ON THE CHANGE OF INDEX FOR SUMMABLE SERIES

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1. Introduction. Assume we have given a series

$$(1.1)\qquad a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

and consider

$$(1.2)\qquad b_0 + b_1 + b_2 + \cdots + b_n + \cdots \text{ with } b_0 = 0 \text{ and } b_n = a_{n-1} \quad (n \geq 1);$$

denote the partial sums by s_n and t_n, respectively. Since $s_n = t_{n+1}$, the convergence of (1.1) is equivalent to that of (1.2). However, if a method of summability V is applied to both series, the statements

$$(1.3)\qquad (\text{a})\ V - \sum a_n = s \qquad\qquad (\text{b})\ V - \sum b_n = s$$ [1]

need not be equivalent (for example, if V is the Borel method; see [**4**, p. 183]). If $V(x; s_\nu)$ and $V(x; t_\nu)$ denote the V-transforms of the sequences $\{s_n\}$ and $\{t_n\}$, respectively, it is therefore interesting to investigate, for which methods V and under what restrictions on $\{a_n\}$ the relations

$$(1.4)\qquad (\text{a})\ V(x; s_\nu) \simeq K \cdot x^q \qquad\qquad (\text{b})\ V(x; t_\nu) \simeq K \cdot x^q$$

$$(x \longrightarrow x_0,\ K \text{ constant};\ q \geq 0, \text{ fixed})$$ [2]

are equivalent.

The cases $V = C_k$ (Cesàro) and $V = A$ (Abel) are quickly disposed of (§ 2), while $V = E$ (general Euler transform) and $V = B$ (Borel) present some interest (§§ 3-5).

2. THEOREM 1. *The statements* (1.4.a) and (1.4.b) *are equivalent for*

[1] We shall always let $\sum_{n=0}^{\infty} a_n = \sum a_n$.

[2] $x \longrightarrow x_0$ through values depending on the method V.

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$V = C_k (k > -1)$ and $V = A$. [3]

Proof. If

$$S_n^{(k)} = C_k(n; s_\nu) \cdot \binom{n+k}{n}$$

and

$$T_n^{(k)} = C_k(n; t_\nu) \cdot \binom{n+k}{n},$$

we have by definition of the Cesàro means

$$(2.1) \qquad (1-x)^{k+1} \sum T_n^{(k)} x^n = \sum b_n x^n = x \cdot \sum a_n x^n = x(1-x)^{k+1} \sum S_n^{(k)} x^n,$$

the series being convergent for $|x| < 1$. The proof of Theorem 1 now follows from the inner equality in (2.1) and the relation

$$\frac{T_n^{(k)}}{\binom{n+k}{n}} = \frac{S_{n-1}^{(k)}}{\binom{n+k}{n}} \cong \frac{S_{n-1}^{(k)}}{\binom{n-1+k}{n-1}} \qquad (n \longrightarrow \infty).$$

3. Let $g(w) = \sum \gamma_n w^n$ be regular and schlicht in $|w| \leq 1$, and assume $g(0) = 0$, $g(1) = 1$. Then the E-transforms of $\sum a_n$ and $\sum b_n$ are obtained by the formal relations [5]

$$(3.1) \qquad \begin{aligned} \sum a_n z^n &= \sum a_n [g(w)]^n = \sum \alpha_n w^n; \quad E(n; s_\nu) = \sum_{\nu=0}^{n} \alpha_\nu \\ \sum b_n z^n &= \sum b_n [g(w)]^n = \sum \beta_n w^n; \quad E(n; t_\nu) = \sum_{\nu=0}^{n} \beta_\nu \end{aligned} \qquad (n = 0, 1, \cdots).$$

THEOREM 2. *The statements* (1.4.a) *and* (1.4.b) *are equivalent for* $V = E$.

