SOME PROPERTIES OF DISTRIBUTIONS ON LIE GROUPS

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1. Introduction. Let G be a separable Lie group and let V be a complete, metrizable, topological vector space. The underlying space of G is a separable real analytic manifold so that we can define, by the methods of L. Schwartz (see [7], [12], [13]), the spaces $\mathscr{E}(V)$ of indefinitely differentiable maps of G into V, and $\mathscr{D}(V)$ which consists of those maps in $\mathscr{E}(V)$ which are of compact carrier. Their duals are $\mathscr{D}'(V)$, the space of distributions on G with values in V' (the dual of V), and $\mathscr{E}'(V)$ which is the space of distributions of compact carrier with values in V'.

By using the group structure in G, we can define the convolution $S*f \in \mathscr{E}(C)$ for any $S \in \mathscr{D}'(V)$, $f \in \mathscr{D}(V)$, where C is the complex plane. The main result of this paper is: Let $S \in \mathscr{D}'(V)$ have the property that $S*f \in \mathscr{D}(C)$ whenever $f \in \mathscr{D}(V)$; then $S \in \mathscr{E}'(V)$. Moreover, the topology of $\mathscr{E}'(V)$ is that obtained by considering each $S \in \mathscr{E}'(V)$ as defining the continuous linear transformation $f \to S*f$ of $\mathscr{D}(V) \to \mathscr{D}(C)$ and then giving this set of transformations the compact-open topology (see [6]). This generalizes the result of [6] in case G is a vector group and $V=C$.

This result is generalized to double coset spaces $L\backslash G/K$ where L and K are compact subgroups of G. In this form, the result will be used by the author and F. I. Mautner to generalize the Paley-Wiener theorem and the theory of mean-periodic functions of Schwartz (see [8]).

The author wishes to express his thanks to Professor F. I. Mautner for helpful discussions.

2. Distributions on G. Instead of using the usual method of defining distributions on G, as for example in de Rham and Kodaira [12], we shall follow another approach which is more akin to the author's thesis [5]. We shall show that the two methods are equivalent.

By "function" we shall mean "complex-valued function" unless the contrary is specifically stated. "Linear" will mean "linear over the complex numbers" always. By 1 we denote the identity in G, and by $\mathfrak{g}$ we denote the Lie algebra of G. For any $Y \in \mathfrak{g}$, we denote by $t \to \exp(tY)$ the unique one parameter subgroup in G whose direction

Received September 22, 1955. Work partially supported by National Science Foundation Grant NSF5-G1010.

at 1 is Y. Let V be a complete metrizable locally convex topological vector space.

The map f of G into V is said to be differentiable in the direction $Y \in \mathfrak{g}$ at $x \in G$ if $\left\{\left(\frac{df}{dt}\right)[(\exp tY)x]\right\}_{t=0}$ exists; if this is the case, we set

$$(D_Y f)(x) = \left\{\left(\frac{df}{dt}\right)[(\exp tY)x]\right\}_{t=0}. \tag{1}$$

If f is a continuous map of G into V, we say that f is in the domain of D_Y if, for any $x \in G$, f is continuously differentiable in the direction Y at x. $D_Y f$ is then defined as the (continuous) map $x \to (D_Y f)(x)$.

By $\mathscr{E}^0$ we denote the space of continuous maps of G into V with the topology of uniform convergence in V on the compact sets of G. By the *carrier* of an $f \in \mathscr{E}^0$ we mean the closure of the set of points where $f \neq 0$. An *operator* on G is a linear mapping of a subspace of $\mathscr{E}^0$ into $\mathscr{E}^0$. The operator D is said to be *closed* if the conditions: $\{f_\alpha\}$ in the domain of D, $f_\alpha \to f$ and $Df_\alpha \to h$ in $\mathscr{E}^0$, imply f is in the domain of D and $Df = H$.

PROPOSITION 1. *For any Y in $\mathfrak{g}$, D_Y is a closed operator.*

Proof. It is clear that D_Y is an operator.

It remains to show that D_Y is closed. Let $\{f_i\}$ be a sequence of functions in the domain of D_Y such that $\{f_i\}$ and $\{D_Y f_i\}$ are Cauchy sequences in $\mathscr{E}^0$; call $f = \lim f_i$, $h = \lim D_Y f_i$, the limits being taken in $\mathscr{E}^0$. Let $Y, X_2, X_3, \cdots, X_n$ be a basis for $\mathfrak{g}$ and N an open neighborhood of 1 in G in which $\exp(t_1 Y)\exp(t_2 X_2)\cdots\exp(t_n X_n)$ form a coordinate system. It is clearly sufficient to prove that f is in the domain of D_Y at 1 and that $(D_Y f)(x) = h(x)$ for any $x \in N$.

Now, $\theta: (t_1, t_2, \cdots, t_n) \to (\exp(t_1 Y), \exp(t_2 X_2), \cdots, \exp(t_n X_n))$ maps a circular neighborhood M of 0 in real Euclidean n-space homeomorphically onto N. It is immediate from the definitions that a continuous map p of G into V is differentiable in the direction Y at 1 if and only if $p\theta$ has a continuous partial derivative in the direction t_1 at 0, and then

$$(D_Y p)(x) = \left(\frac{\partial p\theta}{\partial t_1}\right)(x)$$

for all x in a suitable neighborhood of 1. From this and the known closure of $\partial/\partial t_1$ on Euclidean space, our assertion follows.

