QUOTIENT ALGEBRA OF A FINITE AW*-ALGEBRA

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1. Introduction. In a recent paper [5] Wright proves that *if A is an AW^*-algebra* [2] *having a trace and if M is a maximal ideal of A, then A/M is an AW^*-factor* (that is, *an AW^*-algebra whose center consists of complex numbers*) *having a trace.* The trace enters into his argument in the characterization [5, Theorem 3.1] of the one-to-one correspondence between maximal ideals of A and those of its center Z. This is, in turn, used to verify that A/M satisfies the countable chain condition, namely: every set of mutually orthogonal projections is at most countable, which is crucial to prove that every set of mutually orthogonal projections has a least upper bound (LUB). It is the purpose of this paper to prove the following.

THEOREM. *Let A be a finite AW^*-algebra, and M a maximal ideal of A. Then A/M is a finite AW^*-factor.*

It is not known whether a finite AW^*-factor always has a trace. Since [3] a finite AW^*-algebra of type I always has a trace, our result adds nothing new in this case, and we shall be solely concerned with algebras of type II_1.

Our terminology is that of [2]. We assume familiarity with [2] and [1] (especially [1, pp. 234–242]).

2. Maximal ideal M. We begin with a slightly sharpened version of [5, Theorem 2.5] on p-ideals. A set P of projections is called a p-ideal if

(1) P contains $e \vee f$ whenever it contains e and f

(2) P contains f whenever it contains an $e \succ f$.

It follows from (1) that $e_1 \vee \cdots \vee e_n$ is in P if $e_1, \cdots, e_n$ are in P. For any set S of A let S_p denote the set of projections contained in S.

LEMMA 1. *Let A be an AW^*-algebra. The closed linear subspace M generated by a p-ideal P is an ideal with $M_p = P$. Conversely an ideal M of A is the closed linear subspace generated by the p-ideal M_p.*

Proof. Let P be a p-ideal and M the closed linear subspace generated by P. For M to be an ideal we need to prove that M contains xe for any $x \in A$ and $e \in P$. The left projection [2, p. 244] f of xe, being $\prec e$, is contained in P. Hence P contains $g = e \vee f$. $xe \in gAg \subset M$,

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as gAg is the closed linear subspace generated by all projections $\leqq g$.

Let M_0 denote the linear subspace algebraically generated by P; the elements of M_0 are of the form $x=\sum_{i=1}^{n}\lambda_i e_i$ (λ_i complex numbers, $e_i\in P$). As P contains $e_1\vee\cdots\vee e_n$, the left and right projections of x are in P. Take an f in M_p, and an $\varepsilon>0$. There is an $x\in M_0$ with $\|f-x\|<\varepsilon$. The left projection h of fx, being $\prec$ the right projection of x, is in P. We have $h\leqq f$ and $\|f-h\|=\|(f-h)(f-fx)\|\leqq\|f-fx\|<\varepsilon$. Hence $f=h$. This proves that $M_p=P$.

Assume now that M is an ideal. M_p is [5, Lemma 2.1] a p-ideal. Let M' denote the closed linear subspace generated by M_p. We wish to prove that $M=M'$. Take $x\in M$ and $\varepsilon>0$. There is[1] [2, Lemma 2.1] a projection e, which is a multiple of x, such that $\|x-ex\|<\varepsilon$. Since $ex\in M'$ and M' is closed, $x\in M'$.

Let now A be an AW^*-algebra of type II_1, Z its center. Then [2, p. 247] A admits a dimension function D defined on A_p with values in Z. D has the following properties:

(1) $0\leqq D(e)\leqq 1$ for every e,
(2) $D(e)=e$ if $e\in Z$,
(3) $D(e)=D(f)$ if and and only if $e\sim f$,
(4) $D(\sum e_i)=\sum D(e_i)$ if the e_i's are mutually orthogonal [1, Lemma 6.13].

Moreover, D is uniquely determined by these properties. It is an immediate consequence of (4) that given $0<\lambda<1$ there is a projection e with $D(e)=\lambda$.

