GENERALISED RIEMANN INVARIANTS

G. S. S. LUDFORD

1. Introduction. In a recent paper [5] Martin has considered the Monge-Ampère equation

$$rt - s^2 + \lambda^2 = 0; \quad \lambda = \lambda(x, y), \tag{1}$$

in the special case when $\lambda = X(x)Y(y)$. This equation arises in the treatment of one-dimensional anisentropic flows [4], where λ is a function determined by the distribution of entropy across the flow and the thermodynamic properties of the gas considered. It is of particular interest to find of what form λ must be in order that (1) should possess intermediate integrals, since these integrals play the same role in anisentropic motion as the Riemann invariants play in isentropic motion.

In the present paper the equation (1) is discussed without the restriction that λ is of product form. By means of a very simple piece of analysis it is shown that if and only if λ is of one of two general forms does (1) possess intermediate integrals, or "generalised Riemann invariants". Thus all the results of Martin appear as special cases. Moreover, by means of these more general results, the difficulty of there being only one particular entropy distribution in the case of a polytropic gas, for which generalised Riemann invariants exist, see [3], is overcome.

2. Conditions on λ. The case $\lambda \equiv 0$ is excluded in the following discussion, since it is so well-known.

The equation (1) will have intermediate integrals, see [2], if and only if the equations

$$\begin{cases} L_1(V) \equiv V_x + pV_z + \lambda V_q = 0, \\ L_2(V) \equiv V_y + qV_z - \lambda V_p = 0, \end{cases} \tag{2}$$

or those obtained by changing the sign of λ in (2), have a common integral

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$V(x, y, z, p, q)$. When such an integral exists, the corresponding intermediate integral is $V = \text{const.}$

Consider the equations (2). If they possess a common integral, then this integral must also satisfy

$$(3)\quad \begin{cases} L_{m,n}(V) \equiv (m+n+2)\lambda_{m,n} V_z - \lambda_{m+1,n} V_p - \lambda_{m,n+1} V_q = 0, \\ \lambda_{m,n} = \dfrac{\partial^{m+n}\lambda}{\partial x^m \partial y^n}; \qquad m, n = 0, 1, 2, \cdots, \end{cases}$$

these equations forming the totality of Jacobi conditions[1] associated with (2). Thus if V is not to be identically constant, it is necessary that every three vectors $\mathbf{v}_{m,n} = ((m+n+2)\ \lambda_{m,n},\ \lambda_{m+1,n},\ \lambda_{m,n+1})$ be linearly dependent. This condition may be simplified as follows:

THEOREM. *The necessary and sufficient condition that every three vectors* $\mathbf{v}_{m,n}$ *should be linearly dependent is that one of the following holds*:

(i) $\mathbf{v}_{0,0}$ *and* $\mathbf{v}_{1,0}$ *are independent and* $\mathbf{v}_{0,1}$ *and* $\mathbf{v}_{2,0}$ *depend on them*,

(ii) $\mathbf{v}_{0,0}$ *and* $\mathbf{v}_{0,1}$ *are independent and* $\mathbf{v}_{1,0}$ *and* $\mathbf{v}_{0,2}$ *depend on them*,

(iii) *no two of* $\mathbf{v}_{0,0}, \mathbf{v}_{1,0}, \mathbf{v}_{0,1}$ *are independent.*

Necessity: If $\mathbf{v}_{1,0}$ and $\mathbf{v}_{0,1}$ are independent, then

$$\mathbf{v}_{0,0} = a\mathbf{v}_{1,0} + b\mathbf{v}_{0,1}$$

where a, b are not both identically zero since $\lambda \not\equiv 0$. Hence either (i) or (ii) holds.

Sufficiency: (i) Let $D(i, j;\ k, l;\ m, n)$ denote the determinant whose rows are the vectors $\mathbf{v}_{i,j}, \mathbf{v}_{k,l}, \mathbf{v}_{m,n}$ respectively. Then the following identities hold:

$$D(0,0;\ 1,0;\ m+1, n) \equiv \frac{\partial}{\partial x} D(0,0;\ 1,0;\ m,n) - D(0,0;\ 2,0;\ m,n),$$

$$D(0,0;\ 1,0;\ m, n+1) \equiv \frac{\partial}{\partial y} D(0,0;\ 1,0;\ m,n) - D(0,0;\ 1,1;\ m,n).$$

