DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL n-SPACE

N. D. LANE

Introduction. This paper is a generalization to n dimensions of the classification of the differentiable points in the conformal plane [2], and in conformal 3-space [3]. In the present paper, this classification depends on the intersection and support properties of certain families of tangent $(n-1)$-spheres, and on the nature of the osculating m-spheres at such a point $(m=1, 2, \cdots, n-1)$.

The discussion is also related to the classification [4] of the differentiable points of arcs in projective $(n+1)$-space, since conformal n-space can be represented on the surface of an n-sphere in projective $(n+1)$-space.

1. Pencils of m-spheres. $p, t, P, P_1, \cdots,$ will denote points of conformal n-space and $S^{(m)}$ will denote an m-sphere. When there is no ambiguity, the superscript $(n-1)$ will be omitted in the case of $S^{(n-1)}$; thus an $(n-1)$-sphere $S^{(n-1)}$ will usually be denoted by S alone. Such an $(n-1)$-sphere S decomposes the n-space into two open regions, its *interior* $\underline{S}$, and its *exterior* $\overline{S}$. If $P \not\subset S$, the interior of S may be defined as the set of all points which do not lie on S and which are not separated from P by S; the exterior of S is then defined as the set of all points which are separated from P by S. An m-sphere through an $(m-1)$-sphere $S^{(m-1)}$ and a point $P \not\subset S^{(m-1)}$ will be denoted by $S^{(m)}[P; S^{(m-1)}]$. The m-sphere through $(m+2)$-points $P_0, P_1, \cdots, P_{m+1}$, not all lying on the same $(m-1)$-sphere, will occasionally be denoted by $S^{(m)}(P_0, P_1, \cdots, P_{m+1})$. Such a set of points is said to be *independent*. Most of the following discussion will involve the use of pencils $\pi^{(m)}$ of m-spheres determined by certain incidence and tangency conditions. An $(m-1)$-sphere which is common to all the m-spheres of a pencil $\pi^{(m)}$ is called *fundamental* $(m-1)$-*sphere* of $\pi^{(m)}$. In the pencil $\pi^{(m)}$ through a fundamental $(m-1)$-sphere $S^{(m-1)}$ there is one and only one m-sphere $S^{(m)}(P, \pi^{(m)})$ of $\pi^{(m)}$ through each point P which does not lie on $S^{(m-1)}$. Similarly, in the pencil $\pi^{(m)}$ of all the m-spheres which touch a given m-sphere at a given point Q, there is one and only one m-sphere $S^{(m)}(P, \pi^{(m)})$ through each point $P \neq Q$. The *fundamental point* Q is regarded as a *point* m-*sphere* belonging to $\pi^{(m)}$.

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2. Convergence. We call a sequence of points $P_1, P_2, \cdots$, *convergent* to P if to every $(n-1)$-sphere S with $P \subset \underline{S}$, there corresponds a positive integer $N=N(S)$ such that $P_\lambda \subset \underline{S}$ if $\lambda > N$. We define the convergence of m-spheres to a point in a similar fasion.

We call a sequence of $(n-1)$-spheres $S_1, S_2, \cdots$, *convergent* to S if to every pair of points $P \subset \underline{S}$ and $Q \subset \overline{S}$ there corresponds a positive integer $N=N(P, Q)$ such that $P \subset \underline{S}_\lambda$ and $Q \subset \overline{S}_\lambda$ for every $\lambda > N$.

Finally, a sequence of m-spheres $S_1^{(m)}, S_2^{(m)}, \cdots$, will be called *convergent* to an m-sphere $S^{(m)}$ if to every $S^{(n-m-1)}$ which links [5 ; §77] with $S^{(m)}$ there exists a positive integer $N=N(S^{(n-m-1)})$ such that $S_\lambda^{(m)}$ links with $S^{(n-m-1)}$ whenever $\lambda > N$, $(m=1, 2, \cdots, n-2)$.

3. Arcs. An *arc* A is the continuous image of a real interval. The images of distinct points of this parameter interval are considered to be different points of A even though they may coincide in space. The notation $t \neq p$ will indicate that the points t and p do not coincide. If a sequence of points of the parameter interval converges to a point p, we define the corresponding sequence of image points on the arc A to be *convergent* to the image of p. We shall use the same small italics $p, t, \cdots$, to denote both the points of the parameter interval and their image points on A. The *end-* (*interior*) *points* of A are the images of the end- (interior) points of the parameter interval. A *neighbourhood* of p on A is the image of a neighbourhood of the parameter on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) *one-sided neighbourhoods*.

4. Differentiability. Let p be a fixed point of an arc A, and let t be a variable point of A. Let $1 \leqq m < n$. If $p, P_1, \cdots, P_{m+1}$ do not lie on the same $(m-1)$-sphere, then there exists a unique m-sphere $S^{(m)}(P_1, \cdots, P_{m+1}, p)$ through these points. It is convenient to denote this m-sphere by the symbol $S_0^{(m)}=S^{(m)}(P_1, \cdots, P_{m+1}; \tau_0)$; here τ_0 indicates that this m-sphere passes through p. In the following, the m-sphere $S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)$ is defined inductively by means of the conditions $\Gamma_r^{(m)}$ given below (the τ_r in the symbol $S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)$ indicates that this sphere is a *tangent sphere* of the arc A at the point p meeting A $(r+1)$-times at p). We call A $(m+1)$-*times differentiable* at p if the following sequence of conditions is satisfied.

