# ON HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ARBITRARY CONSTANT COEFFICIENTS

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Let $K$ be an arbitrary ordinary differential field—for our purposes it is sufficient to consider an arbitrary (algebraic) field $K$ which is converted into a differential field by setting $c'=0$ for every $c \in K$. Let $u$ be a differential indeterminate over $K$ and let $u = u_0, u_1, \cdots$ represent the successive derivatives of $u$. Further, let $c_0, \cdots, c_m$ be arbitrary constants over the field $K\langle u\rangle = K(u_0, u_1, \cdots)$, that is, $m+1$ further indeterminates with which we compute in the usual way, setting $c_i' = 0$. In addition to the ring $R = K\{u\} = K[u_0, u_1, \cdots]$, we will also be interested in the rings $R_{t+m} = K[u_0, u_1, \cdots, u_{t+m}]$. Theorems referring to some one of these rings $R_{t+m}$ may, if convenient, be regarded as belonging to ordinary, rather than differential, algebra, but we will still apply the operation of differentiation to elements of $R_{t+m}$ (not involving $u_{t+m}$). This then amounts to a convenience in writing formulas.

Let $l_0 = c_0 u_0 + \cdots + c_m u_m$. This element generates a prime differential ideal $[l_0] = (l_0, l_1, \cdots)$ in $S = K(c)\{u\}$, where $l_i = c_0 u_i + \cdots + c_m u_{i+m}$. *We are interested in having explicitly a basis for* $[l_0] \cap K\{u\}$. If $\Delta(u)$ is the determinant of coefficients of any $m+1$ of the $l_i$ regarded as linear forms in the $c_j$, then clearly $\Delta(u) \in [l_0] \cap K\{u\}$ and Theorem 2 below asserts that the $\Delta(u)$ obtained from all choices of the $l_i$ form the required basis.

Let us confine ourselves to the rings $R_{t+m}$ and $S_{t+m} = K(c)[u_0, \cdots, u_{t+m}]$. In $S_{t+m}$, let $\mathfrak{p} = (l_0, \cdots, l_t)$.

LEMMA 1. *$\mathfrak{p} = (l_0, \cdots, l_t)$ is an m-dimensional prime ideal in $S_{t+m}$.*

*Proof.* Let $G(u_0, \cdots, u_{t+m}) \in S_{t+m}$. Eliminating successively $u_{t+m}$, $u_{t+m-1}, \cdots, u_m$ mod $(l_0, \cdots, l_t)$, we may write $G(u_0, \cdots, u_{t+m}) \equiv G_1(u_0, \cdots, u_{m-1})$ mod $(l_0, \cdots, l_t)$, where $G_1 \in S_{t+m}$ is a polynomial in the indicated variables. Moreover, starting with indeterminate values $\xi_i$ for $u_i$, $i = 0, \cdots, m-1$, we can build up a zero $(\xi_0, \cdots, \xi_{t+m})$ of $\mathfrak{p}$ by defining $\xi_m$ from the condition

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Received December 7, 1953, This paper was written while the author was a Guggenheim Fellow.

*Pacific J. Math.* 5 (1955), 599-606

$l_0(\xi) = 0$, and defining $\xi_{m+i}$ successively from the condition $l_i(\xi) = 0$. Then $(\xi_0, \cdots, \xi_{t+m})$ is clearly a general point of $p$, whence $p$ is prime and $m$-dimensional.

LEMMA 2. *Let* $\mathfrak{p} \cap R_{t+m} = P$; *and let* $t \geq m - 1$. *Then* $P$ *is a* $2m$*-dimensional prime ideal in* $R_{t+m}$.

*Proof.* Consider the equations:

$$c_0 \xi_0 + \cdots + c_m \xi_m = 0$$
$$c_0 \xi_1 + \cdots + c_m \xi_{1+m} = 0$$
$$\vdots$$
$$c_0 \xi_{m-1} + \cdots + c_m \xi_{2m-1} = 0.$$

From these we are going to solve successively for the $c_i$, $i = 0, \cdots, m-1$. Since $\xi_0 \neq 0$, we can solve for $c_0$ and find $c_0 \in K(c_1, \cdots, c_m, \xi_0, \cdots, \xi_m)$. Suppose in this way, solving successively for the $c_i$, we find

$$c_0, \cdots, c_i \in K(c_{i+1}, \cdots, c_m, \xi_0, \cdots, \xi_{m+i}), \qquad i < m-1.$$