Suppose now that every three vectors $\mathbf{v}_{m,n}$ are linearly dependent for $m+n \le p$, where p is a fixed integer greater than or equal to 2. Then since $\mathbf{v}_{0,0}$ and $\mathbf{v}_{1,0}$

[1] See [**1**]. Thus $L_{m+1,n} \equiv (L_1 L_{m,n} - L_{m,n} L_1)$; $L_{m,n+1} \equiv (L_2 L_{m,n} - L_{m,n} L_2)$, and $L_{0,0} \equiv (L_1 L_2 - L_2 L_1)$.

are independent, it is clear from the first identity that $\mathbf{v}_{m+1,n}$ is linearly dependent on these vectors, and from the second that $\mathbf{v}_{m,n+1}$ is linearly dependent on the same vectors. Hence every three vectors $\mathbf{v}_{m,n}$ with $m+n \leq p+1$ are linearly dependent. Furthermore, by writing $m=0$, $n=1$ it is clear that both $\mathbf{v}_{1,1}$ and $\mathbf{v}_{0,2}$ are linearly dependent on $\mathbf{v}_{0,0}$ and $\mathbf{v}_{1,0}$ under the given hypotheses, so that the proposition is proved for $p=2$. Case (ii) follows similarly. Case (iii) can only occur if

$$\lambda = \frac{1}{(\alpha x + \beta y + \gamma)^2}; \qquad \alpha, \beta, \gamma \text{ constants, not all zero}, \tag{4}$$

in which event

$$\mathbf{v}_{m,n} = \frac{(-1)^{m+n}(m+n+2)!\,\alpha^m \beta^n}{(\alpha x + \beta y + \gamma)^{m+n+2}} (\alpha x + \beta y + \gamma, -\alpha, -\beta),$$

and clearly the theorem is true in this case also.

3. The form of λ. For case (i) of the theorem, λ must be a simultaneous solution of two non-linear partial differential equations, one of the second order and the other of the third. The common integrals of these equations may be found in the following manner. Let

$$\begin{cases} f \equiv \lambda_x \lambda_{xy} - \lambda_y \lambda_{xx}, \\ g \equiv 3\lambda_x \lambda_y - 2\lambda\lambda_{xy}, \\ h \equiv 2\lambda\lambda_{xx} - 3\lambda_x^2. \end{cases}$$

Then

$$\begin{cases} f(g_x - f) - f_x g \equiv \lambda_y D(0,0;\,1,0;\,2,0) = 0, \\ \qquad fg_y - f_y g \equiv \lambda_y D(0,0;\,1,0;\,1,1) - \lambda_{xy} D(0,0;\,1,0;\,0,1) = 0, \end{cases}$$

and

$$\begin{cases} \qquad fh_x - f_x h \equiv -\lambda_x D(0,0;\,1,0;\,2,0) = 0, \\ f(h_y - f) - f_y h \equiv \lambda_{xx} D(0,0;\,1,0;\,0,1) - \lambda_x D(0,0;\,1,0;\,1,1) = 0. \end{cases}$$

Provided $f \not\equiv 0$, these two pairs of equations may be integrated to give

$$g/f = x + \alpha, \quad h/f = y + \beta$$

where α, β are arbitrary constants, or

$$-\Lambda\Lambda_{XX} = Y\{\Lambda_X \Lambda_{XX} - \Lambda_Y \Lambda_{XX}\},$$

$$\Lambda\Lambda_{XY} = X\{\Lambda_X \Lambda_{XY} - \Lambda_Y \Lambda_{XX}\},$$

where

$$X = x + \alpha, \quad Y = y + \beta, \quad \Lambda = \lambda^{-\frac{1}{2}}.$$

By division it follows that Λ must be of the form $YF_1(X/Y) + G_1(Y)$, where F_1, G_1 are undetermined functions, and by substituting back into the equations that

$$(5) \qquad \Lambda = YF(X/Y); \qquad F \text{ arbitrary}.$$

If $f \equiv 0$, it follows, since λ_x cannot be identically zero,[2] that $\lambda_y/\lambda_x = F_2'(y)$, where F_2 is undetermined, or $\lambda = F_3(x + F_2(y))$, where F_3 is undetermined. If now this λ is made to satisfy $D(0,0;\ 1,0;\ 0,1) = 0$ and $D(0,0;\ 1,0;\ 2,0) = 0$, it is found that F_2 must be a linear function of y and F_3 is an arbitrary function. Hence