Now, let $Y_1, Y_2, \cdots, Y_n$ be a basis for $\mathfrak{g}$. We set

$$D_1 = D_{Y_1},\ D_2 = D_{Y_2},\ \cdots,\ D_n = D_{Y_n}$$

and we call $\mathfrak{D}$ the family $(D_1, D_2, \cdots, D_n)$ so $\mathfrak{D}$ is a family of closed operators. By means of D we can now define, by the methods of [5], the complete, locally convex, Hausdorff, topological vector spaces $\mathscr{D}$ (or $\mathscr{D}(V)$) of indefinitely differentiable maps of compact carrier of G into V, and $\mathscr{E}$ (or $\mathscr{E}(V)$) of all indefinitely differentiable maps of G into V. $\mathscr{E}$ is a metrizable space; a sequence $\{f_i\}$ converges to zero in $\mathscr{E}$ if and only if for any operator $D^* = D_{j_1}D_{j_2}\cdots D_{j_r}$, $D_{j_m} \in \mathfrak{D}$, $Df \to 0$ uniformly in V on every compact set of G. The topology of $\mathscr{D}$ may be described as follows: For each compact set K, let $\mathscr{D}_K$ be the subspace of $\mathscr{D}$ consisting of those maps of $\mathscr{D}$ which have their carriers in K; the topology of $\mathscr{D}_K$ is that induced by $\mathscr{E}$. Then of all possible locally convex topologies which induce on each $\mathscr{D}_K$ the topology of $\mathscr{D}_K$ that may be given to the set of functions of $\mathscr{D}$, $\mathscr{D}$ is given the strongest (see [4]).

PROPOSITION 2. *The spaces $\mathscr{D}$ and $\mathscr{E}$ are the same as those we would have obtained by considering G as an indefinitely differentiable manifold.*[1]

Proof. Let N be a neighborhood of 1 in G in which $(\exp t_1Y_1 \exp t_2Y_2 \cdots \exp t_nY_n)$ form a coordinate system. Then it is clearly sufficient to prove the theorem for the restrictions of the functions of $\mathscr{E}$ and $\mathscr{D}$ to N. The result now follows by the method of the proof of Proposition 1.

PROPOSITION 3. *$\mathscr{D}$ and $\mathscr{E}$ are reflexive topological spaces.*[2]

Proof. We prove the theorem first for $\mathscr{E}$. Since $\mathscr{E}$ is metrizable, it is sufficient to prove that $\mathscr{E}$ is a Montel space, that is, that the bounded sets of $\mathscr{E}$ are relatively compact (of compact closure). Let then B be a bounded set in $\mathscr{E}$. Let N be a compact neighborhood of 1 in G in which $(\exp t_1Y_1 \exp t_2Y_2 \cdots \exp t_nY_n)$ form a coordinate system. Since G is separable, we can find a sequence of points $a_i \in G$ such that $G = \bigcup(\textit{interior } Na_i)$.

It is easily seen that it is sufficient to show that, for any i, and for any integers $r_1, r_2, \cdots, r_m$, if we set $D^* = D_{r_1}D_{r_2}\cdots D_{r_m}$, then the set $\{D^*f\}_{f\in B}$ is equicontinuous on a_iN. It follows immediately as in the proof of Proposition 1 that the restrictions of the maps D^*f have the property that (if we identify them with maps on a circular neighborhood of zero in Euclidean n-space) their partial derivatives in all direc-

[1] That is, by applying the method of de Rham and Kodaira [**12**].

[2] See [**3**].

tions are uniformly bounded for $f \in B$. As is well-known, this implies the equicontinuity of $\{D^*f\}_{f\in B}$ on a_iN; hence Proposition 3 is established as regards the space $\mathscr{E}$.

If L is a bounded set in $\mathscr{D}$, then all the maps of L have their carriers in a fixed compact set K of G, that is, $L \subset \mathscr{D}_K$. Since the topology induced by $\mathscr{D}$ on $\mathscr{D}_K$ is also the topology induced by $\mathscr{E}$ on $\mathscr{D}_K$, L is bounded in $\mathscr{E}$. Thus, L is relatively compact in $\mathscr{E}$, hence in $\mathscr{D}_K$, hence also in $\mathscr{D}$ which concludes the proof of Proposition 3.

A sequence of open, relatively compact (that is, of compact closure) sets $K_i \subset G$ will be called a *scattered resolution* of G (see [5]) if $\bigcup K_i = G$ and if, given any compact set $K \subset G$, only a finite number of the K_i meet K. Given any scattered resolution $\{K_i\}$ of G, there exists a *partition of unity* $\{h_i\}$ relative to it; by this is meant that the indefinitely differentiable functions h_i have the properties that:

1. For each i, *carrier* $h_i \subset K_i$.

2. For any $x \in G$, $\Sigma\, h_i(x)=1$.

(This sum has meaning because all but a finite number of terms are zero.) To establish the existence of the partition of unity $\{h_i\}$, we have only to note that the scattered resolution $\{K_i\}$ can be "refined" to a scattered resolution $\{L_i\}$ by coordinate neighborhoods (that is, each K_i is contained in a union of a finite number of L_j). The existence of a partition of unity relative to $\{L_i\}$ is readily verified and, in turn, implies immediately the existence of a partition of unity relative to $\{K_i\}$.

