THE PROBLEM OF CONTINUOUS PROGRAMS

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1. Introduction. In a discrete programming problem one selects a policy at specified times which governs the behavior of some process during the succeeding time intervals; the problem is to find that *program*, that is, sequence of choices of policy, which maximizes the value of some pre-assigned functional associated with the process. It is of interest to learn how the values of the functional behave when policy-making decisions are required more and more frequently.

As an example of a discrete programming problem, suppose an investor re-distributes a fixed capital investment among N related businesses once a week. The income rate Q_i^k of the kth business during the ith week $[t_i, t_{i+1})$ depends on the *income* $q_i=(q_i^1, \cdots, q_i^N)$ up to the beginning of the week, where q_i^k is the income of the kth business, on the *policy*, that is, distribution of capital for the week, and on the time t_i. Suppose further that the businesses are risky in that if one fails all fail, and that the probability $P_i(t_{i+1}-t_i)$, of failure during a given week, assuming the businesses exist at the beginning of the week, depends on the policy for the week and the time of year. Setting $Q_i=(Q_i^1, \cdots, Q_i^N)$ and letting p_i represent the probability of survival up to time t_i, it is clear that q_i and p_i satisfy difference equations, stated more explicitly in § 2,

$$q_{i+1}-q_i=Q_i(t_{i+1}-t_i) \tag{1.1}$$

$$p_{i+1}-p_i=-p_iP_i(t_{i+1}-t_i) \ , \tag{1.2}$$

in which the right-hand sides at times t_i depend on q_i, p_i and a policy, which we represent as a point of the set X of all possible distributions of capital. The investor's programming problem is to select a policy for every week of the year which will maximize the expected total income

$$f=\sum_i p_i||Q_i||(t_{i+1}-t_i) \tag{1.3}$$

of all the businesses, where

$$||Q_i||=\sum_{i=1}^{N} |Q_i^k| \ . \tag{1.4}$$

It is assumed that he does not care what happens after the year is over.

Received September 9, 1955, and in revised form December 16, 1955.

A similar example, the "gold-mining problem" is discussed in [**1**–**4**, **7**, **8**]. More realistic examples can easily be constructed in which p is interpreted as the *efficiency* of the process rather than the survival probability.

The question of interest in the present paper is that of the behavior of the l.u.b. of the investor's expected total income as he shifts his capital around more and more frequently rather than just once a week, assuming that Q and P are defined for all t.

The method of investigation is to introduce *continuous programs*, in which the policy-maker is permitted to change his policy at any time he chooses. In this case q and p are computed by differential equations corresponding to (1.1) and (1.2) where Q and P depend in any interval between policy changes on the policy chosen for that interval.

The principal result, given in § 7, is that for functions Q and P satisfying hypotheses given in § 3 the sequence of l.u.b.'s for the approximating discrete programming problems converges, and that the limit is the l.u.b. of the corresponding functional in the continuous case.

2. Definitions. We shall state the programming problem more explicitly, starting with the continuous case. A policy-maker divides the closed-open unit interval $I=[0, 1)$ into a finite number of closed-open sub-intervals $[0, t_1)$, $[t_1, t_2)$, $\cdots$ and associates with each sub-interval a *policy* from an arbitrary set X. Thus a *program on* I is a step-function π on I with values in X. Suppose a real-valued function P on X and I is given such that if the policy-maker has survived during $[0, t)$ the probability that he will not survive $[t, t+\Delta)$ is $P(x, t)\Delta+o(\Delta)$, where x is the policy associated with the sub-interval of I containing t. Let B be a given Banach space, whose points will be called *incomes*. Suppose that a B-valued function Q on B, X, and I is given such that $Q(b, x, t)\Delta+o(\Delta)$ is the income earned during $[t, t+\Delta)$, where b will be an income earned during $[0, t)$ and x is given as before. Let $p(\pi, t)$ represent the probability of survival and $q(\pi, t)$ the income earned during $[0, t)$ for the program π. Then these functions satisfy the equations

$$p(\pi, t)=1-\int_0^t P(\pi(s), s)p(\pi, s)ds \ , \tag{2.1}$$

$$q(\pi, t)=q(0)+\int_0^t Q(q(\pi, s), \pi(s), s)ds \ . \tag{2.2}$$

The continuous programming problem is to determine a program on I which makes the expected total income