$\Gamma_r^{(m)}[r=1, 2, \cdots, m+1]$: If the parameter t is sufficiently close to, but different from, the parameter p, then the m-sphere $S^{(m)}(P_1, \cdots, P_{m+1-r}, t; \tau_{r-1})$ is uniquely defined. It converges if t tends to p. Thus its limit sphere, which will be denoted by

$$S_r^{(m)}=S^{(m)}(P_1, \cdots , P_{m+1-r} ; \tau_r),$$

will be independent of the way t converges to p [condition $\Gamma_{m+1}^{(m)}$ reads: $S^{(m)}(t ; \tau_m)$ exists and converges to $S_{m+1}^{(m)}-S^{(m)}(\tau_{m+1})$].

It is convenient to use the symbols $S_0^{(0)}$ to denote pairs of points P, p, and $S_1^{(0)}$ to denote the point pair p, p (or the point p).

We call A *once differentiable* at p if $\Gamma_1^{(1)}$ is satisfied. The point p is called a *differentiable point* of A if A is n-times differentiable at p.

Let $\tau_r^{(m)}$ denote the family of all the $S_r^{(m)}$'s. Thus $\tau_{m+1}^{(m)}$ consists only of $S_{m+1}^{(m)}$, the *osculating m-sphere* of A at p.

5. The structure of the families $\tau_r^{(m)}$ of m-spheres $S_r^{(m)}$ through p.

THEOREM 1. *Suppose A satisfies condition $\Gamma_1^{(m)}$ at p. Let $S^{(m-1)}$ be any $(m-1)$-sphere. Then there is a neighbourhood N of p on A such that if $t \in N$, $t \neq p$, then $t \not\subset S^{(m-1)}$, $(m=1, 2, \cdots , n-1)$.*

Proof. The assertion is evidently true if $p \not\subset S^{(m-1)}$. Suppose $p \subset S^{(m-1)}$. Choose points $P_1, \cdots , P_m$ on $S^{(m-1)}$ such that $p, P_1, \cdots , P_m$ are independent. If the parameter t is sufficiently close to, but different from, the parameter p, condition $\Gamma_1^{(m)}$ implies that $S^{(m)}$ $(P_1, \cdots , P_m, t; \tau_0)$ is uniquely defined. Thus $t \not\subset S^{(m-1)}$ $(P_1, \cdots , P_m; \tau_0)=S^{(m-1)}$.

COROLLARY. *If A satisfies condition $\Gamma_1^{(m)}$ at p, and $S^{(k)}$ is any k-sphere, then $t \not\subset S^{(k)}$ when the parameter t is sufficiently close to, but different from, the parameter p $(k=0, 1, \cdots , m-1)$.*

In particular, this holds when $m=n-1$.

THEOREM 2. *Let $1<m<n$; $1 \leqq k \leqq m$. If A satisfies $\Gamma_1^{(m)}, \cdots, \Gamma_k^{(m)}$ at p, then $\Gamma_1^{(m-1)}, \cdots , \Gamma_k^{(m-1)}$ will hold there and*

$$(1) \qquad S^{(m-1)}(P_1, \cdots , P_{m-r}; \tau_r)=\prod_P S^{(m)}(P_1, \cdots , P_{m-r}, P; \tau_r) .$$

Conversely, let A satisfy $\Gamma_1^{(m-1)}, \cdots , \Gamma_k^{(m-1)}$ at p, and let $S_m^{(m-1)} \neq p$ if $k=m$. If $P_{m-r+1} \not\subset S^{(m-1)}(P_1, \cdots , P_{m-r}; \tau_r)$, then $\Gamma_r^{(m)}$ will hold for the points $P_1, \cdots , P_{m-r+1}$ and

$$(2) \qquad S^{(m)}(P_1, \cdots , P_{m-r+1}; \tau_r)=S^{(m)}[P_{m-r+1}; S^{(m-1)}(P_1, \cdots , P_{m-r}; \tau_r)] \qquad (r=1, \cdots , k) .$$

REMARK. In general, $\Gamma_1^{(m-1)}, \cdots, \Gamma_k^{(m-1)}$ do not imply $\Gamma_1^{(m)}, \cdots , \Gamma_k^{(m)}$ (see [3], § 7).

Proof. (by induction with respect to k): Suppose $k=1$; $1<m<n$.

Let $\Gamma_1^{(m)}$ hold. If $P_1, \cdots, P_{m-1}, P, p$ are independent points, $S^{(m)}(P_1, \cdots, P_{m-1}, P, t; \tau_0)$ exists when t is sufficiently close to $p, t \neq p, t \in A$. Thus $P_1, \cdots, P_{m-1}, P, t, p$, are also independent, $S^{(m-1)}(P_1, \cdots, P_{m-1}, t; \tau_0)$ exists, and

$$S^{(m-1)}(P_1, \cdots, P_{m-1}, t; \tau_0) = \prod_P S^{(m)}(P_1, \cdots, P_{m-1}, P, t; \tau_0).$$

If $t \to p$, $S^{(m)}(P_1, \cdots, P_{m-1}, P, t; \tau_0)$ converges, and hence $S^{(m-1)}(P_1, \cdots, P_{m-1}, t; \tau_0)$ also converges, $\Gamma_1^{(m-1)}$ is satisfied, and

$$S^{(m-1)}(P_1, \cdots, P_{m-1}; \tau_1) = \prod_P S^{(m)}(P_1, \cdots, P_{m-1}, P; \tau_1).$$