By $\mathscr{D}'$ (or $\mathscr{D}'(V)$ we denote the dual of $\mathscr{D}$ with the topology of uniform convergence on the bounded (compact) sets of $\mathscr{D}$. It can be shown (see [7]) that, $\mathscr{D}'$ can also be described as the space of continuous linear maps of $\mathscr{D}(C) \to V'$, this space of maps being given the compact-open topology. For this reason, $\mathscr{D}'$ is usually called the space of *distributions on G with values in V'*. In this paper, we shall call the elements of $\mathscr{D}'$ *distributions*.

For any distribution S, and any open set O in G, we say that S vanishes on O if $S\cdot f=0$ for any $f \in \mathscr{D}$ whose carrier is contained in O. Because of the existence of partitions of unity, we can easily show that if S vanishes on O_α where O_α are open sets, then S vanishes also on $\bigcup O_\alpha$. Thus there is a largest open set on which S vanishes. The *carrier* of S is defined as the complement of this set.

$\mathscr{E}'$ (or $\mathscr{E}'(V)$) is the dual of $\mathscr{E}$. It can be shown, as in [13], that $\mathscr{E}'$ consists of all distributions of compact carrier.

For any $S \in \mathscr{D}'$, by $\bar{S}$ is meant the distribution $f \to \overline{S\cdot \bar{f}}$ for $f \in \mathscr{D}$, where $\overline{f}(x)=\overline{f(x)}$ for any $x\in G$.

By $G\times G$ we denote the direct product of G with itself; $G\times G$ is again a Lie group whose underlying manifold is the Cartesian product of the underlying manifold of G with itself. By ${}_2\mathscr{D}$, ${}_2\mathscr{E}$, ${}_2\mathscr{D}'$, ${}_2\mathscr{E}'$ we denote the spaces on $G\times G$ corresponding to $\mathscr{D}$, $\mathscr{E}$, $\mathscr{D}'$ $\mathscr{E}'$ respectively.

Let k be a continuous map on $G\times G$ and $x\in G$. Then by $k_{X_1=x}$ we mean the map on G: $y\rightarrow k(x, y)$. Suppose that, for all $x\in G$, $k_{X_1=x}$ is in a space U of mappings on G. Then by k_1 we mean the mapping $x\rightarrow k_{X_1=x}$ of $G\rightarrow U$. Let L be a map defined on U; then we say that k is in the domain of L, and we denote by L_2k the map

$$x\rightarrow Lk_{X_1=x}$$

for $x\in G$. If the range of L is again a space of mappings on G, then we say also that k is in the domain of L_{2l} and we shall denote by $L_{2l}k$ the mapping on $G\times G$:

$$(x, y)\rightarrow Lk_{X_1=x}(y)\ .$$

L_{2l} is called the *lift* of L to $G\times G$. We define $k_{X_2=x}$, k_2, L_1, L_{1l} similarly.

We can now define, as in [5], two products involving distributions and functions:

For any $S\in\mathscr{D}'$, $k\in{}_2\mathscr{D}$, then we have two *inner products*: S_1k and S_2k which are both in $\mathscr{D}$.

For any S, $U\in\mathscr{D}'$ we define the *direct products* $S_1\times U_2$ and $S_2\times U_1$ $\in{}_2\mathscr{D}'$ by

$$S_1\times U_2\cdot k=S\cdot U_2k,\qquad S_2\times U_1\cdot k=S\cdot U_1k$$

for any $k\in{}_2\mathscr{D}$.

The direct products define continuous bilinear maps which are commutative, while the inner products are only separately continuous bilinear maps. (If V, W, X are topological vector spaces and $t: V\times W\rightarrow X$ is a bilinear map, then t is called separately continuous (see [4], [5]) if, for B, B' any bounded sets in V, W respectively, the maps

$$w\rightarrow t(b, w),\ v\rightarrow t(v, b')$$

are, for $b\in B$, $b'\in B'$, equicontinuous linear maps of $W\rightarrow X$ and $V\rightarrow X$ respectively.)

By $\{Q_i\}$ we shall denote an enumeration of the operators $D_{r_1}D_{r_2}\cdots D_{r_m}$ with Q_1=identity.

For f a continuous map defined on G, $\check{f}$ is the map $x\rightarrow f(x^{-1})$.

We shall denote by η the function on G defined by $dxg=\eta(g)dx$, where dx is a left invariant Haar measure. It is known that $\eta\in\mathscr{E}(C)$

and, moreover, η is a homomorphism on G. By ω we denote the function on G defined by $dx^{-1}=\omega(x)dx$. Again, $\omega\in\mathscr{E}(C)$ and ω is a homomorphism on G. It is readily verified that $\omega(y)=\eta(y^{-1})$ for any $y\in G$. For any $S\in\mathscr{D}'$, we write $\check{S}\cdot f=S\cdot\omega\check{f}$ for any $f\in\mathscr{D}$.