$$g(\pi)=\int_0^1 p(\pi, s)\|dq(\pi, s)\|=\int_0^1 \|p(\pi, s)Q(q(\pi, s), \pi(s), s)\|ds \tag{2.3}$$

as large as possible. We set

$$G = \sup_{\pi \text{ on } I} g(\pi) . \tag{2.4}$$

Let E_j represent a finite subset of I with points $t_0=0, t_1, \cdots, t_{n_j}$, $(t_i < t_{i+1} < 1)$. A *program on* E_j is a function on E_j with values in X which is constant in the closed-open sub-intervals of I with initial points on E_j. If the policy-maker has survived during $[0, t_i)$ the probability that he will not survive the interval $[t_i, t_{i+1})$ is taken as $P(x, t_i)(t_{i+1}-t_i)$ in the discrete case, where x is the policy associated with the sub-interval. Let $Q(b, x, t_i)(t_{i+1}-t_i)$ be the income earned during the same period, where b will be an income earned during $[0, t_i)$, with x as before. Let $p_j(\pi, t)$ represent the probability of survival and $q_j(\pi, t)$ the income earned during $[0, t)$ for the program π on E_j. Then these functions satisfy the equation

$$\begin{aligned} p_j(\pi, t_i) &= 1 - \sum_{h=0}^{i-1} P(\pi(t_h), t_h) p_j(\pi, t_h)(t_{h+1}-t_h) \\ &= 1 - \int_0^{t_i} P(\pi(s), s) p_j(\pi, s) dm_j(s) \end{aligned} \tag{2.5}$$

$$\begin{aligned} q_j(\pi, t_i) &= q(0) + \sum_{h=0}^{i-1} Q(q_j(\pi, t_h), \pi(t_h), t_h)(t_{h+1}-t_h) \\ &= q(0) + \int_0^{t_i} Q(q_j(\pi, s), \pi(s), s) dm_j(s) , \end{aligned} \tag{2.6}$$

where the meaning of the *discrete measure* $m_j(s)$ is clear. The discrete programming problem associated with E_j is to determine a program on E_j which makes the expected total income

$$f_j(\pi) = \int_0^1 \| p_j(\pi, s) Q(q_j(\pi, s), \pi(s), s) \| dm_j(s) \tag{2.7}$$

as large as possible. We set

$$F_j = \sup_{\pi \text{ on } E_j} f_j(\pi) . \tag{2.8}$$

Consider a sequence $\{E_j\}$ of sets E_j such that $E_{j+1} \supset E_j$ and $\bigcup_{j=1}^{\infty} E_j = E_*$, where E_* is everywhere dense in I. We shall show in § 6 and § 7 respectively that the sequence of discrete programming problems associated with $\{E_j\}$ approximates the continuous programming problem for any P and Q satisfying the conditions of § 3 in the sense that the sequences $\{f_j(\pi)\}$ and $\{F_j\}$ converge and that

$$\lim_{j \to \infty} f_j(\pi) = g(\pi) \tag{2.9}$$

and

$$\lim_{j\to\infty} F_j = G . \tag{2.10}$$

3. Assumptions. Several restrictions on P and Q are necessary in order to guarantee the existence of appropriate families of solutions to (2.1), (2.2), (2.5) and (2.6). We assume

(3.1)
(i) P is a positive uniformly continuous function on I, the modulus of continuity being uniform in X.
(ii) P is bounded by a constant $\overline{P}$ independent of X.

and

(3.2)
(i) Q is a uniformly continuous function on I, the modulus of continuity being uniform in X.
(ii) Q satisfies the Lipschitz condition $\|Q(b, x, t)-Q(b', x, t)\| \leqq \overline{Q}\|b-b'\|$ where $\overline{Q}$ is a constant.

The Picard theorem and the condition of *uniform* continuity on Q imply that for any program π mapping I into a fixed element X^* of X there exists unique function $q^*(t)=q(\pi, t)$ of I which satisfies (2.2) everywhere in I. We add the restriction

(3.2) (iii) There exists an $X^* \in X$ and a constant C such that

$$\|Q(q^*(t), x, t)-Q(q^*(t), x^*, t)\| \leqq C$$

uniformly for $x \in X$ and $t \in I$.

