MEAN VALUES OF HARMONIC FUNCTIONS
ON HOMOTHETIC CURVES

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1. Introduction. It is well known indeed that if $U(P)$ is harmonic in the plane the mean value of U taken over the perimeter or the area of a circle with center P_0 equals $U(P_0)$. A related result is that of Ásgeirsson [1] which states that mean of U over the area of any one of a family of confocal ellipses equals that over any other. The same is true for the means over the perimeters, provided the means are weighted by integrating with respect to the anomaly angle instead of arc length.

It would be interesting to know if there are any other simple families of curves over which the perimeter or area average is constant. The simplest families to try are homothetic families, and in the following we show that under suitable regularity assumptions, there are none of these except circles.

2. Perimeter means. Let C be a closed simple rectifiable curve containing O in its interior. We suppose that C is smooth enough that its Green's function $g(P)$ with pole at O is continuously differentiable on C, as will be the case if C has a continously turning tangent line. By C_λ we mean the curve obtained from C through the homothetic transformation $x' = \lambda x$, $y' = \lambda y$. If we expect that for $\lambda < 1$, C_λ is inside C, we should assume that C is star-shaped about O, although this is not essential to what follows. A positive continuous weight function $w(P)$ is given, and we suppose that the mean of U with weight w over all C_λ is constant, provided U is harmonic inside and on C.

We immediately note two things. In the first place, letting λ tend to zero, we see that the mean value of U over C_λ must be $U(O)$. In the second place, from obvious continuity considerations, we see that U need only be harmonic inside C and continuous on C for the mean over C_λ to be constant. Since U may be given arbitrary continuous values on C and determined inside so as to be harmonic, it follows that

$$U(O) = \frac{1}{W} \int_C U(P) w(P) ds$$

for every continuous U, where $W = \int_C w(P) ds$. But also

$$U(O) = \frac{1}{2\pi} \int_\sigma U(P) \frac{\partial g}{\partial n}(P) ds ,$$

and so $\int_\sigma U\{\partial g/\partial n - (2\pi/W)w\}ds = 0$ for continuous U.

Thus

$$\frac{\partial g}{\partial n} = \frac{2\pi w}{W} ;$$

that is, the weight function must be the obvious one, $\partial g/\partial n$. In the case of the ordinary perimeter mean, $w=1$ and $\partial g/\partial n =$ constant. This implies that C is a circle. For consider $f(z) = e^{-(g+ih)}$ which maps the interior of C into the unit circle. On C, $f(z)$ remains continuous and $|f'(z)| = \partial h/\partial s = \partial g/\partial n =$ constant. Thus $f' =$ constant and $f(z) =$ constant z and the result is obvious.

3. Area means. Let the equation in polar coordinates of C be $r = f(\theta)$. Saying that the mean of U over the interior of C_λ is constant for all $\lambda \leq 1$ amounts to saying that

$$U(O) = \frac{\int_0^{2\pi} d\theta \int_0^{\lambda f(\theta)} U(r, \theta) r dr}{\int_0^{2\pi} \int_0^{\lambda f(\theta)} r dr} .$$

If the derivative of the right hand member with respect to λ is computed for $\lambda = 1$ and equated to zero, there results

$$U(O) = k \int_0^{2\pi} U(f(\theta), \theta) f^2(\theta) d\theta = k \int_\sigma U f \cos \psi ds,$$

where ψ is the angle between radius vector and normal to C and k is a constant. Thus the problem is reduced to one involving a perimeter mean, and as in § 2 we see that

$$2\pi k f \cos \psi = \partial g/\partial n .$$

This equation says, among other things, that $\partial g/\partial n$ has a lesser value at the nearest than at the farthest points of C from O. In the next section we show that this can be true only if C is a circle about O.

4. A lemma. Let P and Q be a nearest and farthest point, respectively, of C from 0. To complete our discussion of the area means we shall show that $(\partial g/\partial n)_Q \leq 1/OQ < 1/OP \leq (\partial g/\partial n)_P$, which will certainly contradict the result obtained in the previous section unless

$OP=OQ$ and C is a circle. To this end we prove[1] the following lemma, which can be interpreted as a special case of Julia's theorem [2]. The present proof is included because it avoids the use of conformal mapping, and also because its proof is somewhat shorter than the steps required to derive the lemma from the usual statements of Julia's theorem.

LEMMA. *Let D_1 and D_2 be domains with smooth boundaries, with $O \in D_1 \subset D_2$. Let the corresponding Green's functions be g_1 and g_2. If P is on both boundaries, then $\partial g_2/\partial n \geq \partial g_1/\partial n$.*

Let the boundaries be C_1 and C_2, and set $h=g_2-g_1$. Then h is harmonic in D_1, and on the boundary C_1, $h=g_2 \geq 0$. Thus $h \geq 0$ in D_1. Since $h=0$ at P, clearly $0 \leq \partial h/\partial n = \partial g_2/\partial n - \partial g_1/\partial n$, and the lemma follows.

Returning to the matter discussed at the beginning of this section, we see that by comparing C first with its largest inscribed circle with center at O, and then with the smallest circumscribed circle, the stated inequalities follow immediately.

5. **Concluding remarks.** These results have considerable room for improvement in the direction of lightening the restriction on C. One would suppose that in the perimeter case, C should only be required to be rectifiable, and in the area case, no condition at all on C need be put.

The lemma provides an alternate way, avoiding conformal mapping, of completing the discussion of the perimeter means. With this in mind, it would be quite easy to extend the whole discussion to harmonic functions in more than two dimensions.

It is interesting to note another and quite different form in which the area problem can be cast. If the area means of any harmonic function over the homothetic transforms of $r \leq f(\theta)$ are to be constant, this must be true for the simplest ones, $U=r^n e^{\pm in\theta}$. This leads immediately to the equations

$$\int_0^{2\pi} f^{n+2}(\theta) \cos n\theta \, d\theta = \int_0^{2\pi} f^{n+2}(\theta) \sin n\theta \, d\theta = 0$$

for $n=1, 2, \ldots$. Our results show indirectly that if f is positive and sufficiently smooth, the above orthogonality conditions imply that $f=$ constant. It would be interesting to obtain a direct proof of this without the intervention of potential theory.

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[1] The proof given, which is somewhat simpler than that originally devised by the author, is due to Professor David Gilbarg.

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REFERENCES

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2. L. Bieberbach, *Lehrbuch der Funktionentheorie*, Vol. II, pp. 112–114. Teubner., Leipzig, 1921.

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