3. Convolution on G. For any continuous map f of G into V and any $x\in G$ we define the translations

$$(\mathfrak{L}(x)f)(y)=f(x^{-1}y) \qquad (\mathfrak{R}(x)f)(y)=f(yx)$$

for any $y\in G$.

PROPOSITION 4. *$(x, f)\to\mathfrak{L}(x)f$ and $(x, f)\to\mathfrak{R}(x)f$ are continuous maps of $G\times\mathscr{D}\to\mathscr{D}$ and also of $G\times\mathscr{E}\to\mathscr{E}$.*

Proof. We shall establish the theorem for the map $(x, f)\to\mathfrak{L}(x)f$ of $G\times\mathscr{D}\to\mathscr{D}$; the other parts of the proposition may be established by similar methods. By the results of Dieudonné and Schwartz (see [4], [5]) it is sufficient to prove that this is a continuous map of $G\times\mathscr{D}_K\to\mathscr{D}$ for any compact set K of G. Since the map is linear in f and a homomorphism in x, it is sufficient to prove continuity at $f=0$ and $x=1$. Let K be a given compact set in G and choose K' a compact set in G so large that K' contains the carriers of all $\mathfrak{L}(x)f$ for $x\in\mathscr{D}_K$. Let M be a neighborhood of zero in $\mathscr{D}_{K'}$. Then we can find operators $Q_1, Q_2, \cdots, Q_r$, and continuous semi-norms $\rho_1. \rho_2, \cdots, \rho_n$ on V, and a positive number a so that M contains the set of $h\in\mathscr{D}_{K'}$ which satisfy

$$\max_{y\in G,\, i}\rho_i[(Q_jh)(y)]\leqq a$$

for $j=1, 2, \cdots, r$.

For any $p\in\mathscr{D}$, any k, and $x, z\in G$,

$$(D_k\mathfrak{L}(z)p)(x)=\left\{\left[\left(\frac{d}{dt}\right)\mathfrak{L}(z)p\right][(\exp tY_k)x]\right\}_{t=0}$$

$$=\left\{\left[\left(\frac{d}{dt}\right)p\right][z^{-1}(\exp tY_k)x]\right\}_{t=0}$$

$$=\left\{\left(\frac{dp}{dt}\right)[z^{-1}(\exp tY_k)zz^{-1}x]\right\}_{t=0}$$

$$=\left\{\left(\frac{dp}{dt}\right)[(\exp tz^{-1}Y_kz)z^{-1}x]\right\}_{t=0}.$$

Now, write $z^{-1}Y_kz=\Sigma\, c_{kl}(z)Y_l$ where (c_{kl}) is the matrix of the adjoint representation of G on g. Then we have

$$(D_k \mathfrak{L}(z)p)(x) = (D_{z^{-1}Y_k z} f)(z^{-1}x) .$$

We also have

$$D_{z^{-1}Y_k z} p = \Sigma\, c_{kj}(z) D_j p .$$

The functions c_{kj} are continuous and even indefinitely differentiable on G. Hence, we can find an $A > 0$ so that

$$\max_{z \in K} |c_{kj}(z)| \leqq A$$

for all k, j.

It follows immediately from this that we can be assured that, for $q \in \mathscr{D}_K$, $z \in K$,

$$\max_{x \in G,\, i} \rho_i[(D_K \mathfrak{L}(z)q)(x)]$$

will be small by making

$$\max_{x \in G,\, j,\, i} \rho_i[(D_j q)(x)]$$

sufficiently small. Proposition 4 now follows by iteration, since each Q_l is of the form $D_{r_1} D_{r_2} \cdots D_{r_m}$.

For any continuous map f on G, $\mathfrak{L}f$ is the map on $G \times G$: $(x, y) \to f(x^{-1}y)$; $\mathfrak{L}^* f$ is the map on $G \times G$: $(x, y) \to f(xy)$. By the method of proof of Proposition 1, we can establish

PROPOSITION 5. *$f \to \mathfrak{L}f$ and $f \to \mathfrak{L}^* f$ are continuous linear maps of $\mathscr{E} \to_2 \mathscr{E}$.*

We are now in a position to define the convolution product involving distributions and functions. The definition differs slightly from that of Schwartz [13]: For any $S \in \mathscr{D}'$, $f \in \mathscr{D}$, $x \in G$, we set

$$(1) \qquad (S * f)(x) = \bar{S} \cdot \mathfrak{L}(x) \check{f}$$

This formula can also be considered valid if $S \in \mathscr{E}'$ and $f \in \mathscr{E}$.

PROPOSITION 6. *$(S, f) \to S * f$ is a separately continuous map of*

(a) $\mathscr{E}' \times \mathscr{E} \to \mathscr{E}(C)$

(b) $\mathscr{E}' \times \mathscr{D} \to \mathscr{D}(C)$

(c) $\mathscr{D}' \times \mathscr{D} \to \mathscr{E}(C)$.

which is antilinear in S and linear in f.

