# NEIGHBOR RELATIONS ON THE CONVEX OF CYCLIC PERMUTATIONS

I. HELLER

**1. Introduction and summary.** Two vertices of a polyhedron are called neighbors of order $k$ when they have a face of dimension $k$, and none of lower dimension, in common. $K(P)$ denotes the maximum value of $k$ for a given polyhedron $P$. For the convex hull (polyhedron) $P_n$ of all permutations of $n$ elements (represented by square matrices of order $n$ and interpreted as points in $n^2$-space) it was shown [1 and 2] that $K(P)=[n/2]$ (that is, the largest integer not exceeding $n/2$), which is rather small as compared with $\dim P_n=(n-1)^2$. For the convex hull $Q_n$ of all cyclic permutations of $n$ elements that leave no element fixed, *H.* Kuhn performed computations showing that any two vertices of $Q_5$ but not any two vertices of $Q_6$ are neighbors of order 1, which means that $K(Q_5)=1$ and $K(Q_6)>1$. The present note, dealing with general $n$, proves, for $n \geqq 8$:

$$K(Q_n)=K(P_n)-1=\frac{n}{2}-1 \quad \text{if} \quad n=4m+2 \tag{1}$$

$$K(Q_n)=K(P_n)=\left[\frac{n}{2}\right] \quad \text{if} \quad n \neq 4m+2 \tag{2}$$

For $n=1, 2, \cdots 6, 7$, $K(Q_n)=0, 0, 1, 1, 1, 2, 2$ respectively.

**2. A permutation** $p$ of $n$ numbered elements is customarily represented by a matrix $(p_{ij})$, where

$$p_{ij}=\begin{cases} 1 & \text{when } p \text{ sends } i \text{ into } j \\ 0 & \text{otherwise.} \end{cases}$$

To the product of permutations then corresponds the product of the associated matrices under ordinary matrix multiplication, and therefore the same symbol will be used for a permutation and its matrix.

The following facts from [1] and [2] regarding neighbor relations on $P_n$ will be used in the sequal:

Received July 8, 1955. Work done under the sponsorship of the Office of Naval Research.

(2.1) $$K(P_n)=\left[\frac{n}{2}\right]$$

(2.2) $p_1$ and $p_2$ are neighbors of order $k$ on $P_n$ if and only if $p_1^{-1}p_2$ is a product of $k$ disjoint cycles (not counting cycles of length 1)

(2.3) If $c_1, c_2, \cdots, c_k$ are disjoint cycles and $F$ is the face of lowest dimension that contains the two vertices

$$p \text{ and } \bar{p}=pc_1c_2 \cdots c_k ,$$

then $F$ has the $2^k$ vertices

$$pc_{i_1}c_{i_2} \cdots c_{i_s} \qquad (0 \leqq s \leqq k) .$$

3. **If the vertices of a convex polyhedron** $Q$ are a subset of the vertices of a convex polyhedron $P$, let two vertices $q_1$, $q_2$ of $Q$ be neighbors of order $k$ on $P$ and $k^*$ on $Q$:

$$k=k(q_1, q_2 ; P) , \quad k^*=k^*(q_1, q_2 ; Q) .$$

Let

$$F=F(q_1, q_2 ; P) , \quad F^*=F^*(q_1, q_2 ; Q)$$

be the face of lowest dimension of $P$ respectively $Q$ that contains $q_1$ and $q_2$, so that

$$k=\dim A(F) , \quad k^*=\dim A(F^*) ,$$

where $A(F)$ and $A(F^*)$ denote the "affine span" of $F$ and $F^*$ respectively, which is also obtained as the intersection of all hyperplanes that support $P$ respectively $Q$ and contain $q_1$ and $q_2$ (with the understanding that $A$ is the entire space when such hyperplanes do not exist) ; then

(3.1) $$F \supseteqq F^* ,$$

hence

(3.2) $$A(F) \supseteqq A(F^*) ,$$

and therefore

(3.3) $$k \geqq k^* .$$

*Proof of* (3.1). The line segment joining $q_1$ and $q_2$ goes through the interior of $F^*$ (otherwise $q_1$ and $q_2$ would have a face of lower dimension in common). Therefore any hyperplane through $q_1$ and $q_2$ necessarily contains interior points of $F^*$.