Next, suppose that $\Gamma_1^{(m-1)}$ is satisfied, and $P_m \not\subset S^{(m-1)}(P_1, \cdots, P_{m-1}; \tau_1)$. Then $P_m \not\subset S^{(m-1)}(P_1, \cdots, P_{m-1}, t; \tau_0)$ when t is sufficiently close to p, $t \in A$, $t \neq p$, and

$$S^{(m)}(P_1, \cdots, P_m, t; \tau_0) = S^{(m)}[P_m, S^{(m-1)}(P_1, \cdots, P_{m-1}, t; \tau_0)]$$

exists. Hence when $t \to p$, $S^{(m)}(P_1, \cdots, P_m, t; \tau_0)$ converges, $\Gamma_1^{(m)}$ is satisfied relative to the points $P_1, \cdots, P_m$, and

$$S^{(m)}(P_1, \cdots, P_m; \tau_1) = S^{(m)}[P_m; S^{(m-1)}(P_1, \cdots, P_{m-1}; \tau_1)].$$

Thus Theorem 2 is satisfied when $k=1$.

Assume that Theorem 2 holds when k is replaced by $1, 2, \cdots, h$, where $1 \leq h < k \leq m$.

Let $\Gamma_1^{(m)}, \cdots, \Gamma_{h+1}^{(m)}$ hold. Then $S^{(m)}(P_1, \cdots, P_{m-h-1}, P, t; \tau_h)$ exists when t is sufficiently close to $p, t \neq p, t \in A$. Now $\Gamma_1^{(m)}, \cdots, \Gamma_h^{(m)}$ imply $\Gamma_1^{(m-1)}, \cdots, \Gamma_h^{(m-1)}$. If $h = m-1$, $\Gamma_h^{(m-1)} = \Gamma_{m-1}^{(m-1)}$ implies that $S_h^{(m-1)} = S^{(m-1)}(t; \tau_{m-1})$ exists, if $t \neq p$. If $h < m-1$, $\Gamma_1^{(m-1)}, \cdots, \Gamma_h^{(m-1)}$ imply $\Gamma_1^{(m-2)}, \cdots, \Gamma_h^{(m-2)}$. Thus $S^{(m-2)}(P_1, \cdots, P_{m-h-1}; \tau_h)$ exists. Furthermore, $\Gamma_1^{(m-1)}$ and Theorem 1 imply that $t \not\subset S^{(m-2)}(P_1, \cdots, P_{m-h-1}; \tau_h)$. But then Theorem 2, equation (2), with k replaced by h, implies that

$$S^{(m-1)}(P_1, \cdots, P_{m-h-1}, t; \tau_h) = S^{(m-1)}[t; S^{(m-2)}(P_1, \cdots, P_{m-h-1}; \tau_h)]$$

exists. By Theorem 2, equation (1), with k replaced by h,

$$S^{(m-1)}(P_1, \cdots, P_{m-h-1}, t; \tau_h) = \prod_P S^{(m)}(P_1, \cdots, P_{m-h-1}, P, t; \tau_h).$$

When $t \to p$, $S^{(m)}(P_1, \cdots, P_{m-h-1}, P, t; \tau_h)$ converges, hence $S^{(m-1)}(P_1, \cdots, P_{m-h-1}, t; \tau_h)$ also converges, $\Gamma_{h+1}^{(m-1)}$ is satisfied, and

$$S^{(m-1)}(P_1, \cdots, P_{m-h-1}; \tau_{h+1}) = \prod_P S^{(m)}(P_1, \cdots, P_{m-h-1}, P; \tau_{h+1}).$$

Next, suppose $\Gamma_1^{(m-1)}, \cdots, \Gamma_{h+1}^{(m-1)}$ hold, and let $P_{m-h} \not\subset S^{(m-1)}(P_1, \cdots,$

$P_{m-h-1}; \tau_{h+1})$. Then $P_{m-h} \not\subset S^{(m-1)}(P_1, \cdots, P_{m-h-1}, t; \tau_h)$ if t is sufficiently close to p, $t \in A$, $t \neq p$. But Theorem 2, with k replaced by h, then implies that

$$S^{(m)}(P_1, \cdots, P_{m-h-1}, P_{m-h}, t; \tau_h) = S^{(m)}[P_{m-h}; S^{(m-1)}(P_1, \cdots, P_{m-h-1}, t; \tau_h)]$$

exists. Hence when $t \to p$, $S^{(m)}(P_1, \cdots, P_{m-h}, t; \tau_h)$ converges, $\Gamma^{(m)}_{h+1}$ is satisfied for $P_1, \cdots, P_{m-h}$, and

$$S^{(m)}(P_1, \cdots, P_{m-h}; \tau_{h+1}) = S^{(m)}[P_{m-h}; S^{(m-1)}(P_1, \cdots, P_{m-h-1}; \tau_{h+1})] .$$

COROLLARY 1. *Let $1 \leq m < n$. If A is $(m+1)$-times differentiable at p then it is m-times differentiable there.*

COROLLARY 2. *If A satisfies $\Gamma^{(n-1)}_1, \cdots, \Gamma^{(n-1)}_{m+1}$ at p, then it is $(m+1)$-times differentiable there $(0 \leq m < n)$.*

COROLLARY 3.

$$S^{(m-1)}_m \subset S^{(m)}_{m+1} \qquad (m = 1, 2, \cdots, n-1).$$

Proof. By (1),

$$S^{(m)}(t; \tau_m) \supset \prod_P S^{(m)}(P; \tau_m) = S^{(m-1)}_m .$$

Hence $S^{(m)}_{m+1} \supset S^{(m-1)}_m$.