Proof. (a) Let j be fixed and write $A=D_j$. We find from the definitions that, for $S\in\mathscr{E}'$, $f\in\mathscr{E}$, $S*f\in\mathscr{E}(C)$ and, moreover,

$$[A(S*f)](x)=\overline{S}\cdot(A_{1/}\mathfrak{L}\check{f})_{X_1=x}\,. \tag{2}$$

From this it follows by iteration that, for any $Q=Q_s$, we have

$$[Q(S*f)](x)=\overline{S}\cdot(Q_{1/}\mathfrak{L}\check{f})_{X_1=x}\,. \tag{3}$$

Part (a) results immediately from (3) together with Proposition 5.

(b) By a result of Dieudonné and Schwartz (see [4]) it is sufficient to prove that, for K a compact set in G, $(S, f)\to S*f$ is a separately continuous map of $\mathscr{E}'\times\mathscr{D}_K\to\mathscr{D}$. Now, it is obvious that

$$(carrier\ S*f)\subset(carrier\ S)(carrier\ f)\,. \tag{4}$$

Our assertion now follows from (a) above and the fact that $\mathscr{D}_K$ has the topology induced by $\mathscr{E}$.

(c) This is proven by essentially the same reasoning as that employed in the proof of (a) above.

4. $\mathscr{E}$ as a space of linear transformations.

In this section we shall prove our main result.

THEOREM 1. *Let $S\in\mathscr{D}'$ have the property that $S*f\in\mathscr{D}(C)$ whenever $f\in\mathscr{D}$; then $S\in\mathscr{E}'$.*

Proof. Let us suppose that S satisfies the hypotheses of Theorem 1, and let K be a fixed compact set in G. We shall show first that there exists a compact set $K'\subset G$ such that $S*\mathscr{D}_K\subset\mathscr{D}_{K'}$. Assume this is not the case, and let $\{K_i\}$ be a compact exhaustion of G. (That is, each K_i is a compact set which is the closure of a nonempty open set. Moreover, $K_i\subset K_{i+1}$ and $\bigcup K_i=G$.) We shall produce a sequence $\{g_i\}$ with the following properties:

1. Each $g_i\in\mathscr{D}_K$.
2. Σg_i converges in $\mathscr{D}_K$.
3. There is a sequence of positive numbers m_i with $m_{i+1}-m_i\geqq 1$ for all i such that

$$carrier\ (S*g_i)\subset K_{m_i}$$

$$carrier\ (S*g_{i+1})\not\subset K_{m_i}\,.$$

4. There is a sequence of points $a_i\in G$ such that a_i is a point of $K_{m_i}-K_{m_{i-1}}$ (where K_{m_0} is the empty set) for which $(S*g_i)(a_i)\neq 0$ and

$$|(S * g_{i+k})(a_i)| \leqq \frac{1}{3^k} |(S * g_i)(a_i)|$$

for all $k > 0$.

Suppose that the sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ can be found. Then for any $i > 1$,

$$\begin{aligned} |(S * \Sigma g_j)(a_i)| &= |\Sigma(S * g_j)(a_i)| \\ &= |\Sigma_{j \geq i}(S * g_j)(a_i)| \\ &\geqq |(S * g_j)(a_i)| - \sum_{j>i} |(S * g_j)(a_i)| \\ &\geqq |(S * g_i)(a_i)| \left[1 - \sum_{j \geqq 1} \frac{1}{3^j}\right] \\ &= \frac{1}{2} |(S * g_i)(a_i)| \\ &> 0 . \end{aligned}$$

Since the set $\{a_i\}$ is clearly not contained in any compact set of G, we conclude that $S * \Sigma g_j$ is not of compact carrier, which contradicts our hypothesis.

It remains to define the sequences $\{g_i\}$, $\{m_i\}$, and $\{a_i\}$. Let $g_1 \in \mathscr{D}_K$ be chosen so that $S * g_1 \neq 0$. Let a_1 be any point in G for which $(S * g_1)(a_1) \neq 0$, and choose $m_1 > 0$ so that

$$\textit{carrier } (S * g_1) \subset K_{m_1} .$$

Assume that $g_1, \cdots, g_k, a_1, \cdots, a_k, m_1, \cdots, m_k$ have been defined with the required properties; we shall now define $g_{k+1}, a_{k+1}, m_{k+1}$. Now, by our assumption, there is an $f \in \mathscr{D}_K$ such that

$$\textit{carrier } (S * f) \not\subset K_{m_k+1} .$$

Let m_{k+1} be chosen so that *carrier* $(S * f) \subset K_{m_{k+1}}$, and let a_{k+1} be some point in $K_{m_{k+1}} - K_{m_k}$ such that $(S * f)(a_{k+1}) \neq 0$. Define

$$g_{k+1} = \frac{f}{\max(1, \max_{x \in G} |f(x)| 3^{k+1}) \max_{j,\, i \leqq k} (1, \max_{x \in G} {}_j[(Q_i f)(x)], [|(S * g_i)(a_i)|]^{-1})}$$

The sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ are thus defined. It is clear that conditions 1, 3, 4 are satisfied. Further, each $g_i \in \mathscr{D}_K$ and, for R any semi-norm on D_K of the kind used to define the topology of that space, it is clear that

$$\Sigma R(g_1) > \infty .$$

Thus Σg_i converges in $\mathscr{D}_K$.