4. Uniform boundedness. We shall need some kind of uniform bound on Q in §§ 5 and 6. One cannot merely assume that a bound exists for all b, x, and t since Q only satisfies a Lipschitz condition in b, and might be linear, for example. One might assume a uniform bound only for those values of b which can possibly arise in I for any admissible program. This assumption could easily be verified in special cases; the best way of introducing it in general is by imposing the restriction (3.2) (iii). We shall show that this restriction indeed gives the required uniform bound.

First we state a familiar lemma which will be used several times. For a proof see [5].

LEMMA 1. *If $u(t) \geqq 0$ and $v(t) \geqq 0$ and A is a positive constant such that*

$$u(t) \leqq A + \int_0^t u(s)v(s)ds \tag{4.1}$$

then

$$u(t) \leq A \exp \int_0^t v(s)ds \ . \tag{4.2}$$

It should be noted that there is no trouble in verifying the existence of solutions of (2.1). For in between the steps of π the usual existence theorem applies. At the beginning of each new step one merely uses the initial values obtained at the end of the preceding step, which does not affect the convergence. A similar comment holds for (2.2).

LEMMA 2. *If π is any program on I, then there exists a constant* M such that

$$||Q(q(\pi, t), \pi(t), t)|| \leq M \tag{4.3}$$

uniformly in π and t.

Proof. Let x^* be the point of X used in (3.2) (iii). Then for any π on I

$$\begin{aligned} ||q(\pi, t) - q^*(t)|| &\leq \int_0^t ||Q(q(\pi, s), \pi(s), s) - Q(q^*(s), x^*, s)||ds \\ &\leq C + \int_0^t ||Q(q(\pi, s), \pi(s), s) - Q(q^*(s), \pi(s), s)||ds \\ &\leq C + \overline{Q} \int_0^t ||q(\pi, s) - q^*(s)||ds \end{aligned} \tag{4.4}$$

hence

$$||q(\pi, t) - q^*(t)|| \leq Ce^{\overline{Q}} \tag{4.5}$$

by Lemma 1. Thus

$$\begin{aligned} &||Q(q(\pi, t), \pi(t), t) - Q(q^*(t), x^*, t)|| \\ &\quad \leq ||Q(q(\pi, t), \pi(t), t) - Q(q^*(t), \pi(t), t)|| \\ &\qquad\qquad + ||Q(q^*(t), \pi(t), t) - Q(q^*(t), x^*, t)|| \\ &\quad \leq C(\overline{Q}e^{\overline{Q}} + 1) \ . \end{aligned} \tag{4.6}$$

Since $Q(q(t), x, t)$ is clearly bounded in I the lemma follows by the triangle inequality.

5. Programs on E_*. We consider continuous programs in which the policy changes are permitted to occur only on points of E_* and set

$$G_* = \sup_{\pi \text{ on } E_*} g(\pi) \ . \tag{5.1}$$

LEMMA 3. $G_* = G$,

Proof. Trivially $G_* \leqq G$ since E_* is a subset of I. To show the opposite inequality suppose that π and π' are any two programs on I. For convenience we let q, p and q', p' be the functions on I determined by π and π' respectively and omit the dependence on t under integral signs whenever no confusion can arise. Writing q for the value $q(t)$ of the function q at t, etc., suppose that π and π' differ only on a small portion of I. We shall first obtain an estimate of $g(\pi)-g(\pi')$ in terms of the measure of this portion of I, then use this estimate to complete the proof. Let

$$\left\{\begin{aligned} \delta &= \int_0^1 \|Q(q,\pi)-Q(q,\pi')\|dt \\ \delta' &= \int_0^1 \|Q(q',\pi)-Q(q',\pi')\|dt\,. \end{aligned}\right. \tag{5.2}$$

By (3.2) (ii)

$$\begin{aligned} \|q(t)-q'(t)\| &\leqq \int_0^t \|Q(q,\pi)-Q(q',\pi')\|ds \\ &\leqq \int_0^t \|Q(q,\pi)-Q(q,\pi')\|ds + \int_0^t \|Q(q,\pi')-Q(q',\pi')\|ds \\ &\leqq \delta + \overline{Q}\int_0^t \|q-q'\|ds \end{aligned} \tag{5.3}$$

hence

$$\|q(t)-q'(t)\| \leqq \delta e^{\overline{Q}t} \leqq \delta e^{\overline{Q}}\,, \tag{5.4}$$