Proof. First we note that if either

$$E(n; s_\nu) = O(n^q) \text{ or } E(n; t_\nu) = O(n^q) \qquad (n \longrightarrow \infty),$$

[3] For $q = 0$ see [**4**, p. 102].

then the formal relations (3.1) are actually valid for $|w| < 1$ and also

$$(3.2) \qquad \sum \beta_n w^n = \sum b_n [g(w)]^n = g(w) \cdot \sum a_n [g(w)]^n = g(w) \cdot \sum \alpha_n w^n \qquad (|w| < 1).$$

Denote by A_n, B_n, C_n the partial sums of $\sum \alpha_n$, $\sum \beta_n$, $\sum \gamma_n$, respectively. We assume first

$$E(n; s_\nu) = A_n \cong K \cdot n^q \qquad (n \longrightarrow \infty).$$

Then, since by (3.2) $\sum \beta_n$ is the Cauchy product of $\sum \alpha_n$ and $\sum \gamma_n$, we have

$$E(n; t_\nu) = B_n = \gamma_n A_0 + \gamma_{n-1} A_1 + \cdots + \gamma_1 A_{n-1}$$

and for $n \geq 1$

$$(3.3) \qquad \frac{B_n}{n^q} = \frac{\gamma_n}{n^q} A_0 + \gamma_{n-1} \frac{1^q}{n^q} \cdot \frac{A_1}{1^q} + \cdots + \gamma_1 \frac{(n-1)^q}{n^q} \cdot \frac{A_{n-1}}{(n-1)^q}.$$

For the matrix $c_{n\nu}$ in this transformation of the convergent sequence $\{A_n n^{-q}\}$ we have clearly

$$\lim_{n \to \infty} c_{n\nu} = 0 \qquad (\nu = 0, 1, \ldots).$$

Furthermore

$$\sum_\nu |c_{n\nu}| = \sum_{\nu=1}^{n-1} |\gamma_{n-\nu}| \cdot \frac{\nu^q}{n^q} + \frac{|\gamma_n|}{n^q} \leq \sum_{\nu=1}^{n} |\gamma_\nu| \leq \sum_{\nu=1}^{\infty} |\gamma_\nu| = M < \infty;$$

finally we prove

$$\lim_{n \to \infty} \sum_{\nu=0}^{n-1} c_{n\nu} = 1.$$

For $q = 0$ this follows from

$$\sum_{\nu=0}^{n-1} c_{n\nu} = \sum_{\nu=1}^{n} \gamma_\nu \longrightarrow g(1) = 1 \qquad (n \longrightarrow \infty);$$

for $q > 0$

$$\sum_{\nu=0}^{n-1} c_{n\nu} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_{n-\nu} \cdot \frac{\nu^q}{n^q} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_\nu \left(\frac{n-\nu}{n}\right)^q$$

$$= \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} C_\nu \left[\left(\frac{n-\nu}{n}\right)^q - \left(\frac{n-\nu-1}{n}\right)^q\right],$$

and the last term is a positive regular transformation of the sequence $\{C_n\}$ tending to $g(1) = 1$, whence

$$\sum_\nu c_{n\nu} \longrightarrow 1 \qquad (n \longrightarrow \infty).$$

Therefore the transformation (3.3) of $\{A_n n^{-q}\}$ converges to K, which proves $B_n \cong K \cdot n^q$ $(n \longrightarrow \infty)$.

Assume on the other hand $B_n \cong Kn^q$ $(n \longrightarrow \infty)$. Putting $w = 0$ in (3.2), one obtains $\beta_0 = 0$, so that

$$\sum \alpha_n w^n = [g(w)]^{-1} \sum \beta_n w^n = w[g(w)]^{-1} \sum \beta_{n+1} w^n$$

is regular in $|w| < 1$. Furthermore the expansion of the function $w[g(w)]^{-1}$ for $w = 1$ converges absolutely to 1, since $w = 0$ is the only zero of $g(w)$ in $|w| \leq 1$. An argument similar to the one above shows then that $B_{n+1} \cong Kn^q$ $(n \longrightarrow \infty)$ implies $A_n \cong Kn^q$ $(n \longrightarrow \infty)$, which completes the proof of Theorem 2.