To complete the proof of Theorem 1, let us assume that S is not of compact carrier, and let K be a given compact symmetric neighborhood of 1 in G. It is clear that we can choose an open set U in G such that S does not vanish on U and such that $U \cap K'$ is empty, where K' is a compact symmetric set such that $\overline{S} * \mathscr{D}_K \subset \mathscr{D}_{K'}$. It follows easily that we can find a $g \in G$, and an $f \in D$ such that *carrier* $f \subset Kg \subset U$, $S \cdot f \neq 0$.

On the other hand, by definition,

$$S \cdot \check{f} = (\overline{S} * \mathfrak{L}(g)f)(g^{-1}) .$$

But, *carrier* $f \subset Kg$ implies *carrier* $\mathfrak{L}(g)f \subset K$ because K is symmetric. Also, $g \notin K'$ because $1 \in K$ and $K' \cap U$ is empty. Since K' is symmetric, also $g^{-1} \notin K'$. Thus, $S \cdot \check{f} = (\overline{S} * \mathfrak{L}(g)f)(g^{-1}) = 0$; this contradiction completes the proof of Theorem 1.

The set of distributions of $\mathscr{E}'$ forms a vector space of continuous linear mappings of $\mathscr{D} \rightarrow \mathscr{D}$ under convolution; we give this space the compact-open topology (see [6]) and obtain a topological vector space J. A fundamental system of neighborhoods of zero in J consists of all sets N for which we can find a compact set K in $\mathscr{D}$ and a neighborhood of zero M in $\mathscr{D}$ so that N consists of those $S \in \mathscr{E}'$ with $S * h \in M$ for all $h \in K$. By Proposition 1 of § 5 of [6], we would have obtained the same topologies if we had considered the distributions of $\mathscr{E}'$ as defining, under convolution, continuous linear maps of $\mathscr{D}' \rightarrow \mathscr{D}'$.

THEOREM 2. *The natural map* u: $\mathscr{E}' \rightarrow J$ *is a topological isomorphism onto.*

Proof. u is clearly one-to-one, linear, and onto. Moreover, J is given the weakest topology to make the maps

$$S \rightarrow (u^{-1}S) * f$$

of $J \rightarrow \mathscr{D}$ equicontinuous for f in any compact set of $\mathscr{D}$; by Proposition 6 this implies that u is continuous.

Since u^{-1} is linear, we need verify continuity only at zero. Let T be a neighborhood of zero in $\mathscr{E}'$; there is a bounded set $\beta \subset \mathscr{E}$ so that T contains the set of $S \in \mathscr{E}'$ which satisfy $|S \cdot b| \leq 1$ for all of $b \in \beta$.

Let K be an open symmetric neighborhood of 1 in G whose closure is compact. Then it is clear that we can find a sequence of points $a_i \in G$ such that $\{a_i K\}$ is a scattered resolution of G (see § 2). We can also insure that, if a is one of the a_i, so is a^{-1}. Let $\{h\}$ be a partition of unity relative to this scattered resolution (see § 2). It is readily verified by the method of proof of Proposition 1 of § 3 that, for each i, the set B_i of functions $\mathfrak{L}(a_i)(h_i f)$ for $f \in \beta$ is bounded in $\mathscr{D}$. For each j there is a double sequence $s_j = M_{jik}$ of positive numbers so that B_j is contained in the bounded (in $\mathscr{D}$) set L_j of all $g \in D$ whose carriers are contained in K and which satisfy

$$\max_{x \in G} \rho_k[(Q_i g)(x)] \leqq M_{jik}$$

for all i. From the denumerable number of double sequences s_j we construct a double sequence $s = \{M_{ik}\}$ of positive numbers such that, for each j, $M_{jik} \leqq M_{ik}$ for all but a finite number of i, k. Hence, for each j, we can find an $e_j > 0$ so that $e_j M_{jik} \leqq M_{ik}$ for all i, k; we can clearly make $e_j = e_l$ if $a_j = a_l^{-1}$.

Let A be the set of $f \in \mathscr{D}$ for which

1. *carrier* $f \subset K$

2. $\max_{x \in G} \rho_k[(Q_i f)(x)] \leqq M_{ik}$ for all i, k,

so A is bounded in $\mathscr{D}$. Let M be the neighborhood of zero in $\mathscr{D}$ consisting of those $h \in \mathscr{D}$ with

$$\max_{x \in a_j K} \rho_k h(x) \leqq e_j d_j$$

for all j, where d_j are positive numbers which satisfy $\Sigma\, d_j = 1$. Call N the set of $S \in J$ with $S * f \in M$ for all $f \in A$, so N is a neighborhood of zero in J; we claim that $u^{-1}(N) \subset T$.