The last remark implies the following.

COROLLARY 4. *Let $1 \leq m < n$. If $S^{(m)}_{m+1} = p$, then $S^{(r)}_{r+1} = p$ $(r = 0, 1, \cdots, m-1)$. Thus there is an index i, where $1 \leq i \leq n$ such that $S^{(r)}_{r+1} = p$ for $r = 0, 1, \cdots, i-1$, but $S^{(r)}_{r+1} \neq p$, if $r \geq i$.*

COROLLARY 5. *Let $1 \leq m < n$; $1 \leq r \leq m$. Then*

$$S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r) \supset S^{(m-1)}(P_1, \cdots, P_{m+1-r}; \tau_{r-1}) .$$

Proof.

$$S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r) = \lim_{t \to p} S^{(m)}(P_1, \cdots, P_{m+1-r}, t; \tau_{r-1})$$
$$\supset S^{(m-1)}(P_1, \cdots, P_{m+1-r}; \tau_{r-1}) .$$

From Corollary 5, we get the following.

COROLLARY 6. *Let $1 \leq m < n$; $1 \leq r \leq m$. If $P_{m+2-r} \subset S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)$ and $P_{m+2-r} \not\subset S^{(m-1)}(P_1, \cdots . P_{m+1-r}; \tau_{r-1})$ then*

$$S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)=S^{(m)}(P_1, \cdots, P_{m+2-r}; \tau_{r-1}).$$

THEOREM 3. *Let* $1 \leqq r \leqq m < n$. Suppose $\Gamma_1^{(m)}, \cdots, \Gamma_r^{(m)}$ *are satisfied at* p.

(i) If $S_r^{(r-1)} \neq p$, $\tau_r^{(m)}$ consists of all the m-spheres through $S_r^{(r-1)}$.

(ii) Let $S_r^{(r-1)}=p$. Choose any $S_r^{(r)} \in \tau_r^{(r)}$. Then $\tau_r^{(m)}$ is the set of all the m-spheres which touch $S_r^{(r)}$ at p.

Proof of (i). By Theorem 2, equation (1),

$$S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r) \supset S^{(m-1)}(P_1, \cdots, P_{m-r}; \tau_r) \supset \cdots \supset S^{(r)}(P_1; \tau_r) \supset S_r^{(r-1)}.$$

Let $S^{(m)}$ be any m-sphere through $S_r^{(r-1)}$. By Theorem 2, if $P_1 \subset S^{(m)}$, $P_1 \not\subset S_r^{(r-1)}$,

$$S^{(r)}(P_1; S_r^{(r-1)})=S^{(r)}(P_1; \tau_r) \subset S^{(m)}.$$

Suppose $S^{(k)}(P_1, \cdots, P_{k+1-r}; \tau_r) \subset S^{(m)}$, $(r \leqq k < m)$. Choose $P_{k+2-r} \subset S^{(m)}$, $P_{k+2-r} \not\subset S^{(k)}(P_1, \cdots, P_{k+1-r}; \tau_r)$. Then by Theorem 2,

$$S^{(k+1)}(P_1, \cdots, P_{k+2-r}; \tau_r)=S^{(k+1)}[P_{k+2-r}; S^{(k)}(P_1, \cdots, P_{k+1-r}; \tau_r)] \subset S^{(m)}.$$

For $k=m-1$, this yields $S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)=S^{(m)}$. Thus $S^{(m)} \in \tau_r^{(m)}$.

Proof of (ii). Suppose $S_r^{(r-1)}=p$. As above, we have

$$S_r^{(m)}=S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r) \supset \cdots \supset S^{(r)}(P_1; \tau_r).$$

Let $S^{(r)}(Q; \tau_r)$ be any $S_r^{(r)} \in \tau_r^{(r)}$. By Theorem 2, equation (1),

$$S^{(r)}(P, t; \tau_{r-1}) \cap S^{(r)}(Q, t; \tau_{r-1}) \supset S^{(r-1)}(t; \tau_{r-1}).$$

Let P and Q be variable points and let $S^{(r-1)}$ be a variable $(r-1)$-sphere converging to a fixed point. Suppose there is an $(n-1)$-sphere which separates this point from P and Q. Then

$$\lim \measuredangle [S^{(r)}(P; S^{(r-1)}), S^{(r)}(Q; S^{(r-1)})]=0$$

whether or not the spheres $S^{(r)}(P; S^{(r-1)})$ and $S^{(r)}(Q; S^{(r-1)})$ themselves converge. In particular,

$$(3) \qquad \lim_{t \to p} \measuredangle [S^{(r)}(P, t; \tau_{r-1}), S^{(r)}(Q, t; \tau_{r-1})]=0.$$

Thus $S^{(r)}(P; \tau_r)$ touches $S^{(r)}(Q; \tau_r)$ at p. Furthermore, if $S^{(r)}(P; \tau_r)$ and $S^{(r)}(Q; \tau_r)$ have a point $\neq p$ in common, they coincide. Thus $\tau_r^{(r)}$ consists of the family of r-spheres which touch $S^{(r)}(Q; \tau_r)$ at p.

