NON-RECURRENT RANDOM WALKS

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Introduction and Summary. Let $\{X_i\}$ $i=1, 2, \cdots$ be a sequence of independent and identically distributed integral valued random variables such that 1 is the absolute value of the greatest common divisor of all values of x for which $P(X_i=x)>0$. Define

$$S_n=\sum_{i=1}^{n} X_i .$$

Chung and Fuchs [5] showed that if x is any integer, $S_n=x$ infinitely often or finitely often with probability 1 according as $EX_i=0$ or $\neq 0$, provided that $E|X_i|<\infty$. Let $0<EX_i<\infty$, and A denote a set of integers containing an infinite number of positive integers. It will be shown that any such set A will be visited infinitely often with probability 1 by the sequence $\{S_n\}$ $n=1, 2, \cdots$. Conditions are given so that similar results hold for the case where X_i has a continuous distribution and the set A is a Lebesgue measurable set whose intersection with the positive real numbers has infinite Lebesgue measure.

A Theorem about Markov Chains. Let $\{Z_n\}$, $n=0, 1, \cdots$ denote a Markov chain with stationary transition probabilities where each Z_n takes on values in an abstract state space $\boldsymbol{X}$. The distribution of Z_0 is given but arbitrary. Let Ω denote the space of all possible sample sequences w, P the probability measure over Ω and $P(\cdot|\cdot)$ the conditional probability. The following theorem appears in [4].

THEOREM 1. *Let A be any event in $\boldsymbol{X}$. A sufficient condition that*

$$P(Z_n \in A \textit{ infinitely often})=1 \tag{1}$$

is

$$\inf_{z\in X} P(Z_n \in A \textit{ for some } n|Z_0=z)>0 . \tag{2}$$

Since [4] is not readily accessible, we shall prove the theorem here.

Proof.[2] We have with probability 1 that for $j \geqq N$

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[2] The proof given here is a modification of one suggested by J. Wolfowitz.

$$(3)\qquad \begin{aligned} &P(Z_n \in A \text{ for some } n \geqq N \mid Z_0 = z_0, \cdots, Z_j = z_j) \\ &\geqq P(Z_n \in A \text{ for some } n > j \mid Z_0 = z_0, \cdots, Z_j = z_j) \\ &= P(Z_n \in A \text{ for some } n \mid Z_0 = z_j) \end{aligned}$$

using the Markovian and stationarity properties. As $j \to \infty$ the left member of (3) approaches with probability 1 the characteristic function b_N of the event

$$B_N = \{Z_n \in A \text{ for some } n \geqq N\}$$

(see Doob [8, p. 332]). The right member of (3) is bounded below by a positive number on account of (2). Hence $b_N = 1$ with probability 1; that is, $P(B_N) = 1$. This being true for all N we have

$$P(\lim_{N\to\infty} B_N) = \lim_{N\to\infty} P(B_N) = 1 .$$

But $\lim_{N\to\infty} B_N$ is the event that $Z_n \in A$ infinitely often. This proves the theorem.

If X has only a denumerable number of states and if all the states belong to the same class (that is, for every pair of states i and j there exists integers n_1 and n_2 such that $P(Z_{n_1} = j \mid Z_0 = i) P(Z_{n_2} = i \mid Z_0 = j) > 0$) it can be easily seen that (2) is both a necessary and sufficient condition for (1). In fact, the probability in (2) must be 1 for all states z.[3]

Sums of lattice random variables. Let $\{X_i\}$ $i = 1, 2, \cdots$ be a sequence of independent and identically distributed integral valued random variables such that 1 is the absolute value of the greatest common divisor of all values of x for which $P(X_i = x) > 0$. Consider the sequence $\{S_n\}$ $n = 0, 1, \cdots$, where we set $S_0 = 0$ with probability 1 and

$$S_n = S_0 + \sum_{i=1}^{n} X_i .$$

The sequence $\{S_n\}$ is then a Markov chain with stationary transition probabilities and a denumerable state space. Because the transition probabilities are stationary, we shall simply write

$$P(S_{n+m} = i \mid S_n = j) = P(S_m = i \mid S_0 = j)$$

even though $S_0 = 0$ with probability 1.

We now state as lemmas some known results to be used below.

