

$$(2.4) \qquad C_v^{(n)} = (-1)^v n^v + c_{v-1} n^{v-1} + \cdots + c_0 .$$

As for the Euler numbers, we consider the symbolic formula [**1**, p. 124]

$$(2.5) \qquad (E^{(n)} + 1)^{2v} + (E^{(n)} - 1)^{2v} = 2E_{2v}^{(n-1)}$$

in which, after expansion, exponents on the left side are degraded to subscripts. Hence we have

$$(2.6) \qquad E_{2v}^{(n)} + \frac{(2v)(2v-1)}{1 \cdot 2} E_{2v-2}^{(n)} + \cdots = E_{2v}^{(n-1)} .$$

Writing

$$E_{2v}^{(n)} = e_v n^v + e_{v-1} n^{v-1} + \cdots + e_0 ,$$

and

$$E_{2v-2}^{(n)} = e'_{v-1} n^{v-1} + e'_{v-2} n^{v-2} + \cdots + e'_0 ,$$

we see first that

$$E_{2v}^{(n)} - E_{2v}^{(n-1)} = v e_v n^{v-1} + \text{terms of lower degree} .$$

Hence comparing coefficients of n^{v-1} in (2.6) we have

$$e_v = -\frac{(2v)(2v-1)}{2v} e'_{v-1} .$$

Since $E_0^{(n)} = 1$, we obtain recursively

$$(2.7) \qquad E_{2v}^{(n)} = \frac{(2v)!}{(-2)^v v!} n^v + e_{v-1} n^{v-1} + \cdots + e_0 .$$

Next, from [**1**, p. 129]

$$(2.8) \qquad (D^{(n)} + 1)^{2v+1} - (D^{(n)} - 1)^{2v+1} = 2(2v+1) D_{2v}^{(n-1)} ,$$

we find that

$$D_{2v}^{(n)} = \left(-\frac{1}{6}\right)^v \frac{(2v)!}{v!} n^v + d_{n-1} n^{v-1} + \cdots + d_0 . \tag{2.9}$$

We shall employ the difference operator $\Delta_d = \Delta$ for which

$$\Delta f(x) = f(x+d) - f(x) \quad \text{and} \quad \Delta^v = \Delta \cdot \Delta^{v-1} .$$

We recall that if

$$f(x) = a_v x^v + a_{v-1} x^{v-1} + \cdots + a_0 ,$$

then

$$\Delta^v f(x) = a_v d^v v! \tag{2.10}$$

3. Main results. Let

$$f_n(x) = a_{n,n} x^n + a_{n,n-1} x^{n-1} + \cdots + a_{n,0} \qquad (a_{n,n} \neq 0), \tag{3.1}$$

and consider the determinant

$$|f_i(x_j)| \qquad (i, j = 0, 1, \cdots, m). \tag{3.2}$$

This may be written as the product of the two determinants

$$\begin{vmatrix} a_{0,0} & 0 & \cdots 0 \\ a_{1,0} & a_{1,1} & \cdots 0 \\ \cdots & \cdots & \cdots \\ a_{m,0} & a_{m,1} & \cdots a_{m,m} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots 1 \\ x_0 & x_1 & \cdots x_m \\ \cdots & \cdots & \cdots \\ x_0^m & x_1^m & \cdots x_m^m \end{vmatrix} . \tag{3.3}$$

The first determinant in (3.3) reduces simply to the product of the elements on the main diagonal, and the second is the familiar Vandermond determinant. Hence

$$|f_i(x_j)| = \prod_{k=0}^{m} a_{k,k} \prod_{r>s} (x_r - x_s) \qquad (r, s = 0, 1, \cdots, m). \tag{3.4}$$

If we let

$$f_i(x_j) = B_i^{(x_j)} ,$$

then it follows from (3.4) and (2.2) that

$$(3.5)\qquad |B_i^{(x_j)}| = \prod_{k=0}^{m}\left(-\frac{1}{2}\right)^k \prod_{r>s}(x_r - x_s) \qquad (i, j, r, s = 0, 1, \cdots, m).$$

Application of (3.4) to (2.4), (2.7), and (2.9) yields results of a similar nature for the C, D, and E numbers. Consequently we have:

THEOREM 1. *For* $i, j = 0, 1, \cdots, m$,

$$\text{(i)}\qquad |B_i^{(x_j)}| = \prod_{k=0}^{m}\left(-\frac{1}{2}\right)^k \prod_{r>s}(x_r - x_s),$$

$$\text{(ii)}\qquad |C_i^{(x_j)}| = \prod_{k=0}^{m}(-1)^k \prod_{r>s}(x_r - x_s),$$

$$\text{(iii)}\qquad |D_{2i}^{(x_j)}| = \prod_{k=0}^{m}\left(-\frac{1}{6}\right)^k \frac{(2k)!}{k!} \prod_{r>s}(x_r - x_s),$$

$$\text{(iv)}\qquad |E_{2i}^{(x_j)}| = \prod_{k=0}^{m}\left(-\frac{1}{2}\right)^k \frac{(2k)!}{k!} \prod_{r>s}(x_r - x_s).$$

If we take $x_j = a + jd$ then we obtain:

COROLLARY 1. *For* $i, j = 0, 1, \cdots, m$, *a and d constants*,

$$\text{(i)}\qquad |B_i^{(a+jd)}| = \prod_{k=0}^{m}\left(-\frac{d}{2}\right)^k k!,$$

$$\text{(ii)}\qquad |C_i^{(a+jd)}| = \prod_{k=0}^{m}(-d)^k k!,$$

$$\text{(iii)}\qquad |D_{2i}^{(a+jd)}| = \prod_{k=0}^{m}\left(-\frac{d}{6}\right)^k (2k)!,$$

$$\text{(iv)}\qquad |E_{2i}^{(a+jd)}| = \prod_{k=0}^{m}\left(-\frac{d}{2}\right)^k (2k)!$$

If we let

$$f_i(a + xd_i) = g_i(x),$$

$f_i(x)$ defined as in (3.1), then we can readily show by the above method that

$$(3.6) \qquad |f_i(a + jd_i)| = \prod_{k=0}^{m} a_{k,k} d_k^k k! \qquad (i, j = 0, 1, \cdots, m).$$

Hence (3.6) implies

$$|B_i^{(a+jd_i)}| = \prod_{k=0}^{m} \left(-\frac{d_k}{2}\right)^k k!,$$

with like results for the other numbers.

We remark that the determinants of Corollary 1 may also be evaluated by a succession of column subtractions.

4. Applications. We consider first the determinant

$$(4.1) \qquad |B_i^{(a+jd)}(x)| \qquad (i, j = 0, 1, \cdots, m;\ a, d \text{ constants}),$$

where $B_i^{(n)}(x)$ is the Bernoulli polynomial of order n defined by [**1**, p. 145]

$$\left(\frac{t}{e^t - 1}\right)^n e^{xt} = \sum_{r=0}^{\infty} \frac{t^v}{v!} B_v^{(n)}(x).$$

(For $x = 0$, $B_v^{(n)}(0) = B_v^{(n)}$, the Bernoulli number of order n.) Also, by [**1**, p. 143],

$$B_v^{(n)}(x) = \sum_{s=0}^{v} \binom{v}{s} x^{v-s} B_s^{(n)}.$$

Consequently

$$(4.2) \qquad |B_i^{(a+jd)}(x)| = \left| \sum_{s=0}^{i} \binom{i}{s} x^{i-s} B_s^{(a+jd)} \right|.$$

If we define

$$\binom{0}{0} = 1 \text{ and } \binom{i}{j} = 0 \text{ for } j > i,$$

then the right member of (4.2) may be written as the product of the two determinants;

$$\left| \binom{i}{j} x^{i-j} \right| \cdot |B_i^{(a+jd)}|.$$

The first determinant has value 1 and hence, by Corollary 1(i),

$$(4.3) \qquad |B_i^{(a+jd)}(x)| = \prod_{k=0}^{m} \left(-\frac{d}{2}\right)^k k!.$$

The Bernoulli polynomials may also be expressed in terms of the D numbers by [**1**, p. 130]

$$(4.4) \qquad B_v^{(n)}(x) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left(x - \frac{n}{2}\right)^{v-2s} D_{2s}^{(n)}/2^{2s}.$$

If in (4.3) we let $x = hn$, $h \neq 1/2$, then

$$(4.5) \qquad B_v^{(n)}(hn) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left(h - \frac{1}{2}\right)^{v-2s} n^{v-2s} D_{2n}^{(n)}/2^{2s}.$$