Suppose $r < m$ and an m-sphere $S_r^{(m)}=S^{(m)}(P_1, \cdots, P_{m+1-r}; \tau_r)$ of $\tau_r^{(m)}$ has a point $R \neq p$ in common with $S_r^{(m)}(Q; \tau_r)$. From the above,

$S^{(r)}(R; \tau_r)=S^{(r)}(Q; \tau_r)$. If $R \subset S^{(r)}(P_1; \tau_r)$ we have

$$S_r^{(m)} \supset S^{(r)}(P_1; \tau_r)=S^{(r)}(R; \tau_r)=S^{(r)}(Q; \tau_r)$$

while if $R \not\subset S^{(r)}(P_1; \tau_r)$, we have, by Theorem 2,

$$\begin{aligned} S_r^{(m)} &\supset S^{(r+1)}[R; S^{(r)}(P_1; \tau_r)] \\ &=S^{(r+1)}(P_1, R; \tau_r)=S^{(r+1)}[P_1; S^{(r)}(R; \tau_r)] \supset S^{(r)}(R; \tau_r)=S^{(r)}(Q; \tau_r) . \end{aligned}$$

On the other hand, suppose an m-sphere $S^{(m)}$ touches $S_r^{(r)}=S^{(r)}(Q; \tau_r)$ at p. If $S^{(m)} \supset S_r^{(r)}$ it follows, as in the proof of part (i), that $S^{(m)} \in \tau_r^{(m)}$. Suppose $S^{(m)} \cap S_r^{(r)}=p$. Choose an $S^{(r)} \subset S^{(m)}$ such that $S^{(r)}$ touches $S^{(r)}(Q; \tau_r)$ at p. Thus $S^{(r)} \subset \tau_r^{(r)}$. It again follows that $S^{(m)} \in \tau_r^{(m)}$

COROLLARY 1. *Let $\Gamma_1^{(r-1)}, \cdots, \Gamma_r^{(r-1)}$ hold and let $S_r^{(r-1)}=p$. Suppose $\lim_{t \to p} S^{(r)}(P, t; \tau_{r-1})$ exists for a single point P, $P \neq p$. Then $\Gamma_r^{(r)}$ holds at p $(1 < r < n)$.*

Proof. This follows from equation (3).

COROLLARY 2. *There is only one $S_r^{(m)}$ of the pencil $\tau_r^{(m)}$ which contains $(m+1-r)$ points which do not lie on the same $S_r^{(m-1)}$.*

Proof. Such an $S_r^{(m)}$ can be uniquely constructed as in the proof of (i), Theorem 3.

COROLLARY 3. *If two $S_r^{(m)}$'s intersect in an $S^{(m-1)}$ then this $S^{(m-1)} \in \tau_r^{(m-1)}$.*

Proof. The $S_r^{(m)}$'s and hence also $S^{(m-1)}$ contain $S_r^{(r-1)}$. In case $S_r^{(r-1)}=p$, let $R \subset S^{(m-1)}$, $R \neq p$. Then each of the $S_r^{(m)}$'s and hence also $S^{(m-1)}$ contains $S^{(r)}(R; \tau_r)$.

COROLLARY 4.

$$\tau_0^{(m)} \supset \tau_1^{(m)} \supset \cdots \supset \tau_{m+1}^{(m)} .$$

Proof. When $k < m$, or when $k=m$ and $S_m^{(m-1)} \neq p$, Theorem 3 implies that $\tau_k^{(m)}$ is the set of all the m-spheres through $S_k^{(k-1)}$. Hence $S_{k+1}^{(m)}$, being the limit of a sequence of such m-spheres, must itself contain $S_k^{(k-1)}$, and by Theorem 3, $S_{k+1}^{(m)} \in \tau_k^{(m)}$. Suppose $k=m$ and $S_m^{(m-1)}=p$. By Theorem 3, $\tau_m^{(m)}$ is the set of all the m-spheres which touch a given m-sphere $S_m^{(m)} \neq p$ of $\tau_m^{(m)}$ at p. Hence $S_{m+1}^{(m)}$, being the limit of a sequence of such m-spheres, must itself touch $S_m^{(m)}$ at p, and, again by

Theorem 3, $S^{(m)}_{m+1} \in \tau^{(m)}_m$.

THEOREM 4. *Let* $1<m<n$; $1 \leqq k \leqq m$, *and suppose that* $S^{(m-1)}_m \neq p$ *if* $k=m$. *If the conditions* $\Gamma^{(m)}_1, \cdots, \Gamma^{(m)}_k$ *hold at* p, *then* $\Gamma^{(m)}_{k+1}$ *also holds there.*

Proof. By Theorem 2, $\Gamma^{(m-1)}_1, \cdots, \Gamma^{(m-1)}_k$ hold at p. Hence if p, $P_1, \cdots, P_{m-k}$ are independent points $S^{(m-1)}(P_1, \cdots, P_{m-k}; \tau_k)$ is defined. Furthermore, by Theorem 1, we can assume that $t \not\subset S^{(m-1)}(P_1, \cdots, P_{m-k}; \tau_k)$ and by Theorem 2 again,

$$S^{(m)}(P_1, \cdots, P_{m-k}, t; \tau_k)=S^{(m)}[t; S^{(m-1)}(P_1, \cdots, P_{m-k}; \tau_k)].$$

