ON THE SPECTRA OF LINKED OPERATORS

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1. Introduction. Let X, Y be complex linear spaces, and Z a non-void complex linear space contained in both X and Y. Let X be a Banach space X_1, Y a Banach space Y_2 under the norms n_1, n_2 respectively. Let Z be a Banach space Z_N under the norm N defined by $N(z)=\max[n_1(z), n_2(z)]$. (This is equivalent to saying that if $\{z_n\}$ is any sequence with $z_n \in Z$, such that $z_n \to x$ in the topology of X_1 and $z_n \to y$ in the topology of Y_2, then $x=y \in Z$. Our particular method of stating this here will be useful for later purposes.) With the usual uniform norms let T_1, T_2 be bounded distributive operators on X_1, Y_2 respectively, such that $T_1 z = T_2 z \in Z$ when $z \in Z$. Operators satisfying these conditions will be said to be "linked". If, in addition, it is assumed that Z is dense in X_1, T_1 and T_2 will be said to be "linked densely relative to X_1".

We are interested in relationships between the spectra of linked operators. That there are linked, and densely linked operators with different spectra will be shown in § 3. The main result of this paper is the demonstration that, if T_1 and T_2 are linked densely relative to X_1, under certain circumstances any component of the spectrum of T_1 has a non-void intersection with the spectrum of T_2. Sufficient conditions are that if λ belongs to the intersection of the resolvent sets of T_1 and T_2 and $z \in Z$, then $(\lambda I - T_1)^{-1} z = (\lambda I - T_2)^{-1} z \in Z$. With this result we obtain some interesting consequences in the special case where the Banach spaces considered are the sequence spaces l_p.

2. Preliminary definitions and notation. Supposing X to be a complex linear space such that under a norm n_a, $(x \in X,\ n_a(x)=\|x\|_a)$, X becomes a complex Banach space X_a, we let $[X_a]$ denote the set of all operators T that are bounded under the induced norm

$$\|T\|_a = \sup \|Tx\|_a \text{ (for all } x \in X_a,\ \|x\|_a = 1).$$

Such a T will be denoted by T_a when considered as an element of the algebra $[X_a]$. If $T_a \in [X_a]$ we classify all complex numbers into two sets:

(1) The resolvent set $\rho(T_a)$, consisting of all λ such that $\lambda I - T_a$ defines a one-to-one correspondence of X_a onto X_a.

(2) The spectrum $\sigma(T_a)$, consisting of all λ not in $\rho(T_a)$.

The spectrum is divided into three parts:

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(1) The point spectrum $p(T_a)$, consisting of those λ for which $(\lambda I-T_a)^{-1}$ does not exist.

(2) The continuous spectrum $c(T_a)$, consisting of those λ not in $\rho(T_a)$ or $p(T_a)$ for which the range of $\lambda I-T_a$ is dense in X_a; and

(3) The residual spectrum $r(T_a)$, consisting of those λ not in $\rho(T_a)$. $p(T_a)$ or $c(T_a)$.

We shall also have occasion to refer to the so-called "approximate point spectrum," consisting of those λ for which $(\lambda I-T_a)^{-1}$ is not bounded. It is well known that $\sigma(T_a)$ is closed, bounded and nonempty. It is also well known that $R_\lambda(T_a)\equiv(\lambda I-T_a)^{-1}$ is analytic in $\rho(T_a)$ as a function with values in $[X_a]$.

3. An example of linked operators with different spectra. Consider the well known sequence spaces l_1 and l_2. Let T_1 and T_2 be the operation defined as elements of $[l_1]$ and $[l_2]$ respectively by the infinite matrix (t_{ij})

$$t_{ij}=\begin{cases}\dfrac{j}{(i-1)i} & \text{if } i>j\\ 0 & \text{if } i\leqq j.\end{cases}$$

The uniform norm for the operator T defined by such a matrix, when considered as an operator on l_1, can be shown to be the supremum of the l_1 norms of the column sequences of the matrix (t_{ij}):

$$\|T_1\|_1=\sup_j\sum_{i=1}^{\infty}|t_{ij}|$$

([1, pp. 696–697]). From this it is easy to see that $\|T_1\|_1=1$. In fact

$$\sum_{i=1}^{\infty}|t_{ij}|=j\sum_{i=j+1}^{\infty}\frac{1}{(i-1)i}=j\frac{1}{j}=1,$$

the sum being independent of j. Next, considering the powers of T:

$$T^n=(t_{ij})^n=(t_{ij}^{(n)}),$$

we see that

$$\|T_1^2\|_1=\sup_j\sum_{i=1}^{\infty}|t_{ij}^{(2)}|=\sup_j\sum_{i=1}^{\infty}|\sum_{k=1}^{\infty}t_{ik}t_{kj}|=\sup_j\sum_{k=1}^{\infty}(\sum_{i=1}^{\infty}t_{ik})t_{kj}=\sup_j\sum_{k=1}^{\infty}t_{kj}=1.$$