by Lemma 1. Similarly

$$|p(t)-p'(t)| \leqq \eta e^{\overline{P}} \tag{5.5}$$

where

$$\eta = \overline{P}\int_0^1 |p(\pi)-p(\pi')|dt\,. \tag{5.6}$$

Now

$$\begin{aligned} |g(\pi)-g(\pi')| &\leqq \int_0^1 \|pQ(q,\pi)-p'Q(q',\pi')\|dt \\ &\leqq \int_0^1 \|pQ(q,\pi)-pQ(q',\pi)\|dt + \int_0^1 \|pQ(q',\pi)-pQ(q',\pi')\|dt \\ &\qquad + \int_0^1 \|pQ(q',\pi')-p'Q(q',\pi')\|dt \\ &\leqq \delta\overline{Q}e^{\overline{Q}}\int_0^1 |p|dt + \int_0^1 |p|\,\|Q(q',\pi)-Q(q',\pi')\|dt \\ &\qquad + \eta e^{\overline{P}}\int_0^1 \|Q(q',\pi')\|dt\,. \end{aligned} \tag{5.7}$$

Since $\|Q\| \leqq M$ and $|p| \leqq 1$ we may write

$$|g(\pi)-g(\pi')| \leqq \delta\overline{Q}e^{\overline{Q}}+\delta'+\eta e^{\overline{P}}M \ . \tag{5.8}$$

Clearly the boundedness of $\|Q\|$ and $|p|$ implies that each of δ, δ', and η can be made arbitrarily small simply by choosing the set on which π and π' differ to have sufficiently small measure; this choice permits us to make $|g(\pi)-g(\pi')|$ as small as desired.

By definition of G, for any $\varepsilon>0$ there is a program π such that $g(\pi)>G-\varepsilon/2$. But since E_* is dense in I we may choose a program π' on E which differs from π on such a small subset of I that $g(\pi')>g(\pi)-\varepsilon/2$. By definition $G_* \geqq g(\pi')$; hence the combined inequalities give $G_*>G-\varepsilon$ for any $\varepsilon>0$; that is, $G_* \geqq G$. This completes the proof.

6. Convergence for fixed programs. Now suppose that π is a fixed program on E_j and consider the functionals $f_k(\pi)$ for $k \geqq j$. We shall show that the sequence $\{f_k(\pi)\}$ converges to $g(\pi)$. Furthermore, we shall show that there is a modulus of convergence which depends only on k and not on j or π.

First we need to know something about the convergence of $p_k(\pi, t)$ and $Q(q_k(\pi, t), \pi(t), t)$ to $p(\pi, t)$ and $Q(q(\pi, t), \pi(t), t)$ respectively on portions of I for which the program π on E_j is a constant. Let $t_0, \cdots, t_n$ represent the points of E_j. Since the program π remains fixed in the following discussion we write $q(t)$, $q_k(t)$, $p(t)$, $p_k(t)$ for $q(\pi, t)$, $q_k(\pi, t)$, $p(\pi, t)$, $p_k(\pi, t)$ respectively. Let $q(t;\ i, k)$ represent the value at t of the solution $q(i, k)$ of

$$q(t;\ i,\ k)=q_k(t_i)+\int_{t_i}^{t} Q(q(s;\ i,\ k),\ x,\ s)ds \ , \tag{6.1}$$

where the initial value $q_k(t_i)$ is just what one obtains by using the difference equation on E_k up to the point t_i. We note that the Cauchy-Lipschitz existence theorem[1] furnishes a bound on the norm of the difference between $q(t;\ i, k)$ and $q_k(t)$ in the interval $[t_i, t_{i+1})$ and furthermore, that this bound has the form

$$\|q(t;\ i,\ k)-q_k(t)\| \leqq \delta(t-t_i) \tag{6.2}$$

for $t_i \leqq t < t_{i+1}$ and some δ independent of t. Since the continuity and Lipschitz conditions are independent of X it is clear that δ is likewise independent of X. A similar inequaliy holds for p, and we may assume

[1] The author was unable to locate in the literature a proof of the extension of the Cauchy-Lipschitz theorem to differential equations over Banach spaces, even finite-dimensional ones, although it clearly can be obtained with no more difficulty than the extension of the Picard theorem. Such an extension probably does not exist (except for finite-dimensional spaces) for the Cauchy-Peano theorem, since the appropriate analogue of Ascoli's theorem is false; the finite-dimensional case is given in [**6**].

that the same δ appears in both results.