$$(6) \qquad \lambda = F(x + \beta y); \qquad F, \beta \text{ arbitrary}.$$

In a similar manner case (ii) of the theorem may be treated, and it is found that either

$$(7) \qquad \Lambda = XF(Y/X); \qquad F \text{ arbitrary},$$

or

$$(8) \qquad \lambda = F(\alpha x + y); \qquad F, \alpha \text{ arbitrary},$$

where X, Y and Λ have the same meaning as before.

Case (iii) has already been discussed, see (4). Thus one is led to the conclusion that λ must have one of the two forms

$$(9) \qquad \lambda = F(\alpha x + \beta y),$$

[2]This would imply that $f = g = h = 0$, or that $\mathbf{v}_{0,0}$ and $\mathbf{v}_{1,0}$ were dependent.

$$\lambda = \frac{1}{(x+\alpha)(y+\beta)} F\left(\frac{x+\alpha}{y+\beta}\right), \tag{10}$$

which cover all the cases (4), (5), (6), (7) and (8). Here F is an arbitrary function of its argument in each case, and α, β are arbitrary constants. It will now be shown that *whenever* λ has one of these two forms, (1) possesses intermediate integrals.

4. The intermediate integrals: The case (10) will be considered first, then (9) in general, and finally the exceptional cases of (9). In each case a non-singular transformation of the variables x, y, z, p, q will be made, which will reduce the pertinent equations of (2) and (3) to a form in which any common integral may be read off. These transformations can actually be constructed by solution of the original equations treated as an algebraic system in V_x, V_y, V_z, V_p, V_q.

Suppose λ has the form (10). Then assuming $f \not\equiv 0$, since $f \equiv 0$ is covered by (9), make the (non-singular) transformation $\bar{x} = x$, $\bar{y} = y$, $\bar{p} = p$, $\bar{q} = q$ and

$$\bar{z} = z - p(x+\alpha) - q(y+\beta) + \mathfrak{Z}\left(\frac{x+\alpha}{y+\beta}\right); \quad \eta\mathfrak{Z}'(\eta) = F(\eta).$$

Then (2) yields

$$\begin{cases} L_1(V) \equiv V_{\bar{x}} + \lambda V_{\bar{q}} = 0, \\ L_2(V) \equiv V_{\bar{y}} - \lambda V_{\bar{p}} = 0, \end{cases}$$

and (3) yields

$$\begin{cases} L_{0,0}(V) \equiv -\lambda_x V_{\bar{p}} - \lambda_y V_{\bar{q}} = 0, \\ L_{1,0}(V) \equiv -\lambda_{xx} V_{\bar{p}} - \lambda_{xy} V_{\bar{q}} = 0, \end{cases}$$

in the two cases[3] $m = n = 0$ and $m = 1$, $n = 0$. Hence, since $f \not\equiv 0$, V is independent of $\bar{x}$, $\bar{y}$, $\bar{p}$, $\bar{q}$, and there is one and only one intermediate integral $\bar{z} =$ const.. This gives rise to the first entry in the table, see the next section.

Suppose now that λ has the form (9). Then the transformation $\bar{x} = x$, $\bar{y} = y$, $\bar{z} = z$ and

[3] By case (i) of the theorem only these two equations need be considered.

$$\begin{cases} \bar{p} = \beta p - \alpha q + \mathfrak{Z}(\alpha x + \beta y); & \mathfrak{Z}'(\eta) = F(\eta), \\ \bar{q} = \alpha p + \beta q, \end{cases}$$

which is non-singular for α, β not both zero, yields

$$\begin{cases} L_1(V) \equiv V_{\bar{x}} + pV_{\bar{z}} + \beta\lambda V_{\bar{q}} = 0, \\ L_2(V) \equiv V_{\bar{y}} + qV_{\bar{z}} - \alpha\lambda V_{\bar{q}} = 0, \end{cases}$$

and

$$\begin{cases} L_{0,0}(V) \equiv 2F V_{\bar{z}} - (\alpha^2 + \beta^2) F' V_{\bar{q}} = 0, \\ L_{1,0}(V) \equiv 3\alpha F' V_{\bar{z}} - \alpha(\alpha^2 + \beta^2) F'' V_{\bar{q}} = 0, \\ L_{0,1}(V) \equiv 3\beta F' V_{\bar{z}} - \beta(\alpha^2 + \beta^2) F'' V_{\bar{q}} = 0. \end{cases}$$

Thus provided α, β are not both zero and

$$3F'^2 - 2FF'' \neq 0, \tag{11}$$

V is independent of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{q}$ and there is one and only one intermediate integral $\bar{p} = \text{const.}$; this gives rise to the second and third entries in the table, see the next section. The former restriction is covered by the latter (11).