Let us assume this is not the case; then we can find an $S \in N$ with $u^{-1}S \notin T$, that is, $S \in N$ but

$$|u^{-1}S \cdot f| > 1$$

for some $f \in \beta$. Now, $u^{-1}S$ is of compact carrier; thus we can find an r such that

$$\sum_{k=1}^{r} h_k(x) = 1$$

for any $x \in$ *carrier* $(u^{-1}S)$. Hence

$$|u^{-1}S \cdot f| \leqq |u^{-1}S \cdot h_1 f| + |u^{-1}S \cdot h_2 f| + \cdots + |u^{-1}S \cdot h_r f| \,. \tag{5}$$

It is clear from the definitions that, for each i,

$$e_i \mathfrak{L}(a_i)(h_i f) \in A \,.$$

Thus,

$$h_i f = \frac{1}{e_i} \mathfrak{L}(a_i^{-1}) g$$

for some $g \in A$, which gives, for $i=1, 2, \cdots, r$,

$$\begin{aligned} |u^{-1}S \cdot h_i f| &= \frac{1}{e_i} |u^{-1}S \cdot \mathfrak{L}(a_i^{-1}) g| \\ &= \frac{1}{e_i} |(\overline{u^{-1}S} * g)(a_i^{-1})| \\ &\leqq \frac{1}{e_i} e_j d_j \, , \end{aligned}$$

where $a_j = a_i^{-1}$, because $g \in A$ and $u^{-1}S \in N$. Now, since $e_i = e_j$, we have

$$|u^{-1}S \cdot h_i f| \leqq d_j \, .$$

Applying this to equation (5) we obtain

$$|u^{-1}S \cdot f| \leqq d_{1'} + d_{2'} + \cdots + d_{r'} \leqq 1$$

(where we set $a_{j'} = a_j^{-1}$). This contradiction proves the theorem.

5. Extension of the main result. We assumed in §§ 2, 3, 4 that V is metrizable. In case V is not metrizable, then the spaces $\mathscr{E}$ and $\mathscr{D}$ can be defined as before, but E is no longer metrizable, and $\mathscr{D}$ is not an $\mathscr{L}\mathscr{F}$ space in the sence of Dieueonné and Schwartz [**4**]. However, there is no difficulty in extending the definition and continuity properties of the convolution product to this case. Theorem 1 can be extended to this case, but the proof of Theorem 2 does not extend to the case of V not metrizable. All that can be proven (and the proof is much simpler than the proof of Theorem 2 above) is that u is continuous and that u^{-1} is sequentially continuous and takes bounded sets into bounded sets. The continuity of u^{-1} is an open question.

We assume in the following that V is a complete, locally convex, Hausdorff, topological vector space. By V^* we denote the space of continuous linear maps of V into V with the compact-open topology, so V^* is again a complete, locally convex, Hausdorff, topological vector space.

Let K and L denote compact subgroups of G. By a *representation of K on V* we mean a continuous homomorphism U of K into V^*. Let U and W be representations of V of K and L respectively. By ${}_{UW}\mathscr{D}$ we denote the space of those $f \in \mathscr{D}(V^*)$ for which

$$\mathfrak{L}(k^{-1})\mathfrak{R}(l) f = U(k) f W(l) \tag{6}$$

for any $k \in K$, $l \in L$. We give ${}_{UW}\mathscr{D}$ the topology induced by $\mathscr{D}$. ${}_{UW}\mathscr{E}$

is defined similarly.

For any $T \in \mathscr{D}'(V^*)$, $g \in G$, we define $\mathfrak{L}(g)T$ and $\mathfrak{R}(g)T$ as the distributions

$$(7) \qquad \mathfrak{L}(g)T\cdot f = T\cdot \mathfrak{L}(g^{-1})f, \quad \mathfrak{R}(g)T\cdot f = \eta(g)T\cdot \mathfrak{R}(g^{-1})f$$

for any $f \in \mathscr{D}(V^*)$. (η was defined in § 2.) Let us denote by ${}_{UW}\mathscr{D}'$ the space of all $S \in \mathscr{D}'(V^*)$ which satisfy

$$(8) \qquad \mathfrak{L}(k)\mathfrak{R}(l)S\cdot f = S\cdot U(k^{-1})fW(l)$$

for any $f \in \mathscr{D}(V^*)$, $k \in K$, $l \in L$.[3] We shall write $U(k)SW(l^{-1})\cdot f$ for the right side of (8). We give ${}_{UW}\mathscr{D}'$ the topology induced by $\mathscr{D}'(V^*)$. ${}_{UW}\mathscr{E}'$ is defined similarly.

We can easily show

PROPOSITION 7.

$$f \to P_{UW}f = \int_{K\times L} U(k)\mathfrak{L}(h)\mathfrak{R}(l)fV^{-1}(l)dkdl$$

(where dk and dl are the respective Haar measures on K and L so normalized that $\int_K dk = \int_L dl = 1$) defines continuous open projections of $\mathscr{D}(V^*)$ onto ${}_{UW}\mathscr{D}$ and $\mathscr{E}(V^*)$ onto ${}_{UW}\mathscr{E}$. Also

$$S \to P_{UW}S = \int_{K\times L} \mathfrak{L}(k^{-1})\mathfrak{R}(l)[U^{-1}(k)SV(l)]dkdl$$

defines continuous open projections of $\mathscr{D}'(V^*)$ onto ${}_{UW}\mathscr{D}'$ and $\mathscr{E}'(V^*)$ onto ${}_{UW}\mathscr{E}'$.