LEMMA 1. *Let $\{Z_n\}$ $n = 0, 1, \cdots$ be a Markov chain with a denumerable state space. If $\sum_{n=1}^{\infty} P(Z_n = j \mid Z_0 = i) < \infty$ for all i and j, then*

[3] We are indebted to J. Wolfowitz for this remark.

$$(4)\qquad P(Z_n=j \text{ for some } n \mid Z_0=i)=\frac{\sum_{n=1}^{\infty} P(Z_n=j \mid Z_0=i)}{1+\sum_{n=1}^{\infty} P(Z_n=j \mid Z_0=j)}\,.$$

When $EX_i=\mu>0$, a result of Chung and Fuchs [5] implies that

$$(5)\qquad \sum_{n=1}^{\infty} P(S_n=j \mid S_0=i)<\infty$$

for all i and j. Therefore, on replacing Z_n by S_n in (4) and noting that $P(S_n=j \mid S_0=j)=P(S_n=0 \mid S_0=0)$ we have

$$(4')\qquad P(S_n=j \text{ for some } n \mid S_0=i)=\frac{\sum_{n=1}^{\infty} P(S_n=j \mid S_0=i)}{1+\sum_{n=1}^{\infty} P(S_n=0 \mid S_0=0)}$$

Lemma 1 is a special case of a relation given by Doeblin [7] (see Chung [3]). However, we shall sketch a direct proof.

Proof. We define $P(Z_0=j \mid Z_0=j)=1$. Then we have

$$(6)\qquad \begin{aligned} P(Z_n=j \mid Z_0=i) &= \sum_{m=1}^{n} P(Z_m=j,\ Z_r\neq j \text{ for } 1\leqq r<m \mid Z_0=i) P(Z_n=j \mid Z_m=j) \\ &= \sum_{m=1}^{n} P(Z_m=j,\ Z_r\neq j \text{ for } 1\leqq r<m \mid Z_0=i) P(Z_{n-m}=j \mid Z_0=j) \end{aligned}$$

On summing over n in (6) and interchanging summations on the right we get

$$(7)\qquad \begin{aligned} \sum_{n=1}^{\infty} P(Z_n=j \mid Z_0=i) &= \sum_{m=1}^{\infty} P(Z_m=j, Z_r\neq j \text{ for } 1\leqq r<m)\Big(1+\sum_{n=1}^{\infty} P(Z_n=j \mid Z_0=j)\Big) \\ &= P(Z_n=j \text{ for some } n)\Big(1+\sum_{n=1}^{\infty} P(Z_n=j \mid Z_0=j)\Big)\,, \end{aligned}$$

the relation (4).

LEMMA 2. *If* $EX_i=\mu>0$, *then*

$$(8)\qquad \lim_{j\to\infty}\sum_{n=1}^{\infty} P(S_n=j \mid S_0=i) \begin{cases} =\dfrac{1}{\mu}>0\,, & \mu<\infty \\ =0\,, & \mu=+\infty\,. \end{cases}$$

Lemma 2 is due to Chung and Wolfowitz [6]. We now prove the following.

THEOREM 2. (i) *If* $0 < EX_i = \mu < \infty$ *and* A *is any set containing an infinite number of positive integers, then* $S_n \in A$ *infinitely often with probability* 1.

(ii) *If* $EX_i = +\infty$, *then there exists a set* A *containing an infinite number of positive integers such that* $S_n \in A$ *only finitely often with probability* 1.

Proof of (i). Since $0 < \mu < \infty$, by (8) there exists a constant $c > 0$, independent of i, and an integer $J(i)$ such that for all $j > J(i)$

$$\sum_{n=1}^{\infty} P(S_n = j \mid S_0 = i) > c. \tag{9}$$

Therefore by (4′) and (5)

$$P(S_n = j \text{ for some } n \mid S_0 = i) > \frac{c}{1+c'}, \qquad j > J(i) \tag{10}$$

where $c' = \sum_{n=1}^{\infty} P(S_n = 0 \mid S_0 = 0) < \infty$. Since A contains infinitely many positive integers, it always contains an integer greater than $J(i)$ for every i. Therefore (2) holds and part (i) of Theorem 2 follows from Theorem 1.