Further, the vertices of $Q$, hence in particular those of $F^*$, are also vertices of $P$. Therefore any hyperplane that supports $P$ supports $F^*$.

Above establishes that any hyperplane $H$ that supports $P$ and contains $q_1$ and $q_2$ necessarily contains $F^*$, since it supports $F^*$ and contains points interior to $F^*$. Therefore

$$A(F) \supseteqq F^* ,$$

which, in conjunction with

$$P \supset Q \supset F^* ,$$

implies

$$F^* \subseteqq P \cap A(F) .$$

This completes the proof of (3.1), since the right hand side of the last relation equals $F$.

A somewhat sharper form of (3.1) may be noted as

LEMMA 1. *The vertices of $F^*$ are among the vertices of $F$.*

The proof is immediate from (3.1) and the fact that the vertices of $F^*$ are vertices of $P$, and a vertex of $P$ contained in $F$ is vertex of $F$.

From (3.3) it follows that $\max k^* \leqq \max k$, that is

$$K(Q) \leqq K(P) \tag{3.4}$$

**4.** At this point it is convenient to first establish some auxiliary facts. $p$, $q$, $c$ denote permutations of $n$ elements, for fixed $n$.

LEMMA 2. *If*

$$c_1,\ c_2,\ \cdots,\ c_r\ ,\ c_{r+1}\ ,\ \cdots,\ c_s$$

*is a set of $s$ disjoint cycles, and*

$$c' = c_1 c_2 \cdots c_r\ , \quad c'' = c_{r+1} c_{r+2} \cdots c_s$$

*then*

$$c' + c'' = I + c'c'' \tag{4.1}$$

*Proof*. Obvious (note that a cycle of less than $n$ elements is still represented as an $n$ by $n$ matrix, with 1's along the main diagonal for fixed elements).

LEMMA 3. *Under the assumptions of Lemma* 1, *let*

$$q,\ qc',\ qc'',\ qc'c''=\bar{q} \tag{4.2}$$

*be vertices of a polyhedron* $R$. *Then*

*a hyperplane* $H$ *through* $q$ *and* $\bar{q}$ *that supports* $R$ *contains* $qc'$ *and* $qc''$,

*and consequently*

$F(q, \bar{q}; R)$ *contains* $qc'$ *and* $qc''$ (*obviously as vertices*).

*This lemma will be used in the particular case where* $R=Q_n$ *or* $P_n$.

*Proof of Lemma* 3. Using parentheses to denote the inner product, let $H$, given by $(h, x)=\alpha$, contain $q$ and $\bar{q}$ but not contain $qc'$ (say); that is

$$(h, q)=(h, \bar{q})=\alpha\ , \quad (h, qc')=\alpha+\beta, \quad \beta\neq 0\ .$$

By (4.1) and (4.2)

$$qc'+qc''=q+\bar{q}\ ,$$

hence

$$(h, qc'')=(h, q+\bar{q}-qc')=2\alpha-(\alpha+\beta)=\alpha-\beta\ ,$$

so that $H$ separates $qc'$ from $qc''$ and therefore does not support $R$.

LEMMA 4. *If*

$$k=\left[\frac{n}{2}\right], \quad 2s\leqq k$$

$$q=(12\cdots n)$$

$$c_i=(i, i+k) \qquad (i=1, 2, \cdots k)\ ,$$

*then the product of* $q$ *with* $2s$ *distinct* $c_i$,

$$qc_{i_1}c_{i_2}\cdots c_{i_2}$$

is an $n$-cycle.