We add a few remarks about the assumptions on the function $z = g(w)$ by which the E-method is defined.

a. Theorem 2 becomes false if only regularity of $g(w)$ in $|w| < 1$, and continuity and schlichtness in $|w| \leq 1$ are assumed. For there exist such functions $g(w)$ whose power series do not converge absolutely on $|w| = 1$ (cf. [2]). Therefore in (3.2) one could find a convergent $\sum \alpha_n$ whose transform $\sum \beta_n$ diverges.

b. All that was used about the function $g(w)$ in the proof of Theorem 2 was that the power series of $g(w)$ and of $w[g(w)]^{-1}$ converge absolutely to the value 1 for $w = 1$. This can be guaranteed by the weaker *assumption that $g(w)$ with $g(1) = 1$ and $g(0) = 0$ is regular in $|w| < 1$, continuous and schlicht* *in* $|w| \leq 1$, *and that the image of* $|w| = 1$ *under the mapping* $g(w)$ *is a rectifiable Jordan curve.* Because then

$$\int_0^{2\pi} |g'(e^{i\phi})|\, d\phi < \infty$$

and hence $\sum |\gamma_n| < \infty$ [**8**, p. 158]; on the other hand also

$$\int_0^{2\pi} |G'(e^{i\phi})|\, d\phi < \infty,$$

where

$$G'(w) = \left[\frac{w}{g(w)}\right]' = \frac{g(w) - wg'(w)}{[g(w)]^2},$$

so that also the power series of $G(w)$ converges absolutely to the value 1 for $w = 1$.

c. If

$$g(w) = w[(p+1) - pw]^{-1} \qquad (p \geq 0, \text{ fixed})$$

one has $E = E_p$ as the familiar Euler method of order p, for which Theorem 2 is known in the case $q = 0$ [**4**, p. 180].

d. The function

$$g(w) = (2 - w) - 2(1 - w)^{1/2} \qquad (g(0) = 0)$$

leads to the method of Mersman [**6**], as Scott and Wall showed [**7**, p. 270]. Here Theorem 2 is also applicable, since the more general conditions about $g(w)$ in remark (b) are satisfied, as is readily seen.

4. The Borel method is defined by the transformation

$$B(x; s_\nu) = e^{-x} \sum \frac{s_\nu x^\nu}{\nu!} \qquad (x \geq 0),$$

where the power series is assumed to define an entire function. It is known that $B(x; s_\nu) \longrightarrow K$ $(x \longrightarrow \infty)$ implies $B(x; t_\nu) \longrightarrow K$ $(x \longrightarrow \infty)$, but not conversely [**4**, p. 183]. We now prove more generally

THEOREM 3. *The relation*

$$B(x; s_\nu) \cong Kx^q \qquad (x \longrightarrow \infty)$$

implies

$$B(x; t_\nu) \cong Kx^q \qquad (x \longrightarrow \infty).$$

Proof. We have for $x > 0$ [**4**, p. 196]

$$(4.1) \qquad x^{-q} B(x; t_\nu) = x^{-q} e^{-x} \sum \frac{t_\nu x^\nu}{\nu!} = x^{-q} e^{-x} \sum \frac{s_\nu x^{\nu+1}}{(\nu+1)!}$$

$$= x^{-q} e^{-x} \int_0^x \sum \frac{s_\nu t^\nu}{\nu!}\, dt = x^{-q} \int_0^x e^{-(x-t)} t^q \frac{B(t; s_\nu)}{t^q}\, dt.$$

This transformation of the convergent function $B(t; s_\nu) t^{-q}$ $(t \longrightarrow \infty)$ by means of the 'matrix'

$$c(x, t) = e^{-(x-t)} \left(\frac{t}{x}\right)^q \qquad (0 \le t \le x)$$

is regular, since

$$\int_{t_1}^{t_2} |c(x, t)|\, dt \longrightarrow 0 \qquad (x \longrightarrow \infty;\ t_1, t_2 > 0, \text{ fixed})$$

and

$$\int_0^x |c(x, t)|\, dt = \int_0^x c(x, t)\, dt = e^{-x} \int_0^x e^t \left(\frac{t}{x}\right)^q dt \longrightarrow 1 \qquad (x \longrightarrow \infty).$$

Therefore $B(x; t_\nu) \cong Kx^q$ $(x \longrightarrow \infty)$.