Proof of (ii). If $\mu = +\infty$, then from (8) there exists an increasing subsequence $\{i_j\}$ of positive integers such that

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} P(S_n = i_j \mid S_0 = 0) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P(S_n = i_j \mid S_0 = 0) < \infty. \tag{11}$$

Let $A = \{i_j\}$. Now (11) is the expected number of n such that $S_n \in A$. Since this expectation is finite it follows that the number of n such that $S_n \in A$ is finite with probability 1. This completes the proof of the theorem.

Random variables with continuous distribution functions. Consider now a sequence $\{X_i\}$ $i = 1, 2, \cdots$ of independent, identically distributed random variables possessing a common density function $f(x)$. Again let $\{S_n\}$ $n = 0, 1, \cdots$ denote the cumulative sums $S_n = S_0 + \sum_{i=1}^{n} X_i$ where $S_0 = 0$ with probability 1. Our previous remark pertaining to the notation $P(\cdots \mid S_0 = x)$ applies here also. Suppose $EX_i = \mu > 0$. Then a result of Chung and Fuchs [5] implies that $H(x) = \sum_{n=1}^{\infty} P(S_n \leq x) < \infty$ for all x. Since $H(x)$ is non-decreasing, $H'(x)$ exists everywhere except on a set N_0 of Lebesque measure zero. Let

$$\begin{aligned} h(x) &= H'(x) && x \notin N, \\ &= 1, \text{ say}, && x \in N, x \geq 0 \\ &= 0 && x \in N, x < 0 \end{aligned}$$

We shall say that $f(x)$ satisfies condition I if there exist constants K_1 and K_2 such that

$$0 < K_1 \leq \varliminf_{x\to\infty} h(x) \leq \varlimsup_{x\to\infty} h(x) \leq K_2 < \infty \tag{12}$$

and if

$$\lim_{x\to-\infty} h(x) = 0 \tag{13}$$

The behavior of $h(x)$ for large $|x|$ has been investigated in various papers on renewal theory. Smith [10], for example, has shown that if $f(x)=0$ for $x<0$, $f(x)\to 0$ as $|x|\to\infty$ and $f(x)\varepsilon L_{1+\delta}$ for some $\delta>0$, then

$$\begin{aligned} \lim_{x\to\infty} h(x) &= \frac{1}{\mu}, && \mu<\infty \\ &= 0, && \mu=+\infty \end{aligned}$$

More recently, Smith[4] has shown that the condition that $f(x)=0$ for $x<0$ may be dropped, and furthermore (13) holds. We now prove the following.

LEMMA 3. *If* $EX_i=\mu<\infty$, $f(x)$ *satisfies condition I, A is any Lebesgue measurable set of positive real numbers having infinite measure, then*

$$\inf_{-\infty<x<\infty} P(S_n \in A \text{ for some } n \mid S_0=x) > 0 . \tag{14}$$

Proof. For every x, let A_x be a measurable subset of A with $0<c_1<m(A_x)<c_2<\infty$ and such that for a given number L_1 all points in A_x exceed x by at least L_1. Such a set exists since $m(A)=\infty$. For any $\epsilon>0$ it follows from (12) that there exists an $L_1=L_1(\epsilon)$ such that

$$0<(1-\epsilon)K_1c_1<\sum_{n=1}^{\infty} P(S_n\in A_x\mid S_0=x)<(1+\epsilon)K_2c_2<\infty . \tag{15}$$

Let A'_x be any measurable set with $m(A'_x)\leq c_2$ and such that for a given L_2 all points in A'_x are exceeded by x by at least L_2. By (13)[5] there exists an $L_2=L_2(\epsilon)$ such that

$$\sum_{n=1}^{\infty} P(S_n\in A'_x\mid S_0=x)<\varepsilon . \tag{16}$$

[4] Communication by letter.

[5] Added in proof: Condition (13) can be dropped; (16) follows from the fact that $\lim_{x\to-\infty} H(x)=0$ whether (13) holds or not.