*Proof.* Since the $c_i$ are disjoint, they commute, and may be arranged in such manner that

$$i_1<i_2<\cdots<i_{2s}\ ;$$

then

$$\begin{aligned}
&(1\cdots n)(i_1,\ i_1+k)(i_2,\ i_2+k)\cdots(i_{2s-1},\ i_{2s-1}+k)(i_{2s},\ i_{2s}+k)\\
&=(1\cdots i_1,\ i_1+k+1,\ \cdots i_2+k,\ i_2+1,\ \cdots i_3,\ i_3+k+1,\ \cdots i_4+k,\ i_4+1\cdots\\
&\quad\cdots i_{2s-1},\ i_{2s-1}+k+1,\cdots i_{2s}+k,\ i_{2s}+1,\cdots\\
&\quad i_1+k,\ i_1+1,\cdots i_2,\ i_2+k+1,\cdots i_3+k,\ i_3+1,\cdots i_4,\ i_4+k+1,\cdots\\
&\quad\cdots i_{2s-1}+k,\ i_{2s-1}+1,\cdots i_{2s},\ i_{2s}+k+1,\cdots n)\ .
\end{aligned}$$

It is easily verified above relation also holds, with proper changes, for $i_1=1$ and for $2s=k,\ 2k=n$.

In similar straightforward fashion one easily proves:

LEMMA 5. *If $q$ is an $n$-cycle and $d$ is a 3-cycle, then $qd$ is an $n$-cycle if and only if the elements of $d$ occur in $q$ in the same cyclic order as in $d$.*

LEMMA 6. *If $q$ is an $n$-cycle and the 2-cycle $(ij)\neq(km)$, then $q(ij)(km)$ is an $n$-cycle if and only if the pair $i,j$ separates the pair $k$, $m$ in $q$.*

**5. The case** $n=4m,\ n=4m+1;\ m\geqq 2.$

$$K(Q_n)=K(P_n) \qquad (n=4m,\ 4m+1;\ m\geqq 2) \tag{5.1}$$

*Proof.* Because of (3.4), it is sufficient to show that $K(Q_n)\geqq K(P_n)$; this will be achieved by showing that for a particular pair of vertices $q,\ \bar{q}$

$$k(q,\ \bar{q};\ Q_n)\geqq\left[\frac{n}{2}\right]=K(P_n)\ . \tag{5.2}$$

Now let $2m=k$, so that $n\geqq 2k$, choose

$$\begin{cases} q=(12\cdots n) \\ c_s=(i,\ i+k) \qquad (i=1,2\cdots k) \\ \bar{q}=qc_1c_2\cdots c_k=qc\ , \end{cases} \tag{5.3}$$

and denote by $c'$ the product of an even number (including 0 and $k$) of the $c_i$, by $c''$ the product of the remaining $c_i$ (whose number is also even, since $k$ is even):

$$\begin{cases} c'=c_{i_1}c_{i_2}\cdots c_{i_{2s}} \qquad (0\leqq 2s\leqq k) \\ c'c''=c_1c_2\cdots c_k=c\ . \end{cases} \tag{5.4}$$

(It should be noted that the now following proof of $k^*(q, \bar{q}; Q_n) \geqq k$ does not depend on the special assumption $n=4m$, $4m+1$ and $k=2m$, but rather holds in general for any pair $n, k$, where $k$ is even and $n \geqq 2k$; this fact will be used in § 9).

The $qc'$ are vertices of $Q_n$ (by Lemma 4) and therefore (by Lemma 3) they are also vertices of $F^* = F(q, \bar{q}; Q_n)$.

To verify (5.2), that is

$$\dim A(F^*) \geqq k,$$

consider the following subset of $k+1$ vertices of $F^*$:

$$q_1 = qc_1c_1 = q, \quad q_2 = qc_1c_2, \cdots q_k = qc_1c_k, \quad q_{k+1} = qc = \bar{q}. \tag{5.5}$$

The $q_i$ of (5.5) are linearly independent.