Thus $S^{(m)}(P_1, \cdots, P_{m-k}, t; \tau_k)$ exists when t is close to p, $t \in A$, $t \neq p$. Choose $P_{m+1-k} \subset S^{(m-1)}(P_1, \cdots, P_{m-k}; \tau_k)$, $P_{m+1-k} \not\subset S^{(m-2)}(P_1, \cdots, P_{m-k}; \tau_{k-1})$. Then Theorem 2 implies that

$$S^{(m-1)}(P_1, \cdots, P_{m-k}; \tau_k)=S^{(m-1)}(P_1, \cdots, P_{m+1-k}; \tau_{k-1})$$

when $k<m$, or $k=m$ and $S^{(m-2)}_{m-1} \neq p$; if $k=m$ and $S^{(m-2)}_{m-1}=p$, this equation follows from Theorem 3, Corollary 4. Hence

$$\begin{aligned}\lim_{t\to p} S^{(m)}(P_1, \cdots, P_{m-k}, t; \tau_k)&=\lim_{t\to p} S^{(m)}[t, S^{(m-1)}(P_1, \cdots, P_{m+1-k}; \tau_{k-1})]\\ &=\lim_{t\to p} S^{(m)}(P_1, \cdots, P_{m+1-k}, t; \tau_{k-1})=S^{(m)}(P_1, \cdots, P_{m+1-k}; \tau_k).\end{aligned}$$

Thus $\Gamma^{(m)}_{k+1}$ holds at p and

$$S^{(m)}(P_1, \cdots, P_{m-k}; \tau_{k+1})=S^{(m)}(P_1, \cdots, P_{m+1-k}; \tau_k).$$

COROLLARY 1. *If* $\Gamma^{(m)}_1$ *holds at* p, *then* $\Gamma^{(m)}_r$ *holds there,* $r=1, 2, \cdots, m$. *Furthermore, if* $S^{(m-1)}_m \neq p$, A *is* $m+1$ *times differentiable at* p.

COROLLARY 2. *If* $\Gamma^{(n-1)}_1$ *holds at* p, *then* p *is a differentiable point of* A *if and only if* $\lim_{t\to p} S^{(n-1)}(t; \tau_{n-1})$ *exists and converges if* t *tends to* p.

COROLLARY 3. *If* $\Gamma^{(n-1)}_1$ *holds at* p, *and* $S^{(n-2)}_{n-1} \neq p$, *then* p *is a differentiable point of* A.

COROLLARY 4. *If* $\Gamma^{(m)}_1$ *holds at* p, *all the conditions* $\Gamma^{(r)}_k$, *except possibly* $\Gamma^{(m)}_{m+1}$, *automatically hold at* p $(1 \leqq k \leqq r+1 \leqq m+1)$.

Let p be a differentiable point of A. We define the index i of p as in Theorem 2, Corollary 4. Let $P \subset S^{(i)}_{i+1}$, $P \neq p$. Let $S^{(m)}_m=S^{(m)}(P; \tau_m)$, $m=0, 1, \cdots, i$. Then the set of $\tau^{(m)}_r$'s is completely determined by the sequence

$$S_0^{(0)} \subset S_1^{(1)} \subset \cdots \subset S_i^{(i)} = S_{i+1}^{(i)} \subset S_{i+2}^{(i+1)} \subset \cdots \subset S_n^{(n-1)} .$$

Its structure is determined by the single index i.

6. Support and intersection. Let p be an interior point of A. Then we call p a *point of support* (*intersection*) with respect to an $(n-1)$-sphere S if a sufficiently small neighbourhood of p is decomposed by p into two one-sided neighbourboods which lie in the same region (in different regions) bounded by S. S is then called a *supporting* (*intersecting*) $(n-1)$-sphere of A at p. Thus S supports A at p if $p \not\subset S$. By definition, the point $(n-1)$-sphere p always supports A at p.

It is possible for an $(n-1)$-sphere to have points $\neq p$ in common with every neighbourhood of p on A. In this case, S neither supports nor intersects A at p.

7. Support and intersection properties of $\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$. Let p be a differentiable interior point of A. In the following,

$$\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$$

will denote the family of those $(n-1)$-spheres of $\tau_r^{(n-1)}$ which do not belong to $\tau_{r+1}^{(n-1)}$ (cf. Theorem 3, Corollary 4). Our classification of the differentiable points p of A will be based on the index i of p, and on the support and intersection properties of $S_n^{(n-1)}$ and the families $\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$, $r=0, 1, \cdots, n-1$. We shall omit the superscript $(n-1)$ of $\tau_r^{(n-1)}$ when there is no ambiguity; thus $\tau_r = \tau_r^{(n-1)}$.

THEOREM 5. *Every $(n-1)$-sphere $\neq S_n^{(n-1)}$ either supports or intersects A at p.*

Proof. If an $(n-1)$-sphere S neither supports nor intersects A at p, then $p \subset S$ and there exists a sequence of points $t \to p$, $t \subset A \cap S$, $t \neq p$. Suppose $p, P_1, \cdots, P_n$ are independent points on S. Suppose that for some r, $0 \leq r < n-1$, $S = S^{(n-1)}(P_1, \cdots, P_{n-r}; \tau_r)$. By Theorem 2, equation (1),

$$S^{(n-1)}(P_1, \cdots, P_{n-r}; \tau_r) \supset S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r) .$$