In fact, assume we have found inductively that

$$(A_i) \qquad c_0, \cdots, c_i \in K(\xi_0, \cdots, \xi_{2i+1}) \cdot c_{i+1}$$
$$+ K(\xi_0, \cdots, \xi_{2i+2}) \cdot c_{i+2} + \cdots + K(\xi_0, \cdots, \xi_{i+m}) \cdot c_m.$$

Since

$$\text{dt } K(c_0, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K(c_0, \cdots, c_m) = m \text{ and}$$

$$\text{dt } K(c_0, \cdots, c_m)/K = m+1,$$

we have

$$\text{dt } K(c_0, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K = 2m+1$$
$$= \text{dt } K(c_{i+1}, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K,$$

where dt stands for "degree of transcendency". From this we see that $\xi_0, \cdots, \xi_{m+i}$ are algebraically independent over $K$ (since the set $c_{i+1}, \cdots, \xi_{m+i}$ has

$2m+1$ members), in particular they are not zero. The coefficient of $c_{i+1}$ in $l_{i+1}(\xi)$ is $\xi_{2(i+1)}$ plus a term in $K(\xi_0,\cdots,\xi_{2i+1})$ arising from $c_0\,\xi_{i+1}+\cdots+c_i\,\xi_{2i+1}$, and since $i+1<m$, we have $2(i+1)<m+i+1$ and $\xi_{2(i+1)}\notin K(\xi_0,\cdots,\xi_{2i+1})$. Hence $c_{i+1}\in K(c_{i+2},\cdots,\xi_{m+i+1})$; also $A_{i+1}$ holds. Continuing, we have $c_0,\cdots,c_{m-1}\in K(c_m,\xi_0,\cdots,\xi_{2m-1})$. Hence $\xi_0,\cdots,\xi_{2m-1}$ are algebraically independent over $K$. Thus $P$ is at least $2m$-dimensional.

Let $\Delta_i(\xi)$, $i\geq m$, be the determinant of the coefficients of the forms $l_0(\xi),\cdots,l_{m-1}(\xi)$, $l_i(\xi)$ regarded as linear forms in $c_0,\cdots,c_m$; that is,

$$\Delta_i(\xi)=\begin{vmatrix} \xi_0\cdots\xi_m \\ \xi_1\cdots\xi_{1+m} \\ \cdots \\ \xi_{m-1}\cdots\xi_{2m-1} \\ \xi_i\cdots\xi_{i+m}\end{vmatrix}$$

Then one finds $c_j\,\Delta_i(\xi)=0$, so that $\Delta_i(\xi)=0$. The coefficient of $\xi_{i+m}$ in this equation is a polynomial in the indeterminates $\xi_0,\cdots,\xi_{2m-1}$; this coefficient contains the term $\xi_0\,\xi_2\cdots\xi_{2m-2}$ and hence is not zero (therefore also $l_0(\xi),\cdots,l_{m-1}(\xi)$ are linearly independent over $K(\xi)$). Thus $P$ is at most $2m$-dimensional, and hence exactly $2m$-dimensional, Q.E.D.

LEMMA 3. *Let $M=M(u)$ be the matrix:*

$$\left\|\begin{matrix} u_0\cdots u_m \\ u_1\cdots u_{1+m} \\ \cdots \\ u_m\cdots u_{2m} \\ \cdots \\ u_t\cdots u_{t+m}\end{matrix}\right\|,\ t\geq m.$$

*Let $A$ be the ideal generated in $R_{t+m}$ by the $(m+1)\times(m+1)$ subdeterminants of $M(u)$. Then $A\subseteq P$.*

*Proof.* Since $l_0(\xi),\cdots,l_{m-1}(\xi)$ are linearly independent over $K(\xi)$ (and in fact over any field containing $K(\xi)$) but $l_0(\xi),\cdots,l_{m-1}(\xi)$, $l_i(\xi)$ are linearly dependent over $K(\xi)$, the matrix $M(\xi)$ has rank $m$. Hence $A\subseteq P$.

We want to prove $A=P$, in particular that $A$ is prime. Conversely, if we knew that $A$ were prime, we could conclude immediately that $A = P$. In fact, suppose $A$ is prime and let $\eta_0, \cdots, \eta_{t+m}$ be a general point of $A$. Since $A$ has a basis of forms of degree $m+1$, no form of degree $m$ vanishes at $\eta$. Hence all $m \times m$ subdeterminants of $M(\eta)$ differ from zero, and it follows that $A$ is $2m$-dimensional, whence $A = P$.