There remain the cases excluded by (11), for which

$$F(\alpha x + \beta y) = \frac{1}{(\alpha x + \beta y + \gamma)^2}; \qquad \alpha, \beta, \gamma \text{ not all zero}.$$

For $\gamma \neq 0$ one writes $\bar{x} = x$, $\bar{y} = y$, $\bar{z} = z$ and

$$\begin{cases} \bar{p} = \alpha(z - px - qy) - \gamma p - \dfrac{y}{\alpha x + \beta y + \gamma}, \\ \bar{q} = \beta(z - px - qy) - \gamma q + \dfrac{x}{\alpha x + \beta y + \gamma}, \end{cases}$$

and obtains that V is independent of $\bar{x}$, $\bar{y}$, and $\bar{z}$, so that there are two and only two independent intermediate integrals $\bar{p} = \text{const.}$ and $\bar{q} = \text{const.}$ This gives rise to the first half of each of the fourth and fifth entries in the table, see the next section. For $\gamma = 0$ one writes $\bar{x} = x$, $\bar{y} = y$ and

$$\begin{cases} \bar{z} = z - px - qy + \dfrac{\beta x - \alpha y}{(\alpha^2 + \beta^2)(\alpha x + \beta y)}\,, \\[2ex] \bar{p} = \beta p - \alpha q - \dfrac{1}{(\alpha x + \beta y)}\,, \\[2ex] \bar{q} = \alpha p + \beta q, \end{cases}$$

a transformation which is non-singular since α, β cannot be both zero. Then V is found to be independent of $\bar{x}$, $\bar{y}$, $\bar{q}$ so that there are two and only two independent intermediate integrals $\bar{z} = \text{const.}$ and $\bar{p} = \text{const.}$ This gives rise to the second half of each of the fourth and fifth entries in the table, see the next section.

5. Generalised Riemann invariants: In the fluid dynamical application of the theory one replaces, see [**4**], x, y, z, p, q, λ^2 by p, ψ, ξ, t, u, $-\tau_p$ respectively so that the table of generalised Riemann invariants on the following page may be constructed.

In this table $\chi = \alpha p + \beta\psi$ and G, H, ϕ represent in each case arbitrary functions of their arguments: the $\pm$ sign indicates the inclusion of the integrals corresponding to the (second) system associated with (2). The function τ is included in the table since it, rather than λ, is the variable of interest in applications. In each case τ rather than λ has been given in its simplest form. Notice that the inclusion of τ gives rise to the separation of the second and third entries, and the fourth and fifth, since in order to obtain τ from λ an integration has to be made ($\tau_p = -\lambda^2$). As regards λ, the third entry is just a special case ($\alpha = 0$) of the second, and the fifth a special case ($\alpha = 0$) of the fourth. Martin's results appear as particular examples. The group (b) is actually a limiting form of (a).

The forms listed for $\tau(\psi, p)$ provide sufficient generality for a wide range of given functions to be approximated. As *one* example consider the case of a polytropic gas for which, see [**4**].

$$\lambda^2 = \Psi(\psi)p^k; \quad k = -\frac{(\gamma + 1)}{\gamma}\,, \tag{12}$$

where Ψ is a function determined by the (given) distribution of entropy across the particle paths $\psi = \text{const.}$ Only for the particular distribution function

$$\Psi(\psi) = \frac{A}{(\psi + \beta)^{k+4}}\,; \quad A,\ \beta \text{ arbitrary constants}, \tag{13}$$