*Proof.* Assume

$$\lambda qc + \sum_{i=1}^{k} \lambda_i q_i = 0. \tag{5.6}$$

Successive application of (4.1) to

$$c = c_1 c_2 \cdots c_k$$

yields

$$c = c_1[c_2 + \cdots + c_k - (k-2)I], \tag{5.7}$$

and (5.6) becomes

$$\lambda qc_1[c_2 + \cdots + c_k - (k-2)I] + \sum_{i=1}^{k} \lambda_i qc_1c_i = 0$$

that is

$$qc_1[\lambda_1 c_1 - \lambda(k-2)I + \sum_{i=2}^{k} (\lambda_i + \lambda)c_i] = 0$$

or, equivalently, since $q$ and $c_1$ are nonsingular matrices

$$\lambda_1 c_1 - \lambda(k-2)I + \sum_{i=2}^{k} (\lambda_i + \lambda)c_i = 0 \tag{5.8}$$

Since the $c_i$ are disjoint cycles (5.8) implies

$$\lambda_1 = 0; \quad \lambda_i + \lambda = 0 \quad (i=2, \cdots k); \quad \lambda(k-2) = 0$$

which, in conjunction with $k \neq 2$ (following from $m \geqq 2$), further implies

$$\lambda = 0, \quad \lambda_i = 0.$$

This verifies that the $k+1$ $q_i$ of (5.5) are linearly independent, so that the dimension of their linear span is $k+1$, and therefore the dimension of their affine span equal to $k$. This completes the proof of (5.2) and hence of (5.1)

**6. The case** $n=4m$, $n=4m+1$; $m=1$. Removing the restriction $m \geqq 2$ in (5.1) leaves the cases $n=4$ and $n=5$ still to be considered

(6.1) $$K(Q_n)=1 \qquad (n=4, 5)$$

*Proof.* Since, by (3.4) and (2.1), $K(Q_n) \leqq 2$, one only has to show that $K(Q_n) \neq 2$.

Assume there were two vertices $q$ and $\bar{q}$ of $Q_n$ such that

$$k^*(q, \bar{q}\,;\, Q_n)=2\,.$$

Then, by (3.4), (3.3) and (2.1)

$$k(q, \bar{q};\, P_n)=2\,,$$

which by (2.2) implies that $q^{-1}\bar{q}$ is a product of two disjoint cycles, say $c_1$, $c_2$, so that $\bar{q}=qc_1c_2$.

Since $q$ and $\bar{q}$ are cycles of the same length (namely $n$), $c_1c_2$ is necessarily an even permutation, so that $c_1$ and $c_2$ are both of length 2.

Now let $F$ be the lowest dimensional face of $P_n$ containing $q$ and $\bar{q}$. Then, by (2.3), $F$ has the 4 vertices

$$q,\, \bar{q},\, qc_1,\, qc_2\,.$$

of which the last two are not $n$-cycles and therefore not vertices of $F^*$. Hence, by Lemma 1, $F^*$ has only the two vertices $q$ and $\bar{q}$, which implies $k^*=1$ in contradiction to the assumption that $k^*=2$. This completes the proof of (6.1).

**7. The case** $n=4m+3$; $m \neq 1$.

(7.1) $$K(Q_n)=K(P_n) \qquad (n=4m+3,\ m\neq 1)\,,$$

including $m=0$.

*Proof.* Because of (3.4) it is again sufficient to point out two vertices, $q$, $\bar{q}$, of $Q_n$, such that

(7.2) $$k^*(q, \bar{q};\, Q_n) \geqq K(P_n)=2m+1\,.$$

For $k=2m$, let $q$, $c_i$, $c$, $c'$, $c''$ be defined as in (5.3) and (5.4), let $d=(2k+1,\ 2k+2,\ 2k+3)$, and $\bar{q}=qcd$.