Let C be a commutative AW^*-subalgebra [3] of A. C is the closed linear subspace generated by C_p. We shall extend D to a linear transformation T_C of C into Z. First define T_C on the linear combinations of projections by setting

$$T_C\left(\sum_{i=1}^{n}\lambda_i e_i\right)=\sum_{i=1}^{n}\lambda_i D(e_i)\,.$$

We must show that T_C is uniquely defined, i.e., if $x=y$ then $T_C(x)=T_C(y)$. If $x=\sum_{i=1}^{n}\lambda_i e_i$, there are orthogonal projections $f_1,\cdots,f_m$ such that each e_i is a sum of the f's:

$$e_i=\sum_{j=1}^{m}\alpha_{ij}f_j \quad\text{where}\quad \alpha_{ij}=\begin{cases}1 & \text{if}\quad e_i f_j=f_j\\ 0 & \text{if}\quad e_i f_j=0\,.\end{cases}$$

$$x=\sum_{i=1}^{n}\lambda_i e_i=\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\lambda_i\alpha_{ij}\right)f_j\,.$$

[1] To use [2, Lemma 2.1] we first imbed xx^* in a maximal commutative self-adjoint subalgebra of A. Working in this subalgebra we get a projection e with $\|xx^*-exx^*\|<\varepsilon^2$. Then $\|x-ex\|=\|(x-ex)(x-ex)^*\|^{1/2}=\|xx^*-exx^*\|^{1/2}<\varepsilon$.

It follows from $D(e_i)=\sum_{j=1}^{m}\alpha_{ij}D(f_j)$ that

$$T_C(\sum_{i=1}^{n}\lambda_i e_i)=T_C(\sum_{i,j}\lambda_i\alpha_{ij}f_j)\,.$$

Hence to prove the uniqueness of T_C we may restrict ourselves to the linear combinations of mutually orthogonal projection, $\sum_{i=1}^{n}\lambda_i e_i$. Moreover, as D is additive on orthogonal projections, we may assume that all the coefficients λ_i are unequal. Suppose therefore

$$x=\sum_{i=1}^{n}\lambda_i e_i=\sum_{j=1}^{m}\mu_i f_j\,,$$

where the e's and f's are mutually orthogonal and the λ's and μ's are all different. Then $xf_j=\mu_j f_j=(\sum_{i=1}^{n}\lambda_i e_i)f_j$. Since the λ's are all different, to each j there is exactly one i such that $e_i f_j=f_j$ and $\lambda_i=\mu_j$. By symmetry $e_i f_j=e_i$. Hence $\sum_{j=1}^{m}\mu_j f_j$ is merely a rearrangement of $\sum_{i=1}^{n}\lambda_i e_i$. This proves the uniqueness of T_C. If $x=\sum_{i=1}^{n}\lambda_i e_i$ where the e's are mutually orthogonal and $\lambda_i>0$, then

$$\begin{aligned}\|T_C(x)\|&=\|\sum_{i=1}^{n}\lambda_i D(e_i)\|\leqq\max\lambda_i\|\sum_{i=1}^{n}D(e_i)\|\\&\leqq\max\lambda_i=\|x\|\,.\end{aligned}$$

Hence T_C is a bounded linear operator defined on a dense subset of C, therefore can be extended to all of C. T_C is positive because D is. If $x\in A$ is normal, x can be imbedded in a maximal commutative self-adjoint subalgebra C' of A. Let C be the intersection of all such C', C and C' are AW^*-subalgebras of A. As x can be approximated within both C and C', $T_{C'}(x)=T_C(x)$. Let $T(x)$ denote their common value. T is unitarily invariant (i.e. $T(uxu^{-1})=T(x)$ for every unitary u), because if $\sum\lambda_i e_i$ is an approximation of x then $\sum\lambda_i ue_i u^{-1}$ is one of uxu^{-1} and D is unitarily invariant. T is also linear on each commutative AW^*-subalgebra of A. We shall use this T to play the role of trace.

THEOREM 1. *Let A be an AW^*-algebra of type II_1, Z its center. Let N be a maximal ideal of Z. Then the unique maximal ideal M of A containing N is that generated by the p-ideal P consisting of all projections e with $T(e)\in N$. Or, equivalently, M is the set of elements x with $T(x^*x)\in N$.*

Proof. Consider Z as functions on its structure space of maximal ideals. Then N contains $b\geqq 0$ whenever it contains $a\geqq b$; therefore P

satisfies (2) of a p-ideal. (1) follows from $T(e\vee f)=T(e)+T(f)-T(e\wedge f)$ because [2, Theorem 5.4] $e\vee f-e\sim f-e\wedge f$. Thus M is an ideal by Lemma 1. Moreover $M\neq A$ as $1\notin P$. Let M' be a maximal ideal containing M. Then M is maximal if and only if $M_p=M'_p$. Take an $e\in M'_p$. If $e\notin P$ then $T(e)\equiv\lambda$ (mod N) with $\lambda>0$. Choose an integer n and a projection f such that $T(f)=1/n<\lambda$. f is a simple projection with central carrier 1, that is, there exist mutually orthogonal projections $f=f_1, \cdots, f_n$ with $f_1+\cdots+f_n=1$. Compare e and f; there exists [2, Theorem 5.6] a central projection g with $ge\succ gf$ and $(1-g)e\prec(1-g)f$. Then gf and, therefore, g are in M'. As

$$0\leqq T((1-g)f)-T((1-g)e)\equiv(1/n-\lambda)(1-g)\ (\text{mod } N)$$

and $1/n-\lambda<0$, $1-g$ is also in M'. Hence $1\in M'$, contradicting the choice of M'. Hence $e\in P$ and $M=M'$ is maximal. The uniqueness follows from [5, Theorem 2.5].