By induction it is easy to show that $\sum_{i=1}^{\infty}|t_{ij}^{(n)}|=1$ for any j, and hence $\|T_1^n\|_1=1$. Now it is well known that the spectral radius of T_1,

$$|\sigma(T_1)|\equiv\sup|\lambda|,\ (\text{for } \lambda\in\sigma(T_1)),$$

is given by the formula

$$|\sigma(T_1)| = \lim_{n\to\infty} (\|T_1^n\|_1)^{1/n};$$

hence the spectral radius of T_1 is 1.

On the other hand, by making use of an inequality due to Schur [2, p. 6], we can estimate the norm of T as an operator on l_2:

$$\|T_2\|_2 \leqq [(\sup_i \sum_{j=1}^{\infty} |t_{ij}|)(\sup_j \sum_{i=1}^{\infty} |t_{ij}|)]^{\frac{1}{2}}.$$

In this way we see that

$$\|T_2\|_2 \leqq \sqrt{1\cdot\tfrac{1}{2}} < 1,$$

since

$$\sum_{j=1}^{\infty} |t_{ij}| = \sum_{j=1}^{i-1} \frac{j}{(i-1)i} = \tfrac{1}{2},$$

the sum being independent of i. Since it is always true that $|\sigma(T_2)| \leqq \|T_2\|_2$, it is now clear that $|\sigma(T_2)| < |\sigma(T_1)|$, whence we immediately infer that there exists a λ such that $\lambda \in \sigma(T_1)$ and $\lambda \notin \sigma(T_2)$.

4. The projection corresponding to a spectral set. For the proof of our main theorem we need the concepts of spectral set and the projection associated with a spectral set. For this purpose we introduce the following definitions.

Suppose X is a complex Banach space, and T an element of $[X]$. A set σ in the complex plane is called a spectral set of T if $\sigma \subset \sigma(T)$ and if σ is both open and closed in the relative topology of $\sigma(T)$.

If σ is a spectral set of T, the corresponding projection is the operator defined by

$$E_\sigma[T] = \frac{1}{2\pi i}\int R_\lambda(T)d\lambda,$$

the integral being extended in the positive sense around the boundary of a suitable bounded open set D such that $\sigma \subset D$ and the closure of D does not intersect the rest of $\sigma(T)$. It is easy to see that if Δ is a closed set which does not intersect σ, the set D may be chosen to satisfy the additional requirement that its closure does not intersect Δ.

We now proceed to the proof of our main theorem.

5. Relations between the spectra of linked operators. Let X and Y be complex linear spaces such that X becomes a Banach space X_1 and Y becomes a Banach space Y_2 under the norms n_1 and n_2, respectively.

THEOREM. *Let $T_1 \in [X_1]$ and $T_2 \in [Y_2]$ be linked densely relative to X_1 and let $Z \subset X \cap Y$ be a complex linear space that becomes a Banach space Z_N under the norm N defined by $N(z) = max\,[n_1(z), n_2(z)]$. Let $R_\lambda(T_1)z = R_\lambda(T_2)z \in Z$ for every $z \in Z$, provided that $\lambda \in \rho(T_1) \cap \rho(T_2)$. Then if C is any component of $\sigma(T_1)$, $C \cap \sigma(T_2)$ is non-void.*

Proof. We shall first prove that if σ is any non-void spectral set of $\sigma(T_1)$, then $\sigma \cap \sigma(T_2)$ is non-void.