By Theorem 1, $t \not\subset S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r)$ and again by Theorem 2, equation (2),

$$S = S^{(n-1)}[t; S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r)] = S^{(n-1)}(P_1, \cdots, P_{n-r-1}, t; \tau_r)$$

for each t. Condition $\Gamma_{r+1}^{(n-1)}$ now implies that

$$S = S^{(n-1)}(P_1, \cdots, P_{n-r-1}; \tau_{r+1}).$$

Thus we get, in this way,

$$S = S^{(n-1)}(P_1; \tau_{n-1}).$$

By Theorem 2, $S \supset S_{n-1}^{(n-2)}$, and by Theorem 1, $t \not\subset S_{n-1}^{(n-2)}$ when the parameter t is close to, but different from, the parameter p. If $S_{n-1}^{(n-2)} \neq p$, Theorem 2, equation 2, implies that $S = S^{(n-1)}[t; S_{n-1}^{(n-2)}] = S^{(n-1)}(t; \tau_{n-1})$, while if $S_{n-1}^{(n-2)} = p$, Theorem 3 implies that $S = S^{(n-1)}(t; \tau_{n-1})$. Applying condition $\Gamma_n^{(n-1)}$, we are led to the conclusion $S = S_n^{(n-1)}$.

THEOREM 6. *If $S_n^{(n-1)} = p$, then the $(n-1)$-spheres of $\tau_{n-1} - \tau_n$ all intersect A at p, or they all support.*

Proof. Let S and S' be two distinct $(n-1)$-spheres of $\tau_{n-1} - \tau_n$. Since $S_n^{(n-1)} = p$, Theorem 2, Corollary 4 implies that $S_{n-1}^{(n-2)} = p$, and Theorem 3 implies that S and S' touch at p. Thus we may assume that $S' \subset (p \cup S)$ and $S \subset (p \cup S')$. Suppose now, for example, that S supports A at p while S' intersects. Then $A \cap \overline{S}'$ is not void and $A \subset (p \cup \overline{S})$. Let $t \to p$ in $A \cap \underline{S}'$. Hence $S^{(n-1)}(t; \tau_{n-1}) \subset (\underline{S}' \cap \overline{S}) \cup p$. Consequently, $S(t; \tau_{n-1})$ can not converge to $S_n^{(n-1)} = p$, as t tends to p. Thus S and S' must both support, or both intersect A at p.

THEOREM 7. *If $S_{r+1}^{(r)} \neq p$ while $S_r^{r-1} = p$, then every $(n-1)$-sphere of $\tau_r - \tau_{r+1}$ supports A at p $(1 \leqq r \leqq n-1)$.*

Proof. Suppose $S_r^{(r-1)} = p$, so that by Theorem 3, the r-spheres of $\tau_r^{(r)}$ all touch any $(n-1)$-sphere of τ_r. Let $S \in \tau_r - \tau_{r+1}$, $S \neq p$. If a sequence of points t exists such that $t \subset A \cap \overline{S}$, $t \to p$, then each $S^{(r)}(t; \tau_r)$ lies in the closure of $\overline{S}$. Hence $S_{r+1}^{(r)}$ will also lie in the same closed domain. Since $S_{r+1}^{(r)} \in \tau_r^{(r)}$, either $S_{r+1}^{(r)} = p$, or it touches S at p. Since $S \notin \tau_{r+1}$, $S_{r+1}^{(r)}$ must lie in $p \cup \overline{S}$. Similarly, the existence of a sequence $t' \subset \underline{S} \cap A$, $t' \to p$, implies that $S_{r+1}^{(r)} \subset p \cup \underline{S}$. Thus if S intersects A at p, $S_{r+1}^{(r)} \subset (p \cup \overline{S}) \cap (p \cup \underline{S}) = p$; that is, $S_{r+1}^{(r)} = p$.

THEOREM 8. *All the $(n-1)$-spheres of $\tau_r - \tau_{r+1}$ support A at p, or they all intersect; $r = 0, 1, \cdots, n-1$.*

Proof. Let S' and S'' be two distinct $(n-1)$-spheres of τ_r. Suppose, for the moment, that the intersection $S' \cap S''$ is a proper $(n-2)$-sphere $S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r)$. Suppose, for example, that S' intersects, while S'' supports A at p. Thus $A \cap \underline{S}'$ and $A \cap \overline{S}'$ are not void.

With no loss in generality, we may assume that $A \subset \overline{S}'' \cup p$. If t is close to p, $t \neq p$, Theorem 1 implies that $t \not\subset S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r)$ and Theorem 2, equation 2, implies that

$$S^{(n-1)}[t; S^{(n-2)}(P_1, \cdots, P_{n-r-1}; \tau_r)] = S^{(n-1)}(P_1, \cdots, P_{n-r-1}, t; \tau_r) .$$

If $t \subset A \cap \underline{S}'$, then $S^{(n-1)}(P_1, \cdots, P_{n-r-1}, t; \tau_r)$ lies in the closure of

$$(\underline{S}' \cap \overline{S}'') \cup (\overline{S}' \cap \underline{S}'') .$$

Letting t tend to p, we conclude that $S^{(n-1)}(P_1, \cdots, P_{n-r-1}; \tau_{r+1})$ lies in the same closed domain. By letting t converge to p through $\overline{S}' \cap A$, we obtain symmetrically that $S^{(n-1)}(P_1, \cdots, P_{n-r-1}; \tau_{r+1})$ also lies in the closure of

$$(\overline{S}' \cap \overline{S}'') \cup (\underline{S}' \cap \underline{S}'')$$

Hence $S^{(n-1)}(P_1, \cdots, P_{n-r-1}; \tau_{r+1})$ lies in the intersection $S' \cup S''$ of these two domains, that is, $S^{(n-1)}(P_1, \cdots, P_{n-r-1}; \tau_{r+1})$ is either S' or S'', in other words, one of the $(n-1)$-spheres S' and S'' belongs to τ_{r+1}. Thus if S' and S'' belong to $\tau_r - \tau_{r+1}$ and have a proper $S^{(n-2)}$ in common, they both support or both of them intersect.