Finally we assert that $x\in M$ if and only if $T(x^*x)\in N$. It is well known that $x\in M$ if and only if $x^*x\in M$. Thus we need only to prove that $0<x\in M$ if and only if $T(x)\in N$. Suppose $0<x\in M$. Given $\varepsilon>0$ there is a projection e, which is a multiple of x, such that $\|x-ex\|<\varepsilon$. $T(e)\in N$ because $e\in M$. Then $T(ex)\leqq\|x\|T(e)$ is also in N. Therefore $T(x)\in N$. Conversely, assume $T(x)\in N$, $x>0$. Imbed x in a maximal communtative subalgebra C. Given $\varepsilon>0$ there are projections $e_1, \cdots, e_n$ in C and positive real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$0\leqq x-\sum_{i=1}^{n}\lambda_i e_i<\varepsilon\,.$$

$T(e_i)\in N$ $(i=1, \cdots, n)$ for $\lambda_i T(e_i)\leqq T(x)$. Hence $e_i\in M$ $(i=1, \cdots, n)$, and $x\in M$.

3. The quotient algebra A/M.

LEMMA 2. *Let $\bar{e}_1, \bar{e}_2, \cdots$ be a countable set of mutually orthogonal projections in A/M. There exist mutually orthogonal projections $e_1, e_2, \cdots$ in A such that $\bar{e}_n=e_n+M$, $(n=1, 2, \cdots)$.*

Proof. By [5, Theorem 3.2] we can find a projection e_1 representing $\bar{e}_1$. If x is a representative of $\bar{e}_2$, so is $(1-e_1)x(1-e_1)$. Hence the proof of [5, Theorem 3.2] shows that $\bar{e}_2$ admits a projection representative e_2 orthogonal to e_1. A straight forward induction yields Lemma 2.

LEMMA 3. $e\equiv f$ (mod M) *if and only if* $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$ (mod $N=M\cap Z$). *Consequently A/M satisfies the countable chain condition.*

Proof. If $e\equiv f$ (mod M) then $0\leqq e-efe\in M$. Hence $T(e)\equiv T(efe)$

(mod N). Similarly $T(f)\equiv T(fef)$ (mod N). But [6, Corollary to Lemma 2.1] $efe=u(fef)u^*$ for some unitary u. Hence $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$ (mod N). Conversely, if $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$, then $e\equiv efe$ and $f\equiv fef$ (mod M). As $(e-fe)^*(e-fe)=e-efe\equiv 0$ (mod M) we have, $e\equiv fe$ and $e\equiv f$ (mod M). The above result permits us to define an "additive" function $\bar{D}$ on the projections of A/M by setting $\bar{D}(\bar{e})$ to be the common value of $T(e)$ at N where $e+M=\bar{e}$. $D(\bar{e})\neq 0$ if $\bar{e}\neq 0$. Hence A/M satisfies the countable chain condition.

LEMMA 4. *Any set of mutually orthogonal projections in A/M has a least upper bound.*

Proof. By Lemma 3 such a set is countable. Let $\bar{e}_1, \bar{e}_2, \cdots$ be mutually orthogonal projections. We first prove a sharpened version of Lemma 2:

(*) *there exist mutually orthogonal projections $e_1, e_2, \cdots$ representing $\bar{e}_1, \bar{e}_2, \cdots$, respectively, such that $T(e_n)=\bar{D}(\bar{e}_n)$ for $n=1, 2, \cdots$.*