Suppose that $\sigma \cap \sigma(T_2)$ is void. Let $E_\sigma[T_1]$ be the projection in $[X_1]$ associated with σ, that is

$$E_\sigma[T_1] = \frac{1}{2\pi i}\int_{+B(D)} R_\lambda(T_1)\,d\lambda,$$

where $B(D)$ is the boundary of a bounded Cauchy domain such that $\sigma \subset D$ while the closure of D intersects neither $\sigma(T_2)$ nor the rest of $\sigma(T_1)$. We know that $E_\sigma[T_1] \neq 0$ by a theorem [3, p. 210] which states that the spectral set σ is empty if and only if $E_\sigma[T_1] = 0$. Now consider the operator (an element of $[Y_2]$)

$$F = \frac{1}{2\pi i}\int_{+B(D)} R_\lambda(T_2)\,d\lambda.$$

Since D and $B(D)$ lie in $\rho(T_2)$, $R_\lambda(T_2)$ is analytic inside and on $B(D)$; therefore the integral defining F is the zero element of $[Y_2]$, by Cauchy's theorem.

If $\lambda \in \rho(T_1) \cap \rho(T_2)$, then by hypothesis $R_\lambda(T_1)z = R_\lambda(T_2)z$ for $z \in Z$, and from this we see that

$$Fz = E_\sigma[T_1]z \text{ for } z \in Z,$$

since the integrals defining Fz and $E_\sigma[T_1]z$ can be regarded as limits, in Y_2 and X_1 respectively, of the same sequence in Z. However, since $E_\sigma[T_1] \neq 0$ and is continuous, and Z is dense in X_1, there exists a z, $z \in Z$, such that $E_\sigma[T_1]z \neq 0$. But $Fz = 0$, which is a contradiction. Thus any non-void spectral set of $\sigma(T_1)$ has a non-void intersection with $\sigma(T_2)$.

Let C be any component of $\sigma(T_1)$. To show that $C \cap \sigma(T_2)$ is non-void we will need the following theorem [4, p. 15]: *If A and B are disjoint closed subsets of a compact set K such that no component of K intersects both A and B, there exists a separation $K = K_1 \cup K_2$, where K_1 and K_2 are disjoint compact sets containing A and B respectively.* Now suppose that $C \cap (\sigma(T_1) \cap \sigma(T_2))$ is void. Then, since C and $\sigma(T_1) \cap \sigma(T_2)$ are non-void disjoint closed subsets in $\sigma(T_1)$ and as the only component of $\sigma(T_1)$ intersecting C is C itself, we have $\sigma(T_1) = K_1 \cup K_2$, where $K_1 \supset C$, $K_2 \supset \sigma(T_1) \cap \sigma(T_2)$, and K_1, K_2 are disjoint compact sets. But K_1 is closed, being compact, and also relatively open, since it is the relative complement of the closed set K_2. Thus K_1 is a spectral set of $\sigma(T_1)$, and $K_1 \cap (\sigma(T_1) \cap \sigma(T_2))$ is void, which is in contradiction to what we have shown above. Thus if C is any component of $\sigma(T_1)$, then $C \cap \sigma(T_2)$ is non-void, as was to be proved.

We note that if in the hypotheses of the theorem we only require T_2 to be a closed distributive operator on Y_2, such that $\sigma(T_2)$ is non-void, the conclusion and proof of the theorem will be unaltered. Also, if we replace the hypotheses that $T_1 \in [X_1]$ and $T_2 \in [Y_2]$ by "*T_1 and T_2 are closed distributive operators on X_1 and Y_2 respectively, such that $\sigma(T_2)$ is nonvoid*", and retain the remaining hypotheses, we can conclude, using the same reasoning as before, that any non-void bounded spectral set of $\sigma(T_1)$ has a non-void intersection with $\sigma(T_2)$.

A very special case of our theorem, but one of considerable practical importance, is given in the following corollary.

COROLLARY 1. *In addition to the hypotheses of the preceding theorem let Z be dense in Y_2, and let $\sigma(T_1)$ and $\sigma(T_2)$ be such that all of their components are single points. Then $\sigma(T_1)=\sigma(T_2)$.*

In the special case where $X \subset Y$, the operators $T_1 \in [X_1]$, $T_2 \in [Y_2]$ are linked, and X plays the role of Z, we have the following two corollaries.

COROLLARY 2. *If C is any component of $\sigma(T_1)$, then $C \cap \sigma(T_2)$ is non-void.*

Proof. This follows from the theorem, since if $\lambda \in \rho(T_1) \cap \rho(T_2)$, then $R_\lambda(T_1)x = R_\lambda(T_2)x$ for $x \in X$.

COROLLARY 3. *If T_1 and T_2 are linked densely relative to Y_2 and C is any component of $\sigma(T_2)$, then $C \cap \sigma(T_1)$ is non-void.*

This should be clear from the proof of the theorem in view of the remark following the statement of Corollary 2.