By Lemmas 4 and 5 the $qc'$ and $qc'd$ are vertices of $Q_n$ for all $c'$ of (5.4), and by Lemma 3 they are also vertices of $F^*(q, \bar{q}; Q_n)$. To prove that

$$\dim A(F^*) \geqq 2m+1 ,$$

it is shown that the dimension of the linear span of $F^*$ is $\geqq 2m+2 = k+2$, in verifying that the $k+2$ vertices of $F^*$

$$q_1 = q = qc_1c_1,\ q_2 = qc_1c_2,\ \cdots,\ q_k = qc_1c_k,\ q_{k+1} = qd, \quad q_{k+2} = \bar{q} = qcd \tag{7.3}$$

are linearly independent.

Assume

$$\sum_{i=1}^{k+2} \lambda_i q_i = 0 \tag{7.4}$$

or, equivalently, substituting for $q_i$ their expressions from (7.3), omitting the non singular common factor $qc_1$, and writing $\mu_i$ for $\lambda_{k+i}$,

$$\sum_{i=1}^{k} \lambda_i c_i + \mu_1 c_1 d + \mu_2 c_2 c_3 \cdots c_k d = 0 . \tag{7.5}$$

Application of (4.1) yields for the left hand side of (7.5)

$$\sum_{i=1}^{k} \lambda_i c_i + \mu_1(c_1 + d - I) + \mu_2[c_2 + \cdots + c_k + d - (k-1)I] ,$$

so that (7.4) is equivalent to

$$(\lambda_1 + \mu_1)c_1 + \sum_{i=2}^{k} (\lambda_i + \mu_2)c_i + (\mu_1 + \mu_2)d - [\mu_1 + (k-1)\mu_2]I = 0 \tag{7.6}$$

Since the $c_i$ and $d$ are disjoint cycles, (7.6) implies

$$\begin{cases} \lambda_1 + \mu_1 = 0 & \\ \lambda_i + \mu_2 = 0 & (i = 2, 3, \cdots k) \\ \mu_1 + \mu_2 = 0 & \\ \mu_1 + (k-1)\mu_2 = 0 & \end{cases} \tag{7.7}$$

The last two relations of (7.7) imply (because of the assumption $m \neq 1$, hence $k \neq 2$, $k-1 \neq 1$)

$$\mu_1 = \mu_2 = 0 ,$$

which in conjunction with the first two relations of (7.7) implies

$$\lambda_i = 0 \qquad (i = 1, 2, \cdots k) ,$$

so that all coefficients of (7.4) vanish; this proves that the $q_i$ of (7.4) are linearly independent, and completes the proof of (7.2) and hence (7.1).

**8. The case** $n=7$ (excepted in § 7).

(8.1) $$K(Q_7)=K(P_7)-1=2$$

*Proof*. By (3.4) and (2.1)

$$K(Q_7)\leqq 3\,.$$

To see that equality cannot hold, let $q=(12\cdots 7)$.

Because of (2.1) and (3.3), only such $\bar{q}$ must be considered where

$$k(q,\,\bar{q}\,;\,P_7)=3\,.$$

By (2.2) the last relation is only possible for

$$\bar{q}=qc_1c_2d\,,$$

where $c_1$, $c_2$, $d$ are disjoing cycles.

For $\bar{q}$ to be a 7-cycle it is necessary (not sufficient) that $c_1c_2d$ be even, that is, that two of them, say $c_1$ and $c_2$, be transpositions and $d$ a 3 cycle.

For the same reason, among the 8 vertices of $F(q,\,\bar{q}\,;\,P_7)$ determined by (2.3), at most 4 are 7-cycles, namely

(8.2) $$q_1=q,\ q_2=qc_1c_2,\ q_3=qd,\ q_4=\bar{q}=qc_1c_2d\,,$$

so that, by Lemma 1, $F^*(q,\,\bar{q}\,;\,Q_7)$ has at most the 4 vertices (8.2).