Let f_1 be a projection representing $\bar{e}_1$ and g_1 a projection with $T(g_1)=\bar{D}(\bar{e}_1)$. Compare f_1 and g_1; there is a central projection h_1 such that $h_1g_1>h_1f_1$ and $(1-h_1)g_1<(1-h_1)f_1$. There are projections e_1' and e_1'' such that $h_1g_1\sim e_1'\geqq h_1f_1$ and $(1-h_1)g_1\sim e_1''\leqq(1-h_1)f_1$. From

$$0\leqq T(e_1'-h_1f_1)=T(e_1')-T(h_1f_1)=h_1(T(g_1)-T(f_1))\equiv 0 \pmod{M\cap Z}$$

it follows that $e_1'\equiv h_1f_1$ (mod M). Similarly $e_1''\equiv(1-h_1)f_1$ (mod M). Hence $e_1=e_1'+e_1''\equiv f_1$ (mod M) and $T(e_1)=\bar{D}(\bar{e}_1)$. Next let f_2, g_2 be projections $<1-e_1$ and be such that $\bar{e}_2=f_2+M$ and $T(g_2)=\bar{D}(\bar{e}_2)$. Repeat the argument applied to f_1 and g_1 we can find the desired $e_2=e_2'+e_2''$; (Since $1-e_1>h_2g_2$ and h_2g_2, e_2' can be taken inside $1-e_1$. So is e_2'', therefore $e_1e_2=0$). A simple induction yields (*).

Let $e=\mathrm{LUB}_n e_n$. We wish to prove that $\bar{e}=e+M$ is the LUB of $\bar{e}_n$. Or, equivalently, $\bar{f}\bar{e}=0$ if $\bar{f}\bar{e}_n=0$ for all n. Choose representatives f, $f_1, f_2, \cdots$ of $\bar{f}$ so that $f_ne_m=0$ for all $m\leqq n$. Consider efe. We have

$$efe\equiv ef_nfe\equiv g_nf_nfe\equiv g_nfe\equiv g_nefe \pmod{M}$$

where $g_n=e-e_1-\cdots-e_n$. Imbed efe in a maximal commutative self-adjoint subalgebra C and apply [2, Lemma 2.1] which states (in C): given $\varepsilon>0$ there exists a projection h, which is a multiple of efe, such that $\|efe-hefe\|<\varepsilon$. efe is in M if all such h's are.

$$h=efey\equiv g_nefey\equiv g_nh \pmod{M}.$$

Hence $h\equiv hg_nh$ (mod M) and $T(h)\equiv T(hg_nh)$ (mod $M\cap Z$). But $T(hg_nh)=T(g_nhg_n)\leqq T(g_n)$ and $T(g_n)$ can be made arbitrarily small when n is large enough. Hence $T(h)\equiv 0$ (mod $M\cap Z$) and $h\in M$. This completes the proof.

THEOREM 2. *A/M is a finite AW^*-factor.*

Proof. To show that A/M is an AW^*-algebra we need to verify two things: (1) every set of mutually orthogonal projections has a LUB and (2) any maximal commutative self-adjoint subalgebra is generated by its projections. (1) is the context of Lemma 4. (2) is equivalent to that every element of A/M has a left and a right projection, or the left (right) annihilator of every element is a principal left (right) ideal generated by a projection. This last can be easily verified following the argument used in [**2**, Lemmas 2.1, 2.2, and Theorem 2.3]. As A/M is simple it must be factorial. It remains to prove the finitness. This will be the case if we show that $\bar{D}(\bar{e})=\bar{D}(\bar{f})$ if $\bar{e}\sim\bar{f}$, since $\bar{D}$ is non-zero on non-zero projections. This is a consequence of the following lemma, a special case of [**4**, Proposition 2] if A is a W^*-algebra.

LEMMA 5. *Suppose $\bar{e}\sim\bar{f}$. Then there exists equivalent projections e, f representing $\bar{e}$, $\bar{f}$ respectively.*

Proof. Let $\bar{x}^*\bar{x}=\bar{e}$ and $\bar{x}\bar{x}^*=\bar{f}$. Let x, e_1 and f_1 respectively be the representative of $\bar{x}$, $\bar{e}$ and $\bar{f}$. Then

$$e_1\equiv x^*x\equiv e_1x^*xe_1\equiv e_1x^*(xx^*)xe_1\equiv e_1x^*f_1xe_1\equiv(e_1x^*f_1)(f_1xe_1)\ ,$$

and

$$f_1\equiv xx^*\equiv f_1xx^*f_1\equiv(f_1xe_1)(ex^*f_1)\ .$$

Let e be the left projection of $e_1x^*f_1$ and f the right projection of $e_1x^*f_1$. e and f are the desired projections, for

$$e=ee_1\equiv e(e_1x^*f_1)(f_1xe_1)\equiv(e_1x^*f_1)(f_1xe_1)\equiv e_1$$

and, similarly, $f\equiv f_1$.

REMARK. If an AW^*-factor always possesses a trace, then any AW^*-algebra of type II_1 will admit a trace, for $T(x+y)-T(x)-T(y)$ takes the value 0 at every maximal ideal of Z.

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