Since $D_{2n}^{(n)}$ may be written as a polynomial in n of degree s, and $D_0^{(n)} = 1$, it follows readily from (4.4) that, expressed as a polynomial in n,

$$(4.6) \qquad B_v^{(n)}(hn) = \left(h - \frac{1}{2}\right)^v n^v + \text{terms of lower degree}.$$

Consequently, using the same procedure that gave (3.4), we can show for a, d fixed constants, $i, j = 0, 1, \cdots, m$, that

$$(4.7) \qquad |B_i^{(a+jd)}(h(a+jd))| = \prod_{k=0}^{m} \left(h - \frac{1}{2}\right)^k d^k k!.$$

For $h = 0$, (4.7) reduces to the case of Corollary 1(i). If $h = 1/2$ and v is odd, then it follows from (4.4) that

$$B_v^{(n)}(n/2) = 0.$$

Therefore for $m \geq 1$, the value of the determinant in (4.6) is zero. However, if v is even, then

$$B_v^{(n)}(n/2) = D_v^{(n)}/2^{2v},$$

and

$$(4.7)' \qquad \left| B_{2i}^{(a+jd)}\left(\frac{a+jd}{2}\right) \right| = |D_{2i}^{(a+jd)}/2^{2i}| = \prod_{k=0}^{m} \left(-\frac{d}{24}\right)^k (2k)!,$$

where in evaluating the second determinant we have applied Corollary 1(iii).

Finally, it is of interest to point out that [1, p. 4]

$$\Delta^v f(x) = \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} f(x + jd)$$

together with (2.2), (2.4), (2.7), (2.9), and (2.10) yield the recursion formulas

$$(4.8) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} B_v^{(a+jd)} = \left(-\frac{d}{2}\right)^v v!,$$

$$(4.9) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} C_v^{(a+jd)} = (-d)^v v!,$$

$$(4.10) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} E_{2v}^{(a+jd)} = \left(-\frac{d}{2}\right)^v (2v)!$$

and

$$(4.11) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} D_{2v}^{(a+jd)} = \left(-\frac{d}{6}\right)^v (2v)!.$$

5. Some additional results. The above methods may also be applied to the evaluation of determinants involving the classic orthogonal polynomials. We consider first the Laguerre polynomials defined by [2, p. 97]

$$(5.1) \qquad L_n^{(\alpha)} = \prod_{v=0}^{n} \binom{n+\alpha}{n-v} \frac{(-x)^v}{v!} .$$

Setting $\alpha = a + jd$ and writing (5.1) as a polynomial in j we have

$$L_n^{(a+jd)}(x) = j^n \frac{d^n}{n!} + \text{terms of lower degree}.$$

Consequently, as in § 3, we obtain

$$(5.2) \qquad |L_i^{(a+jd)}(x)| = \prod_{k=0}^{m-1} d^k = d^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

For the Jacobi polynomials defined by [2, p. 67]

$$(5.3) \qquad P_n^{(a,\beta)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v}\binom{n+\beta}{v}\left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v}$$

we set $\alpha = a + jd$ and hold β fixed. Then, as a polynomial in j

$$P_n^{(a+jd,\beta)}(x) = j^n \frac{d^n}{2^n} \frac{(x+1)^n}{n!} + \text{terms of lower degree}.$$

Hence, we find

$$(5.4) \qquad |P_i^{(a+jd,\beta)}(x)| = \left\{\frac{(x+1)d}{2}\right\}^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

Similarly

$$(5.5) \qquad |P_i^{(\alpha,b+je)}(x)| = \left\{\frac{(x-1)e}{2}\right\}^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

We consider next, as a polynomial in j,

$$P_n^{(a+jd,b+je)}(x)$$
$$= j^n \sum_{v=0}^{n} \frac{\alpha^{n-v}}{(n-v)!} \frac{e^v}{v!} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v} + \text{terms of lower degree}$$

$$= \frac{j^n}{n!}\left[\frac{(d+e)x+d-e}{2}\right]^n + \text{terms of lower degree},$$

which yields

$$(5.6) \qquad |P_i^{(a+jd,b+je)}(x)| = \left[\frac{(d+e)x+d-e}{2}\right]^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

Finally, for $\alpha = \beta$, the Jacobi polynomials reduce to the ultraspherical polynomials $P^{(a)}(x)$. It follows from (5.6) that

$$(5.7) \qquad |P_i^{(a+jd)}(x)| = (dx)^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

REFERENCES

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