τ	λ	Generalised Riemann Invariants
(a) $G(\psi) - \dfrac{1}{(\psi+\beta)^3} \quad H\left(\dfrac{p+\alpha}{\psi+\beta}\right)$	$\dfrac{1}{(\psi+\beta)^2} \quad \sqrt{H'\left(\dfrac{p+\alpha}{\psi+\beta}\right)}$	$\xi - t(p+\alpha) - u(\psi+\beta) \pm \mathcal{H}\left(\dfrac{p+\alpha}{\psi+\beta}\right)$; $\mathcal{H}'(\eta) = \sqrt{H'(\eta)}$
(b) $G(\psi) - \dfrac{1}{\alpha} H(\chi)$	$\sqrt{H'(\chi)}$	$\beta t - \alpha u \pm \mathcal{H}(\chi)$; $\mathcal{H}'(\eta) = \sqrt{H'(\eta)}$
The following occur as special cases of these:		
$G(\psi) - pH(\psi)$	$\sqrt{H(\psi)}$	$t \pm \mathcal{H}(\psi)$; $\mathcal{H}'(\eta) = \sqrt{H(\eta)}$
$G(\psi) + (\chi+\gamma)^{-3}/3\alpha$	$(\chi+\gamma)^{-2}$	$\gamma \neq 0$: $\phi\left(\alpha\,\overline{\xi - tp - u\psi} - \gamma t \mp \dfrac{\psi}{\chi+\gamma},\ \beta\,\overline{\xi - tp - u\psi} - \gamma u \pm \dfrac{p}{\chi+\gamma}\right)$ $\gamma = 0$: $\phi\left(\xi - tp - u\psi \pm \dfrac{\beta p - \alpha\psi}{(\alpha^2+\beta^2)\chi},\ \beta t - \alpha u \mp \dfrac{1}{\chi}\right)$
$G(\psi) - \dfrac{p}{(\beta\psi+\gamma)^4}$	$(\beta\psi+\gamma)^{-2}$	As above with $\alpha = 0$.
Lastly, the case excluded in this paper:		
$G(\psi)$	0	$\phi(t, u)$

does (12) fall into the group (a) in the preceding table.[4] However, any λ^2 may be approximated near an arbitrary point (ψ_0, p_0) by means of the function

$$\bar{\lambda}^2 = \frac{p_0^k(p_0+\alpha)^4}{(p+\alpha)^4}\,\Psi\left[\frac{(p_0+\alpha)(\psi+\beta)}{(p+\alpha)} - \beta\right],$$

where $\beta = -\psi_0 + \tilde{\beta}$ and $\tilde{\beta}$ is a root of the quadratic

$$\left[k\,\Psi_0\Psi_0'' - (k-1)\Psi_0'^{\,2}\right]\tilde{\beta}^2 + 2(k+4)\Psi_0\Psi_0'\tilde{\beta} + 4(k+4)\Psi_0^2 = 0,$$

$\Psi_0, \Psi_0', \Psi_0''$ being the values of Ψ, Ψ', Ψ'' respectively for $\psi = \psi_0$, and

$$\alpha = \frac{-p_0[(k+4)\psi_0 + \tilde{\beta}\Psi_0']}{k\Psi_0}.$$

The function $\bar{\lambda}^2$ is identical to λ^2 when $p = p_0$, and has the same first derivatives and unmixed second derivatives as λ^2 at the point (ψ_0, p_0). Furthermore the function $\bar{\lambda}^2$ is in the group (a) of the table and is identical with λ^2 for the special Ψ given by (13). The generalised Riemann invariant corresponding to $\bar{\lambda}$ may easily be written down from the table. The function $G(\psi)$ in $\bar{\tau}$ is still disposable, and in this case is best fixed by ensuring that $\bar{\tau}$ agrees with τ for $p = p_0$.

Once having determined a suitable approximation function, the generalised Riemann invariants can be used as independent variables, and the solution of (1) reduced to the solution of a linear equation of the second order as in [5]. Finally, it may be noted that *any* motion for which the relation between τ, p, ψ is of the form

$$(14) \qquad p = \frac{A(\psi)}{\rho} + B(\psi); \quad \tau = \frac{1}{\rho}; \qquad A, B \text{ arbitrary},$$

possesses generalised Riemann invariants. The relation (14) corresponds to "anisentropic" flows of the Kàrmàn-Tsien gas.

[4] With $H(\eta) = A^2\eta^{2k+1}/(2k+1)$, and $\alpha = 0$.

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UNIVERSITY OF MARYLAND