We discuss now the converse of Theorem 3.

THEOREM 4. *The relation*

$$B(x; t_\nu) \cong Kx^q \qquad (x \longrightarrow \infty)$$

implies

$$B(x; s_\nu) \cong Kx^q \qquad (x \longrightarrow \infty),$$

if

$$(4.2) \qquad \limsup |a_n|^{1/n} < \infty,$$

that is, if the series $\sum a_n z^n$ *has a positive radius of convergence.*

Proof. Using (4.1) we have for $x > 0$

$$F(x) = x^{-q} B(x; t_\nu) = x^{-q} e^{-x} \int_0^x e^t B(t; s_\nu)\, dt.$$

Consider now $F(x)$ as function of the complex variable x for $\Re(x) \geq 1$. Then (4.2) implies $|t_n| \leq M^n$ for some constant $M > 0$ and hence in $\Re(x) \geq 1$

$$|B(x; t_\nu)| \leq e^{-1} \sum \frac{M^n |x|^n}{n!} = e^{-1+M|x|},$$

and also

$$(4.3) \qquad |F(x)| \leq \alpha e^{\beta|x|} \qquad \Re(x) \geq 1$$

for positive constants α and β. Hence one knows that

$$F(x) \longrightarrow K \qquad (x \longrightarrow +\infty)$$

implies

$$F'(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),\ ^4$$

that is,

$$x^{-q} B(x; s_\nu) + \int_0^x e^t B(t; s_\nu)\, dt \left[-1 - \frac{q}{x}\right] e^{-x} x^{-q}$$

$$= x^{-q} B(x; s_\nu) - K + o(1) = o(1) \qquad (x \longrightarrow +\infty),$$

from which the result follows.

5. We now show that Theorem 4 is best possible in a certain sense.

[4] If $F(x)$ is regular in $\Re(x) \geq 1$ and (4.3) holds, then $F(x) \longrightarrow A$ $(x \longrightarrow +\infty)$ implies $F'(x) \longrightarrow 0$ $(x \longrightarrow +\infty)$. This lemma was used also in [3], where Theorem 4 was proved for $q = 0$.

THEOREM 5. *In Theorem 4 the Condition (4.2) cannot be replaced by*

$$\limsup n^{-\epsilon} |a_n|^{1/n} < \infty \qquad (\epsilon > 0). \tag{5.1}$$

For the proof we need the following

LEMMA. *For every $\beta > 1$, there exists an entire function $f(z)$ of order β satisfying*

$$f(x) \longrightarrow 0 \ \ (x \longrightarrow +\infty),\ f'(x) \not\longrightarrow 0 \ \ (x \longrightarrow +\infty) \qquad (z = x + iy). \tag{5.2}$$

Proof. Put $\alpha = \beta^{-1}$ and consider the Mittag-Leffler function

$$E_\alpha(z) = \sum \frac{z^n}{\Gamma(1+\alpha n)},$$

which is an entire function of order $\alpha^{-1} = \beta$. Let m be the integer with

$$\frac{\alpha}{1-\alpha} \le m < \frac{\alpha}{1-\alpha} + 1.$$

We first study the derivatives of $E_\alpha(z)$ of order $1, 2, \cdots, m$ on the line $\arg z = \alpha\pi/2$ for large $|z|$. For these z (assume for definiteness $|z| > 2$) one has [1, pp. 272-275]

$$E_\alpha(z) = \frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{dt}{t-z} + \frac{1}{\alpha} e^{z^{1/\alpha}}, \tag{5.3}$$