DEFINITION. *If A, B, C are sets such that any component of C has a non-void intersection with both A and B we shall say that A and B are "linked by C". If in addition every component of A has a non-void intersection with C we shall say that A is "totally linked to B by C".*

Now suppose that neither X nor Y is necessarily contained in the other and let $T \in [Z_N]$ be the operator defined by $Tz = T_1 z$ for $z \in Z$. Then we have the following results for $T_1 \in [X_1]$ and $T_2 \in [Y_2]$.

COROLLARY 4. *If T_1 and T_2 are linked (not necessarily densely linked), then $\sigma(T_1)$ and $\sigma(T_2)$ are linked by $\sigma(T)$.*

This follows immediately from Corollary 2.

COROLLARY 5. *If T_1 and T_2 are linked densely relative to X_1, then $\sigma(T_1)$ is totally linked to $\sigma(T_2)$ by $\sigma(T)$.*

This follows from Corollary 3.

COROLLARY 6. *If T_1 and T_2 are linked, then*

$$\sigma(T)-(\sigma(T_1)\cup\sigma(T_2))$$

is contained in that portion of the residual spectrum of T for which $(\lambda I-T)^{-1}$ is bounded.

Proof. Clearly $p(T)$ is contained in both $p(T_1)$ and $p(T_2)$. If λ belongs to the approximate point spectrum of T then there exists a sequence $\{z_n\}$, $z_n \in Z$, such that

$$\lim_{n\to\infty} \|(\lambda I-T)z_n\|_N=0 \text{ and } \|z_n\|_N=1.$$

But either 1°: Infinitely many z_n are such that $\|z_n\|_{n_1}=1$, or 2°: Infinitely many z_n are such that $\|z_n\|_{n_2}=1$. If 1° holds there exists a subsequence $\{x_n\}$ of $\{z_n\}$ such that

$$\lim_{n\to\infty} \|(\lambda I-T)x_n\|_{n_1}=0 \text{ and } \|x_n\|_{n_1}=1,$$

and thus λ belongs to the approximate point spectrum of T_1. If 2° holds similar reasoning shows that λ belongs to the approximate point spectrum of T_2. From these results it follows that the only possibility for an element λ, $\lambda \in \sigma(T)$, to be such that $\lambda \notin \sigma(T_1)\cup\sigma(T_2)$ is for λ to be an element of the residual spectrum of T with $(\lambda I-T)^{-1}$ bounded.

The following is a corollary concerning the sequence spaces l_n, which we considered earlier.

COROLLARY 7. *Suppose that $1\leq r<s$, and suppose that the infinite matrix (t_{ij}) defines operators T_r and T_s on l_r and l_s, respectively, such that $T_r \in [l_r]$ and $T_s \in [l_s]$. Then $C\cap\sigma(T_s)$ is non-void for any component C of $\sigma(T_r)$. Moreover, $C\cap\sigma(T_r)$ is non-void for any component C of $\sigma(T_s)$.*

Proof. These are special cases of Corollaries 2 and 3, for it is well known that, for the classes l_r and l_s, we have $l_r\subset l_s$; that $\|x\|_s\leq\|x\|_r$ for $x\in l_r$; and that l_r is dense in l_s.

Corollary 7 is true even if $s=\infty$. (We recall that l_∞ is the set of all sequences $x=\{\xi_i\}$ such that $\sup_i |\xi_i|<\infty$, and such that if $x\in l_\infty$, $\|x\|_\infty=\sup_i |\xi_i|$.) For, although in this case it is not true that l_r is dense in l_∞, the following *is* true: if an element of $[l_\infty]$ is defined by an infinite matrix, and if the operator is 0 when restricted to l_r, then it is the zero operator on l_∞. The reasoning of the main theorem now applies with only slight modifications for the case in which $X_1=l_\infty$, $Y_2=l_r (1\leqq r<\infty)$, $Z=l_r$ and T_1 and T_2 are defined by the same matrix.

Before stating the final corollary we recall the following facts.

If $1<p\leqq\infty$ and $1/p+1/p'=1$ (with $p'=1$ if $p=\infty$), we can identify the conjugate space $(l_{p'})^*$ with l_p. If (t_{ij}) is an infinite matrix defining a bounded linear operator T on $l_{p'}$, we can identify the adjoint operator T^* with the bounded linear operator T^t defined on l_p by the transposed matrix (t_{ij}^t), where $t_{ij}^t=t_{ji}$. Since $\sigma(T)=\sigma(T^*)$, as is well known [5, pp. 304 and 306], we have $\sigma(T_{p'})=\sigma(T_p^t)$, where the subscripts serve to remind us on what space the operator is defined.