Let $L=\max(L_1, L_2)$. For a given $y \in A_x$ let $A^1_{xy}=A_x \cap [y-L, y+L)$, $A^2_{xy}=A_x \cap [y+L, \infty)$ and $A^3_{xy}=A_x \cap (-\infty, y-L)$.
Then from (15) and (16)

$$\begin{aligned}(17) \qquad \sum_{n=1}^{\infty} P(S_n \in A_x \mid S_0=y) &= \sum_{n=1}^{\infty} P(S_n \in A^1_{xy} \mid S_0=y) \\ &+ \sum_{n=1}^{\infty} P(S_n \in A^2_{xy} \mid S_0=y) + \sum_{n=1}^{\infty} P(S_n \in A^3_{xy} \mid S_0=y) \\ &\leqq \sum_{n=1}^{\infty} P(-L<S_n<L \mid S_0=0) + K_2 c_2(1+\varepsilon) + \varepsilon .\end{aligned}$$

The first term on the right of (17) is finite by the result of Chung and Fuchs [5]. Therefore, since (17) is true for all $y \in A_x$ we have

$$(18) \qquad \sup_{y \in A_x} \sum_{n=1}^{\infty} P(S_n \in A_x \mid S_0=y) < c_3 < \infty$$

Let $F^{(v)}_x(B)=P(S_v \in B, S_{v'} \notin A_x \text{ for } 1 \leqq v' < v \mid S_0=x)$ where B is any measurable subset of A_x. Define $P(S_0 \in A_x \mid S_0=y)=1$ if $y \in A_x$ and $=0$ otherwise. Then we have

$$\begin{aligned}\sum_{n=1}^{N} P(S_n \in A_x \mid S_0=x) &= \sum_{n=1}^{N} \sum_{v=1}^{n} \int_{A_x} P(S_n \in A_x \mid S_v=y) F^{(v)}_x(dy) \\ &= \sum_{v=1}^{N} \int_{A_x} \sum_{n=v}^{N} P(S_n \in A_x \mid S_v=y) F^{(v)}_x(dy) \\ &\leqq \sum_{v=1}^{N} \int_{A_x} \sum_{n=0}^{\infty} P(S_n \in A_x \mid S_0=y) F^{(v)}_x(dy) \\ &\leqq \sum_{v=1}^{N} F^{(v)}_x(A_x) \sup_{y \in A_x} \sum_{n=0}^{\infty} P(S_n \in A_x \mid S_0=y) \\ &\leqq P(S_n \in A_x \text{ for some } n \mid S_0=x)(1+c_3) .\end{aligned}$$

This being true for all N the lemma follows on account of (15).

We now state the following.

THEOREM 3. (i) *If* $0<EX_i=\mu<\infty$, *Condition I is satisfied, and* A *is any Lebesgue measurable subset of the positive real numbers, then* $S_n \in A$ *infinitely often or finitely often with probability* 1 *according as* $m(A)=\infty$ *or* $<\infty$.

(ii) *If* $\mu=\infty$, *then there exists a measurable subset* A *of the positive real numbers with* $m(A)=\infty$ *such that* $S_n \in A$ *for only finitely many* n *with probability* 1.

Proof of (i). If $m(A)=\infty$, the result follows from Theorem 1 and Lemma 3. If $m(A)<\infty$ it follows from (15) that $\sum_{n=1}^{\infty} P(S_n \in A) < \infty$.

Since that is the expected number of n such that $S_n \in A$, the assertion follows immediately.

Proof of (ii). A result due to Blackwell [1] asserts that for any fixed $d > 0$.

$$\lim_{y\to\infty} \sum_{n=1}^{\infty} P(y \leq S_n \leq y+d) = 0 .$$

Using this result the rest of the proof is similar to that of part (ii) Theorem 2.

Unsolved problems. Let $\{X_i\}$ be a sequence of independent and identically distributed r-dimensional random vectors, $S_n = \sum_{i=1}^{n} X_i$, B be any Borel set in the r-dimensional Euclidean space R^r. It has been recently proved by Hewitt and Savage [9] (in the lattice case also by Blackwell [2]) that the probability that $S_n \in B$ infinitely often is necessarily either 0 or 1. It would be of interest to determine for which sets the probability is 0, and for which the probability is 1. Our results give a criterion for this dichotomy in certain cases in R^1, namely in the lattice case where EX_i exists and is finite (Theorem 2) and in the continuous case under more restrictive conditions (Theorem 3).

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