In proving $A = P$, we proceed by induction on $m$, the assertion being clearly true for $m = 0$. For given $m$, we proceed by induction on $t (t \geq m)$. For $t = m$, we have to prove the following lemma.

LEMMA 4. *Let $D$ be the determinant*

$$\begin{vmatrix} u_0 \cdots u_m \\ u_1 \cdots u_{1+m} \\ \cdot \quad \cdot \quad \cdot \\ u_m \cdots u_{2m} \end{vmatrix} .$$

Then $D$ is different from zero and is irreducible in $R_{2m}$.

*Proof.* By induction on $m$, being trivial for $m = 0$. $D$ is linear in $u_0$, the coefficient $\delta$ of $u_0$ being different from zero and irreducible by induction: in particular, therefore, $D \neq 0$. Also $D$ is linear in $u_{2m}$ and the coefficient $\delta'$ of $u_{2m}$ is irreducible. $D$ is reducible if and only if $\delta$ is a factor of $D - u_0\delta$, hence of $D$. Similarly for $\delta'$. Now $\delta$ and $\delta'$ are not associates, since they are of different degree in $u_0$. So $D$ is reducible if and only if it is divisible by $\delta\delta'$. For $m = 1$, this means if and only if $u_0 u_2 - u_1^2$ is divisible by $u_0 u_2$. This is not the case. For $m > 1$, $D$ is reducible only if it is of degree at least $2m$, whereas it is of degree $m + 1$. Hence for every $m$, $D$ is irreducible.

DEFINITION. An ideal is called homogeneous if it has a basis of forms. Similarly we call an ideal *isobaric* if it has a basis of isobaric polynomials.

LEMMA 5. *$A$ and $P$ are homogeneous and isobaric.*

*Proof.* $A$ is clearly homogeneous. Moreover consider one of the $(m+1) \times (m+1)$ subdeterminants of $M(u)$, say one involving the $i$th and $j$th rows, $i < j$. Then $u_{i+k-2}$ is the element in the $i$th row and $k$th-column and $u_{j+l-2}$ is the element in the $j$th row and $l$th column. Suppose $k > l$. The determinant in question has together with a term $\pi \cdot u_{i+k-2} u_{j+l-2}$ also a term $\pm\pi \cdot u_{i+l-2} \cdot u_{j+k-2}$, which is of the same weight. Hence if rows $i_0, \cdots, i_m$ are involved, each term has the weight of the term $u_{i_0} u_{i_1+1} u_{i_2+2} \cdots u_{i_m+m}$, that is, the determinant is isobaric. Thus $A$ is isobaric. As for $P$, we know that $\mathfrak{p}$ is homogeneous, and from this and the fact that $P = \mathfrak{p} \cap R_{t+m}$ one concludes immediately that $P$ also is homogeneous. To see that $P$ is isobaric, let $g(u) \in P$ and write $g(u) = g_r(u) + g_{r+1}(u) + \cdots$, where $g_j(u)$ is zero or isobaric of weight $j$. It is clearly sufficient to prove $g_r(u) \in P$, assuming $g_r \neq 0$. Since $g(u) \in P$, we have

$$h(c)g(u) = \sum A_i(c,u)\, l_i(c,u),$$

where $h(c)$ is a polynomial in the $c_i$ alone, and the $A_i$ are polynomials in the $c_i$ and $u_j$. We assign to $c_i$ the weight $m-i$. Let $h(c) = h_s(c) + h_{s+1}(c) + \cdots$, where $h_j(c)$ is zero or isobaric of weight $j$ and $h_s(c) \neq 0$. Observe that the $l_i(c,u)$ are isobaric. Comparing terms of like weight on both sides of the above equation we see that $h_s(c)g_r(u) = \sum A_i'(c,u)\, l_i(c,u)$. Hence $g_r(u) \in \mathfrak{p}$.