Now we proceed to prove convergence for fixed programs. The preceding comments indicate that no trouble occurs between points of E_j, so that convergence for any given fixed program is to be expected. The chief problem is to show that there is a modulus of convergence which is independent of the number or location of the steps of the program π.

LEMMA 4. *Given any $\varepsilon > 0$ there exists a positive number h such that $|f_k(\pi) - g(\pi)| < \varepsilon$ for any $k \geqq h$, uniformly for all programs π on E_k.*

Proof. Consider any fixed point t^* in the closure of I and let $t_0 = 0, t_1, \cdots, t_{n-1}$ be the points in the intersection of $[0, t^*)$ and E_j; for convenience we write $t^* = t_n$. For any $k \geqq j$ we note that the functions q and $q(i-1, k)$ are both solutions of the same differential equation, but with different initial values at the point $t = t_{i-1}$. Hence,

$$q(t) - q(t; i-1, k) = q(t_{i-1}) - q_k(t_{i-1}) + \int_{t_{i-1}}^{t} [Q(q(s), \pi(s), s) - Q(q(s; i-1, k), \pi(s), s)]ds \; . \tag{6.3}$$

Taking norms and applying the Lipschitz condition on Q, we have

$$\begin{aligned} &\|q(t) - q(t; i-1, k)\| \\ &\leqq \|q(t_{i-1}) - q_k(t_{i-1})\| + \int_{t_{i-1}}^{t} \bar{Q}\|q(s) - q(s; i-1, k)\|ds \; . \end{aligned} \tag{6.4}$$

Now apply Lemma 1 and set $t = t_i$ to find

$$\|q(t_i) - q(t_i; i-1, k)\| \leqq \|q_k(t_{i-1}) - q(t_{i-1})\|(i, i-1) \tag{6.5}$$

where we write $(i, i-1)$ for $e^{\bar{Q}(t_i - t_{i-1})}$. For convenience set

$$\begin{aligned} A_i &= \|q(t_i) - q(t_i; i-1, k)\| \\ B_i &= \|q(t_i; i-1, k) - q_k(t_i)\| \\ C_i &= \|q(t_i) - q_k(t_i)\| \; . \end{aligned} \tag{6.6}$$

Then (6.5) is written

$$A_i \leqq C_{i-1}(i, i-1) \tag{6.7}$$

and (6.2) implies

$$B_i \leqq \delta(t_i - t_{i-1}) \tag{6.8}$$

where δ depends only on k. Finally the triangle inequality implies

$$C_i \leqq A_i + B_i \ . \tag{6.9}$$

Now we apply (6.7), (6.8), (6.9) repeatedly to obtain a bound on C_n. In the following calculation we also use the obvious results $(n, m)(m, l) = (n, l)$ and $(m, l) \leqq e^{\overline{Q}}$ for $1 \leqq l < m < n$.

$$\begin{aligned} C_n &\leqq A_n + B_n \\ &\leqq C_{n-1}(n, n-1) + \delta(t_n - t_{n-1}) \\ &\leqq [A_{n-1} + B_{n-1}](n, n-1) + \delta(t_n - t_{n-1})e^{\overline{Q}} \\ &\leqq [C_{n-2}(n-1, n-2) + \delta(t_{n-1} - t_{n-2})](n, n-1) + \delta(t_n - t_{n-1})e^{\overline{Q}} \\ &\leqq C_{n-2}(n, n-2) + \delta(t_n - t_{n-2})e^{\overline{Q}} \\ &\quad \vdots \\ &\leqq C_{n-m}(n, n-m) + \delta(t_n - t_{n-m})e^{\overline{Q}} \\ &\quad \vdots \\ &\leqq C_1(n, 1) + \delta(t_n - t_1)e^{\overline{Q}} \ . \end{aligned} \tag{6.10}$$

Finally $q(t; 0, k) = q(t)$ since $q(t; 0, k)$ is defined by the same differential equation and the same initial values as $q(t)$; hence $A_1 = 0$ so that as a special case of (6.9) we have $C_1 \leqq \delta(t_1 - t_0) = \delta t_1$. Thus

$$C_n \leqq \delta t_n e^{\overline{Q}} \leqq \delta e^{\overline{Q}} \ , \tag{6.11}$$

that is

$$\|q(t^*) - q_k(t^*)\| \leqq \delta e^{\overline{Q}} \tag{6.12}$$

uniformly for all $t^* \in I$ where δ depends only on k. Furthermore, this inequality is uniform in the programs π on E_k since the right-hand side of (6.12) is independent of the distribution or number of points $t_0, \cdots, t_n$. In a similar fashion we have

$$|p(t^*) - p_k(t^*)| \leqq \delta e^{\overline{P}} \tag{6.13}$$

uniformly for $t^* \in I$ and the programs on E_k.