Suppose now that $S' \cap S'' = p$. Theorem 3 implies that $S_r^{(r-1)} = p$. In view of Theorems 6 and 7, there remain to be considered only the cases where $r < n-1$, and, indeed, when $r \leqq n-2$, we have only to consider those cases for which $S_{r+1}^{(r)} = p$.

By Theorem 3, any $S^{(n-1)}$ which touches an $S_r^{(r)}$, but which does not touch an $S_{r+1}^{(r+1)}$ belongs to $\tau_r - \tau_{r+1}$. Hence there exists an $(n-1)$-sphere S of $\tau_r - \tau_{r+1}$ which intersects S' and S'' respectively in a proper $(n-2)$-sphere. From the above, S and S', and also S and S'' both support or both intersect A at p. Thus S' and S'' both support or both intersect A at p in this case also.

8. Characteristic and classification of the differentiable points. The characteristic $(a_0, a_1, \cdots, a_n; i)$ of a differentiable point p of an arc A is defined as follows:

$a_r = 1$ or 2 when $r < n$; $a_n = 1, 2$, or ∞. The index $i = 1, 2, \cdots, n$.

$a_0 + \cdots + a_r$ is even or odd according as every $S_r^{(n-1)}$ of $\tau_r - \tau_{r+1}$ supports or intersects A at p; $r = 0, 1, \cdots, n-1$.

$a_0 + \cdots + a_n$ is even if $S_n^{(n-1)}$ supports, odd if $S_n^{(n-1)}$ intersects, while $a_n = \infty$ if $S_n^{(n-1)}$ neither supports nor intersects A at p.

Finally the characteristic of p has index i if and only if $S_i^{(i-1)} = p$, while $S_{i+1}^{(i)} \neq p$.

Theorem 7, and the convention that $S_n^{(n-1)}$ supports A at p when $S_n^{(n-1)} = p$, lead to the following restriction on the characteristic $(a_0, a_1, \cdots, a_n; i)$:

$$\sum_{k=0}^{i} a_k \equiv 0 \pmod 2 .$$

As a result of this restriction, the number of types of differentiable points corresponding to each value of $i<n$ is $3(2)^{n-1}$, and there are 2^n types when $i=n$. Thus there are $(3n-1)2^{n-1}$ types altogether.

If we introduce a rectangular Cartesian coordinate system into the conformal n-space, examples of each of the $(3n-1)2^{n-1}$ types are given by the curves

$$\text{(I)} \qquad x_1=t^{m_1},\ x_2=t^{m_2},\ \cdots ,\ x_n=t^{m_n}$$

in the cases $a_n=1$ or 2, and

$$\text{(II)} \qquad x_1=t^{m_1},\ x_2=t^{m_2},\ \cdots ,\ x_n=\begin{cases} t^{m_n}\sin t^{-1}, & \text{if } 0<|t|\leq 1 \\ 0 & , \text{ if } t=0 \end{cases}$$

for the cases in which $a_n=\infty$, all relative to the point $t=0$. The m_r are positive integers and $m_1<m_2<\cdots<m_n$. The different types are determined by the parities of the m_i and by the relative magnitudes of the m_r and $2m_1$. In each of these examples, the $S_1^{(m)}$ touch the x_1-axis at the origin; $m=1, 2, \cdots , n-1$.

When $m_i<2m_1<m_{i+1}$, the point $t=0$ has a characteristic of the form $(a_0, a_1, \cdots , a_n; i)$ where a_n can be 1, 2, or ∞, and $i<n$.

When $m_n<2m_1$, the point $t=0$ has a characteristic of the form $(a_0, a_1, \cdots , a_n; n)$ where a_n is either 1 or 2. The following table lists some of the properties of a differentiable point p having the characteristic $(a_0, a_1, \cdots , a_n; i)$:

$(a_0, a_1, \cdots , a_n; i)$

Index	a_n	Osculating		Supporting family	Restriction	Example	
		$(i-1)$-sphere	i-sphere				
$i<n$	$a_n=1$ or 2	$S_i^{(i-1)}=p$	$S_{i+1}^{(i)}\neq p$	$\tau_i-\tau_{i+1}$	$\sum_{r=0}^{i} a_r\equiv 0$ (mod 2)	I	$m_i<2m_1<m_{i+1}$
	$a_n=\infty$					II	
$i=n$	$a_n=1$ or 2			τ_n		I	$m<2m_1$

References

1. J. Hjelmslev, *Introduction à la théorie des suites monotones*, Oversigt Kgl. Danske Vidensk. Selsk. Forh., No. 1 (1914).

2. N. D. Lane and P. Scherk, *Differentiable points in the conformal plane*, Canad. J.

Math., **5** (1953), 512–518.
3. N. D. Lane and F. A. Sherk, *Differentiable points of arcs in conformal 3-space*, Canad. J. Math., **8** (1956), 105–118.
4. P. Scherk, *Über differenzierbare Kurven und Bögen I. Zum Begriff der Characteristik*, Časopis Pěst. Mat., **66** (1937), 165–171.
5. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Chelsea, 1947.

HAMILTON COLLEGE,
MCMASTER UNIVERSITY.