THEOREM 1. *$A = P$. In particular, therefore, for $m > 0$, $A : u_0 = A$.*

*Proof.* We proceed by induction on $m$ and $t$, and first show that $A : u_0 = A$. Let $\xi_0, \cdots, \xi_{t+m}$ be the general zero of $P$ introduced above. Let $D(u)$ be the determinant occurring in Lemma 4. From $D(\xi) = 0$ we see that $\xi_{2m}$ can be written as a quotient of two polynomials in the indeterminates $\xi_0, \cdots, \xi_{2m-1}$, with the denominator being

$$\begin{vmatrix} \xi_0 & \cdots & \xi_{m-1} \\ \cdot & \cdot & \cdot \\ \xi_{m-1} & \cdots & \xi_{2m-2} \end{vmatrix}$$

which is irreducible by Lemma 4. Hence we see that

$$\begin{vmatrix} \xi_2 & \cdots & \xi_{m+1} \\ \cdot & \cdot & \cdot \\ \xi_{m+1} & \cdots & \xi_{2m} \end{vmatrix} \neq 0,$$

(for were it zero, then $\xi_{2m}$ could be written as a quotient of two irreducible polynomials in $\xi_1, \cdots, \xi_{2m-1}$, the denominator this time not being an associate of the other denominator). Hence $\xi_0$ is algebraic over $K(\xi_1, \cdots, \xi_{t+m})$. Hence $\xi_1, \cdots, \xi_{t+m}$ defines a $2m$-dimensional prime ideal $P_1$ in $K[u_1, \cdots, u_{t+m}]$; and $P_1$ is generated by the $(m+1) \times (m+1)$ subdeterminants of $M(u)$ which do not involve the first row of $M(u)$. Designating also by $P_1$, the extension of $P_1$ to $K[u_0, \cdots, u_{t+m}]$, we see that $P_1 \subseteq A$. Let now $u_0 g(u) \in A$. We write

$u_0 g(u) = \sum A_i(u)\Delta_i(u)$, where the $\Delta_i(u)$ are the $(m+1)\times(m+1)$ subdeterminants of $M(u)$, and the $A_i$ are polynomials. We write $A_i = A_i' + u_0 A_i''$, where $A_i'$ does not involve $u_0$. We then have $u_0(g(u) - \sum A_i''\Delta_i(u)) = \sum A_i'\Delta_i(u)$. The right hand side here is of degree at most one in $u_0$, hence $g_1 = g(u) - \sum A_i''\Delta_i(u)$ does not involve $u_0$: $g_1 = g_1(u_1, \cdots, u_{t+m})$. Now $g(u)$ and $\Delta_i(u)$ vanish at $\xi_0, \cdots, \xi_{m+t}$, hence so does $g_1$; that is, $g_1$ vanishes at $\xi_1, \cdots, \xi_{m+t}$. Hence, $g_1 \in P_1$, whence $g \in A$. Hence $A: u_0 = A$.

As a corollary to the above we get that $A : f = A$ for any polynomial $f \in R_{m+t}$ containing a term $du_0^r$, $d \in K$, $d \neq 0$ $(m > 0)$. For suppose $fg \in A$: to prove $g \in A$. We may suppose $f$ and $g$ isobaric; and also homogeneous. We then get $du_0^r\, g \in A$, whence $g \in A$.

We proceed to prove that $A$ is prime. Let $\overline{l}_i = l_i/u_0 = c_0 v_i + \cdots + c_m v_{i+m}$, where $v_i = u_i/u_0$. We pass to the rings $\overline{R}_{t+m} = K[v_1, \cdots, v_{t+m}]$ and $\overline{S}_{t+m} = K(c)[v]$. Observe that $v_1, \cdots, v_{t+m}$ are algebraically independent over $K$. Let $\overline{M}$ be the matrix of the coefficients of the $\overline{l}_i$, that is, the matrix:

$$\left\|\begin{matrix} 1 & v_1 & v_2 & \cdots & v_m \\ v_1 & v_2 & v_3 & \cdots & v_{1+m} \\ \cdot & & \cdot & & \cdot \\ v_t & v_{t+1} & v_{t+2} & \cdots & v_{t+m} \end{matrix}\right\|,$$

and let $A$ be the ideal generated in $R_{t+m}$ by the $(m+1)\times(m+1)$ subdeterminants of $M(v)$. Each such subdeterminant is a power of $u_0$ times an $(m+1)\times(m+1)$ subdeterminant of $M(u)$; and vice-versa. It would therefore be sufficient to prove $\overline{A}$ prime, in fact it would be sufficient to prove that the extension of $A$ to the quotient ring $Q$ of $\overline{R}_{t+m}$ relative to the ideal $(v_1, \cdots, v_{t+m})$ is prime. For suppose this proved and $g(u)\,h(u) \in A$, where we assume without loss of generality that $g(u)$, $h(u)$ are homogeneous. Dividing by appropriate powers of $u_0$ and setting