COROLLARY. *${}_{UW}\mathscr{D}'$ is the dual of ${}_{UW}\mathscr{D}$ and ${}_{UW}\mathscr{E}'$ is the dual of ${}_{UW}\mathscr{E}$.*

Proof. This is an immediate consequence of Proposition 6 and the fact that, for $S \in \mathscr{D}'$, $f \in \mathscr{D}$ (or for $S \in \mathscr{E}'$, $f \in \mathscr{E}$), we have $P_{UW}S\cdot f = S\cdot P_{UW}f$.

Suppose that $K=L$; then we see easily that the convolution defined in § 3 defines a separately continuous bilinear map of ${}_{UW}\mathscr{D}' \times {}_{WZ}\mathscr{D} \to {}_{UZ}\mathscr{E}(C)$ (where U, W, Z are representations of K on V). The method of proof of Theorems 1 and 2 can be used to show.

THEOREM 3. *${}_{UW}\mathscr{E}'$ consists of all $S \in {}_{UW}\mathscr{D}'$ such that $S * f \in {}_{UW}\mathscr{D}$ for any $f \in {}_{WW}\mathscr{D}$. The topolagy of ${}_{UW}\mathscr{E}'$ is sequentially the same as that obtained by considering the elements of ${}_{UW}\mathscr{E}'$ as defining (by convolution) continuous linear maps of ${}_{WW}\mathscr{D} \to {}_{UW}\mathscr{D}$ and giving this set the*

[3] Note that since L is compact, the restriction of η to L is 1.

compact-open topology τ. *Moreover, the bounded sets of* ${}_{UW}\mathscr{E}'$ *are the same as those of* τ.

REMARK 1. We do not know whether the topologies τ and that of ${}_{UW}\mathscr{E}'$ are the same. The difficulty is that, for $f \in {}_{WW}\mathscr{D}$, $g \in G$, $\mathfrak{L}(g)f$ is no longer in ${}_{WW}\mathscr{D}$.

REMARK 2. In case that $K=L$, V is finite dimensional, and U, W, X, Z are irreducible unitary representations of K on V, then it follows easily from the Schur orthogonality relations that $S*f=0$ for any $S \in {}_{UW}\mathscr{E}'$ $f \in {}_{XZ}\mathscr{D}$ if W is not equivalent to X.

REMARK 3. The conclusion of Theorem 3 does not necessarily hold if the space ${}_{WW}\mathscr{D}$ in the hypothesis of the theorem is replaced by ${}_{WZ}\mathscr{D}$ where Z is different from W, even if V is finite dimensional and U, W, Z are irreducible unitary representations of K. An example will be given in a forthcoming paper of the author and F. I. Mautner. (G can be taken as the complex unimodular group.)

6. General remarks. We have assumed that G is a separable Lie group. In the general case, the spaces $\mathscr{E}$ and $\mathscr{D}$ can be defined as before, but $\mathscr{E}$ will not be metrizable and $\mathscr{D}$ will not be an $\mathscr{L}\mathscr{F}$ space in the sense of Dieudonné and Schwartz [4] because $\mathscr{D}$ will be the inductive limit of a non-denumerable number of spaces, For this reason, the topology of $\mathscr{D}$ is best defined as follows: Let $\{f_{ij}\}=\sigma$ be a family of continuous functions on G such that

(a) For each i, only a finite number of j appear.

(b) Only a finite number of f_{ij} are different from zero on any compact set of G.

Then we define N_σ as the set of $h \in \mathscr{D}$ for which

$$\max_{x\in G} \rho_j[(f_{ij}(x)Q_jh)(x)] \leqq 1$$

for all i, j, where the Q_j are as in § 2, and $\{\rho_j\}$ denotes an enumeration of semi-norms which are sufficient to define the topology of V. The sets N_σ are seen to form a fundamental system of neighborhoods of zero of a locally convex topological vector space which we shall call $\mathscr{D}$. In case $\mathscr{D}$ is separable it is easily verified that the two definitions agree.

The advantage of the above definition is that it implies immediately the completeness of $\mathscr{D}$. For, the completion of $\mathscr{D}$ obviously consists of indefinitely differentiable maps. Moreover, if h is any map in the completion of $\mathscr{D}$, then, for any continuous function f on G, and any

k, it is easily seen that $\rho_k(fh)$ is a bounded function. This implies immediately that h is of compact carrier, hence $h \in \mathscr{D}$.

The properties of convolution can be extended to the nonseparable case and there is no difficulty in extending part of our main results. We can, as in § 5, prove only that the topology of $\mathscr{E}'$ is sequentially, and in regard to bounded sets, the same as the compact-open topology of the space of linear transformations of $\mathscr{D} \to \mathscr{D}$ (under convolution).

The results of § 5 on double coset spaces $K\backslash G/L$ can also be extended to functions invariant under a compact group of automorphisms of G (the group of automorphisms of G is given the compact-open topology).

In addition, the main results of this paper can be extended to locally compact groups. There $\mathscr{E}$ is replaced by the space of continuous functions, $\mathscr{D}$ the space of continuous functions of compact carrier, $\mathscr{D}'$ the space of measures and $\mathscr{E}'$ the space of measures of compact carrier.

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