COROLLARY 8. *Suppose the matrix* (t_{ij}) *defines* $T_p\in[l_p]$ *and* $T_{p'}\in[l_{p'}]$, *where* $1<p\leqq\infty$. *Then* $C\cap\sigma(T_p^t)$ *is non-void for any component* C *of* $\sigma(T_p)$, *and* $C\cap\sigma(T_p)$ *is non-void for any component* C *of* $\sigma(T_p^t)$.

Proof. This follows from Corollary 7 and the foregoing remarks, by taking p and p' to be r and s or s and r, depending on whether $p\leqq2$ or $2<p$.

6. Further comments. The referee made some suggestions concerning the condition which was imposed in the main theorem of § 5, namely that

$$\text{(R)}\qquad R_\lambda(T_1)z=R_\lambda(T_2)z\in Z \text{ if } z\in Z \text{ and } \lambda\in\rho(T_1)\cap\rho(T_2).$$

We shall refer to this as Condition (R). We add some discussion of this condition, guided in part by the suggestions of the referee.

As in § 5, let us denote by T the member of $[Z_N]$ defined by $Tz=T_1z=T_2z$ when $z\in Z$. It is then easy to see that $R_\lambda(T)z=R_\lambda(T_k)z\in Z$ if $z\in Z$ and $\lambda\in\rho(T)\cap\rho(T_k)$, $k=1, 2$. Consequently $R_\lambda(T_1)z=R_\lambda(T_2)z\in Z$ if $z\in\rho(T)\cap\rho(T_1)\cap\rho(T_2)$. The intersection of these three resolvent sets certainly contains all sufficiently large values of λ. Now let D be the set of those $\lambda\in\rho(T_1)\cap\rho(T_2)$ for which $R_\lambda(T_1)z=R_\lambda(T_2)z\in Z$ if $z\in Z$. This set is evidently closed relative to $\rho(T_1)\cap\rho(T_2)$ (by the continuity of the resolvents and the way in which the norm of Z is defined). It is also open relative to $\rho(T_1)\cap\rho(T_2)$, as we may see by using the expansion

$$R_\lambda=\sum_{n=0}^{\infty}(\mu-\lambda)^n R_\mu^{n+1}$$

for the resolvent of an operator in the neighborhood of a point μ in the resolvent set. Consequently D contains all of any particular component of $\rho(T_1)\cap\rho(T_2)$ if it contains any point of that component. In particular D contains all of the unbounded component of $\rho(T_1)\cap\rho(T_2)$. This shows that we can omit the Condition (R) if $\rho(T_1)\cap\rho(T_2)$ has only one component. In particular this will be true if $\sigma(T_1)$ and $\sigma(T_2)$ are totally disconnected. From what was said previously it is clear that $\rho(T_1)\cap\rho(T_2)-D$ lies in $\sigma(T)-(\sigma(T_1)\cup\sigma(T_2))$, and hence, by Corollary 6, in that part of $\sigma(T)$ for which $(\lambda I-T)^{-1}$ exists and is bounded. It is not very difficult to prove that a point of this latter kind is not in the closure of $\rho(T)$. (The argument uses the functional equation of the resolvent, $R_\lambda-R_\mu=(\mu-\lambda)R_\lambda R_\mu$, to show that if $\alpha\in\overline{\rho(T)}$ then $\lim_{\lambda\to\alpha}R_\lambda$ exists and is necessarily R_α.) Consequently we see that D contains the set $\overline{\rho(T)}\cap\rho(T_1)\cap\rho(T_2)$. This shows, for example, that Condition (R) is superfluous if $\rho(T)$ is everywhere dense in the plane.

References

1. L. W. Cohen and Nelson Dunford, *Transformations on sequence* spaces, Duke Math. J., **3**, (1937), 689–701.

2. J. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. für Math., **140** (1911), 1–28.

3. A. E. Taylor, *Spectral theory of closed distributive operators*, Acta. Math., **84** (1950), 189–224.

4. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, **28**, New York, 1942.

5. A. C. Zaanen, *Linear analysis*, Interscience, New York, 1953.

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