$$g(u)/u_0^r = \overline{g}(v), \quad h(u)/u_0^s = \overline{h}(v),$$

we get $\overline{g}(v)\overline{h}(v) \in \overline{A}$, whence by assumption $\overline{f}(v)\overline{g}(v)$ or $\overline{f}(v)\overline{h}(v)$, say $\overline{f}\,\overline{g}$ is in $\overline{A}$ for some $\overline{f}(v) \in \overline{R}_{t+m}$, $\overline{f} \notin (v_1, \cdots, v_m)$. Multiplying by a power of $u_0$ we find $u_0^\rho f(u) g(u) \in A$, where $f(u)$ contains a term $d u_0^\sigma$. Hence $g(u) \in A$.

The ideal $\overline{A}$ in $\overline{R}_{t+m}$ has $\xi_1/\xi_0, \cdots, \xi_{t+m}/\xi_0$ as a zero, hence is at least $(2m-1)$-dimensional. Also $\overline{A}$ remains at least $(2m-1)$-dimensional upon extension to $Q$. In fact, if $\xi_1/\xi_0, \cdots, \xi_{t+m}/\xi_0$ determines $\overline{P}$ in $\overline{R}_{t+m}$, then

$\overline{P} \subseteq (v_1, \cdots, v_{t+m})$, as one sees from the fact that $\xi_0, \cdots, \xi_{t+m}$ determines a homogeneous and isobaric ideal $P$ and $u_0 \notin P$.

Subtracting $v_i$ times the first row from the $(i+1)$th row of $\overline{M}$, we get the matrix

$$\begin{Vmatrix} 1 & v_1 & v_2 & \cdots v_m \\ 0 & v_2 - v_1 v_1 & v_3 - v_1 v_2 & \cdots v_{m+1} - v_1 v_m \\ \cdot & \cdot & \cdot & \cdot \quad\quad \cdot \\ 0 & v_{t+1} - v_t v_1 & v_{t+2} - v_t v_2 & \cdots v_{t+m} - v_t v_m \end{Vmatrix}.$$

Each $(m+1) \times (m+1)$ subdeterminant of this matrix is also an $(m+1) \times (m+1)$ subdeterminant of $M$. Hence one sees that every $m \times m$ subdeterminant of the matrix

$$\begin{Vmatrix} v_2 & v_3 & \cdots v_{1+m} \\ \cdot & \cdot & \cdot \quad \cdot \\ v_{t+1} & v_{t+2} & \cdots v_{t+m} \end{Vmatrix}$$

is a leading-form of an element in $Q \cdot \overline{A}$. These $m \times m$ subdeterminants generate, by induction, a $2(m-1)$-dimensional prime ideal in $K[v_2, \cdots, v_{t+m}]$, and hence a $(2m-1)$-dimensional prime ideal $\overline{q}$ in $K[v_1, \cdots, v_{t+m}]$. The leading form ideal of $\overline{A}$ contains or equals $\overline{q}$. If it contained $\overline{q}$ properly, it would be of dimension less than $2m-1$. But an ideal and its leading form ideal have the same dimension [**1**; Satz 8]. Hence $\overline{q}$ is the leading-form ideal of $\overline{A}$ and $\overline{A}$ is $(2m-1)$-dimensional.

Moreover $A$ is prime. For quite generally in a local ring, if an ideal $\overline{A}$ has a prime ideal $\overline{q}$ as leading form ideal, it must itself be prime. In fact, suppose $gh \in \overline{A}$, $g \notin \overline{A}$, $h \notin \overline{A}$. Then the leading form ideal $LFI(\overline{A}, g)$ of $(\overline{A}, g)$ contains $\overline{q}$ properly, and likewise for $(\overline{A}, h)$. But $LFI(\overline{A}, g) \times LFI(\overline{A}, h) \subseteq LFI((\overline{A}, g) \times (\overline{A}, h)) \subseteq LFI\overline{A} = \overline{q}$, a contradiction. Hence $\overline{A}$ is prime, and the proof is complete.

The following theorem is an immediate consequence of Theorem 1.

THEOREM 2. *A basis for $[l_0] \cap K\{u\}$ is given by the $(m+1) \times (m+1)$ subdeterminants of the $\infty \times (m+1)$ matrix*

$$\begin{Vmatrix} u_0 & u_1 & \cdots u_m \\ u_1 & u_2 & \cdots u_{1+m} \\ \cdot & \cdot & \cdot \end{Vmatrix}.$$

## Reference

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