However, application of (4.1) yields

$$q_1+q_4=q(I+c_1c_2d)=q(I+c_1c_2+d-I)=q_2+q_3$$

which is a relation

$$\Sigma\,\lambda_ic_i=0 \quad \text{with} \quad \Sigma\,\lambda_i=0\,,$$

therefore

$$\dim A(F^*)\leqq 2\,.$$

It has thus been established that

$$K(Q_7)\leqq 2\,.$$

To complete the proof of (8.1), choose

(8.3) $$q=(12\cdots 7),\ c_1=(13),\ c_2=(24),\ d=(567)\,.$$

Then each $q_i$ of (8,2) is a 7-cycle (by Lemmas 4 and 5) and a vertex of $F^*(q, \bar{q}; Q_7)$ (by Lemma 3.) The last 3 of these $q_i$ are linearly independent. This establishes, for this particular face $F^*$,

$$\dim A(F^*)=2\,,$$

and completes the proof of (8.1).

**9. The case** $n=4m+2$.

$$K(Q_n)=K(P_n)-1=2m \qquad (n=4m+2). \tag{9.1}$$

The proof is achieved in showing

$$K(Q_n)\leqq K(P_n)-1=2m \tag{9.2}$$

$$K(Q_n)\geqq K(P_n)-1=2m\,. \tag{9.3}$$

To verify (9.2), assume $K(Q_n)>K(P_n)-1$, which, by (3.4) and (2.1), implies $K(Q_n)=K(P_n)=2m+1$.

Then there must be a pair of vertices $q$ and $\bar{q}$ on $Q_n$ such that

$$k^*(q, \bar{q}; Q_n)=2m+1\,,$$

and hence, by (3.3) and (2.1),

$$k(q, \bar{q}; P_n)=2m+1\,,$$

which, by (2.2) implies

$$\bar{q}=qc_1c_2\cdots c_{2m+1}\,,$$

where the $c_i$ are disjoint cycles, and therefore necessarily transpositions, because of $n=2(2m+1)$. Then however, the product of the $c_i$ is an odd permutation, and $\bar{q}$ cannot be an $n$-cycle if $q$ is one. This proves (9.2).

To verify (9.3), consider first the case $m\geqq 2$. Setting $2m=k$, the construction from (5.3) through the end of § 5 proves the existence of $q, \bar{q}$ with $k^*(q, \bar{q}; Q_n)=k$, which implies $K(Q_n)\geqq k$.

For $m=1$, that is, $n=6$, choose

$$q=(12\cdots 6),\ d_1=(123),\ d_2=(456),\ \bar{q}=qd_1d_2\,.$$

Then, by Lemma 5, the 4 points

$$q,\ qd_1,\ qd_2,\ \bar{q}=qd_1d_2$$

are 6-cycles, and therefore, by Lemma 3, vertices of

$$F^*(q, \bar{q}; Q_6)\,.$$

This implies $\dim A(F^*)\geqq 2$ (since not more than two vertices can be on a line), that is,

$$k^*(q, \bar{q};\ Q_6) \geqq 2\,.$$

Finally (if one wants to split hairs) for $m=0$, that is, $n=2$, (9.3) amounts to asserting the existence of at least one 2-cycle; for $q=\bar{q}=$ (12), $F^*(q, \bar{q};\ Q_2)=q$, $k^*=0$, hence $K(Q_2)\geqq 0$. This completes the proof of (9.1).

The relations (5.1), (6.1), (7.1), (8.1), and (9.1) constitute the statement at the end of § 1.

## REFERENCES

1. I. Heller, *Geometric characterization of cyclic permutations*, (Abstract), Bull. Amer. Math. Soc., 1955.

2. ———, *On the travelling salesman's problem*, Proc. Second symposium on Linear Programming, Washington, D.C., January 29, 1955.

3. T. S. Motzkin, *The assignment problem*, Proc. 6th Appl. Math. Symposium, McGraw-Hill, 1955.

THE GEORGE WASHINGTON UNIVERSITY