the path L being

$$t = re^{-i\phi_0} \left(\infty > r \ge 1,\ \alpha\pi > \phi_0 > \frac{\pi\alpha}{2}\right),\ t = e^{i\phi}(-\phi_0 \le \phi \le +\phi_0),$$

$$t = re^{i\phi_0} \ \ (1 \le r < \infty);$$

$t^{1/\alpha}$ is the branch which is positive for $t > 0$. The kth derivative of the integral part in (5.3) can then be estimated as follows

$$\left| \frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{k!}{(t-z)^{k+1}} dt \right| \le \frac{k!}{2\pi\alpha |z|^{k+1}} \int_L |e^{t^{1/\alpha}}| \frac{|dt|}{|1-(t/z)|^{k+1}}$$

$$= O(|z|^{-k-1}) = o(1) \quad (|z| \longrightarrow \infty),$$

since for our values of z one has $|1-(t/z)| \geq \delta > 0$ and on the straight line segments of L

$$|e^{t^{1/\alpha}}| = e^{|t|^{1/\alpha} \cdot \cos \phi_0/\alpha} \quad \text{with} \quad \cos \frac{\phi_0}{\alpha} < 0.$$

Therefore

$$E_\alpha'(z) = o(1) + \frac{1}{\alpha^2} e^{z^{1/\alpha}} z^{1/\alpha - 1}$$

$$E_\alpha''(z) = o(1) + \frac{1}{\alpha^3} e^{z^{1/\alpha}} z^{(1/\alpha - 1)2}$$

(5.4)

$$E_\alpha^{(m-1)}(z) = o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{(1/\alpha - 1)(m-1)}$$

$$E_\alpha^{(m)}(z) = o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{(1/\alpha - 1)m}.$$

Now we consider the function

$$F(z) = \frac{1}{z} [E_\alpha^{(m-1)}(z) - E_\alpha^{(m-1)}(0)],$$

which is again an entire function of order α^{-1}. For $|z| \longrightarrow \infty$ on $\arg z = \alpha\pi/2$ we have by (5.4)

$$F(z) = o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{(1/\alpha - 1)(m-1) - 1} = o(1);$$

however

$$F'(z) = o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{(1/\alpha - 1)m - 1},$$

and herein $|e^{z^{1/\alpha}}| = 1$ and $((1/\alpha) - 1)m - 1 \geq 0$, so that $F'(z) \not\rightarrow 0$ ($|z| \longrightarrow \infty$ on $\arg z = \alpha\pi/2$). For the lemma it is therefore sufficient to take

$$f(z) = F(ze^{i\,\alpha\pi/2}).$$

Proof of Theorem 5. Define the $\{a_n\}$ of (1.1) by

$$f(x) = \int_0^x e^{-t} \sum \frac{a_\nu t^\nu}{\nu!}\, dt = \int_0^x e^{-t} a(t)\, dt,$$

with the $f(x)$ of the above lemma and $\beta = (1-\epsilon)^{-1}$. Since $f(x)$ is of order $\beta > 1$, so is $a(t)$, and therefore [**1**, p. 238][5]

$$\limsup n^{1/\beta} \left| \frac{a_n}{n!} \right|^{1/n} = e \limsup n^{-\epsilon} |a_n|^{1/n} < \infty,$$

that is, (5.1) is fulfilled. Furthermore

$$f(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

which is equivalent to

$$B(x; t_\nu) \longrightarrow 0 \qquad (x \longrightarrow +\infty).$$

However, in order that

$$B(x; s_\nu) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

it would be necessary and sufficient to have [**4**, pp. 182-183]

$$e^{-x} a(x) = f'(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

which by our lemma is not fulfilled. So we have given an example of a series $\sum a_n$ for which $B(x;t_\nu) \longrightarrow 0$ $(x \longrightarrow +\infty)$ does not imply $B(x;s_\nu) \longrightarrow 0$ $(x \longrightarrow +\infty)$ and for which (5.1) holds.

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[5] Prof. Lösch (Stuttgart) suggested to me the relation to the coefficient problem for entire functions.

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