Suppressing the dependence on t we have

$$\begin{aligned} &\Big| |p| \, \|Q(q)\| - |p_k| \, \|Q(q_k)\| \Big| \\ &\leqq \|Q(q)\| \Big| |p| - |p_k| \Big| + |p_k| \Big| \|Q(q)\| - \|Q(q_k)\| \Big| \\ &\leqq M|p - p_k| + \|Q(q) - Q(q_k)\| \\ &\leqq \delta(Me^{\overline{P}} + \overline{Q}e^{\overline{Q}}) \ , \end{aligned} \tag{6.14}$$

hence, writing $Q(q)$ for $Q(q(\pi, t), \pi(t), t)$, we have

$$\begin{aligned}
|f_k(\pi)-g(\pi)| &= \left|\int_0^1 |p|\,\|Q(q)\|dt-\int_0^1 |p_k|\,\|Q(q_k)\|dm_k(t)\right| \\
(6.15) \qquad &\leqq \int_0^1 |p|\,\|Q(q)\|\,|dt-dm_k(t)|+\int_0^1 \Big|\,|p|\,\|Q(q)\|-|p_k|\,\|Q(q_k)\|\,\Big|\,dm_k(t) \\
&\leqq M\int_0^1 |dt-dm_k(t)|+\delta(Me^{\bar{P}}+\bar{Q}e^{\bar{Q}}) .
\end{aligned}$$

None of the terms in the last line of (6.15) depends on the program, and clearly each of them can be made arbitrarily small by choosing k large enough. This completes the proof.

7. The convergence theorem.

THEOREM. $\lim_{j\to\infty} F_j=G.$

Proof. Given an $\varepsilon>0$ there exists a program π_* on E_* such that $g(\pi_*)>G_*-\varepsilon/2=G-\varepsilon/2$. But since π_* is a step function of E_* into X, the points t at which steps actually occur must lie in some E_j for j sufficiently large, by definition of E_*; hence π_* is a program on E_j for some j. For sufficiently large k Lemma 4 guarantees that $f_k(\pi_*)>g(\pi_*)-\varepsilon/2$. Hence since

$$F_k=\sup_{\pi \text{ on } E_k} f_k(\pi)\geqq \sup_{\pi \text{ on } E_j} f_k(\pi)\geqq f_k(\pi_*) ,$$

we have

$$(7.1) \qquad F_k>G-\varepsilon .$$

On the other hand, Lemma 4 also asserts that for k sufficiently large $g(\pi)>f_k(\pi)-\varepsilon/2$, independently of π, assuming, of course, that π is on E_k. Let π be a program on E_k such that $f_k(\pi)>F_k-\varepsilon/2$. Then

$$(7.2) \qquad G=G_*\geqq g(\pi)>f_k(\pi)-\frac{\varepsilon}{2}>F_k-\varepsilon$$

for k sufficiently large. Combining (7.1) and (7.2) completes the proof.

REFERENCES

1. R. Bellman, *An introduction to the theory of dynamic programming*, The RAND Corporation, R-245, June, 1953.

2. R. Bellman and R. S. Lehman, *On a functional equation in the theory of dynamic programming and its generalizations*, The RAND Corporation, P-433, January, 1954.

3. ———, *On the continuous goldmining equation*, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 115–119.

4. R. Bellman, *Dynamic programming of continuous processes*, The RAND Corporation, R-271, July, 1954.

5. ———, *Stability theory of differential equations*, McGraw-Hill, New York, 1953, 35–36.
6. F. J. Murray, and K. S. Miller, *Existence theorems for ordinary differential equations*, New York Univ. Press, New York, 1954, 13-20.
7. H. A. Osborn, *On the convergence of discrete stochastic decision processes to their continuous analogues-I*, The RAND Corporation, RM–1368, Nov. 8, 1954.
8. ———, *On the convergence of discrete stochastic decision processes to their continuous analogues-II*, The RAND Corporation, RM–1414, Jan. 19, 1955.

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