ON MAPPINGS FROM THE FAMILY OF WELL ORDERED SUBSETS OF A SET

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A simply ordered set E is called a *k-set* if there exists a simply ordered extension of the family of nonempty well ordered subsets of E, ordered by initial segments, into E. If E is not a k-set then it is called a *k′-set*. Kurepa [1;2] first discussed these sets. He showed that if E is a subset of the reals and if the smallest ordinal number α such that E does not contain a subset of order type α is ω_1, then E is a k'-set. In particular the rationals and the reals, denoted by R and R^+ respectively, are both k'-sets. In this paper the existence of k-sets and k'-sets is discussed further. Theorem 7 states that each simply ordered set E is a terminal segment of some k-set $F(E)$. It is not true, however, that each simply ordered set E is similar to an initial section of some k-set $F(E)$ (Theorem 2). Finally, in Theorem 10 it is shown that each infinite simply ordered group is a k'-set.

Following the symbolism in [1 ; 2] let E be a simply ordered set and ωE the family of all nonempty well ordered subsets of E, partially ordered as follows: For A and B in ωE, $A <_k B$ if and only if A is a proper initial segment of B.[1]

Definition. A function f from ωE to E is called a *k-function* on E, if $A <_k B$ implies that $f(A) < f(B)$.

If there exists a k-function on E, that is, from ωE to E, then E is called a *k-set*. If not, then E is called a *k′-set*.

THEOREM 1. *If f is a k-function on E, then for each nonempty well ordered subset W of E, there exists an element x in W such that $f(W) \leq x$.*

Proof. Suppose that the theorem is false, that is, suppose that there exists an element W_1 in ωE with the property that $x < f(W_1)$ for each x in W_1. Let $W_2 = W_1 \cup f(W_1)$. It is easily seen that W_2 is well ordered, $W_1 <_k W_2$, $x < f(W_2)$ for each element x in W_2, and the order type of W_2 is ≥ 2. Suppose that for each $0 < \xi < \alpha$, W_ξ is an element

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[1] A is a (proper) initial segment of B if A is a (proper) subset of B and if, for each element z in A, $\{x | x \leq z, x \in B\}$ is a subset of A. A is a terminal segment of B if A is a subset of B and if, for each element z in A, $\{x | z \leq x, x \in B\}$ is a subset of A.

of ωE such that

(1) $x < f(W_\xi)$ for each x in W_ξ,

(2) $W_\xi <_k W_\upsilon$ for $\xi < \upsilon < \alpha$,

and (3) the order type of W_ξ is $\geqq \xi$.

Two possibilities arise.

(a) If $\alpha = \beta + 1$ let $W_\alpha = W_\beta \cup f(W_\beta)$. By (1) and the fact that W_β is well ordered, it follows that W_α is well ordered. Clearly $W_\beta <_k W_\alpha$. Thus $f(W_\beta) < f(W_\alpha)$. It is now easy to verify that (1), (2), and (3) are satisfied for $\xi \leqq \alpha$.

(b) Suppose that α is a limit number. Let $W_\alpha = \bigcup_{\xi < \alpha} W_\xi$. Since $W_\xi <_k W_\upsilon$ for $\xi < \upsilon$, W_α is well ordered. It is obvious that (2) and (3) are satisfied for $\xi \leqq \alpha$. Let x be any element of W_α. Then x is in W_ξ for some $\xi < \alpha$, thus $x < f(W_\xi) < f(W_\alpha)$. Hence (1) is also satisfied.

In this way W_ξ becomes defined for each ordinal number ξ. Thus W_δ is defined, where δ is the smallest ordinal number such that E contains no subset of order type δ. This is a contradiction since W_δ is of order type $\geqq \delta$.

We conclude that no such set W_1 exists, that is, the theorem is true.

Suppose that E is a k'-set and that the ordered sum[2] $E+F$ is a k-set for some simply ordered set F. Let f be a k-function on $E+F$. Since E is a k'-set, for some well ordered subset W of E, $f(W)$ is not in E, thus is in F. Then $f(W) \leqq x$ for some x in W is false. By Theorem 1, therefore, f is not a k-function on $E+F$. Hence we have

THEOREM 2. *If E is a k'-set then so is $E+F$ for every simply ordered set F.*

The simplest example of a k'-set E is any infinite well ordered set. This is an immediate consequence of the following observation, whose proof is by a straightforward application of transfinite induction.

'The initial segments of an infinite well ordered set of order type α form a set of order type $\alpha+1$'.

Another consequence of this observation is the following: For any infinite k-set E, the smallest ordinal number δ having the property that E contains no subset of order type δ, is a limit number.

Suppose that E is a k-set and has an initial segment of n-elements, say $x_0 < x_1 < \cdots < x_{n-1}$. Letting $A_j = \{x_i | i < j\}$, by a simple application of Theorem 1, it is easily seen that $f(A_j) = x_{j-1}$ for each k-function f on E. In other words, there is no element x of A_j such that $f(A_j) < x$.

[2] The ordered sum $\sum_\upsilon E_\upsilon$, or $\cdots + E_{\upsilon_1} + \cdots + E_{\upsilon_2} + \cdots$, of a family of pairwise disjoint simply ordered sets is the set $E = \bigcup_\upsilon E_\upsilon$ ordered as follows: If x and y are in the same E_υ, then $x < y$ or $y < x$ according as $x < y$ or $y < x$ in E_υ. If x is in E_υ and y is in E_ν and $\upsilon < \nu$ in V, then $x < y$.

This result cannot occur if E has no first element. To be precise we have:

THEOREM 3. *If E is a k-set without a first element, then there exists a k-function g such that $g(W) < x$ for each element W in ωE and for some element x in W.*

Proof. Let f be a k-function on E. Well order the elements of ωE into the sequence $\{W_\xi\}$, $\xi < \delta$. Suppose that g is already defined for each W_ξ, $\xi < \theta$ (possibly other W_ξ also) such that

(1) $g(W_\lambda) \leqq f(W_\lambda)$ for each W_λ for which g is defined;
(2) g is not defined for W_θ;
(3) if g is defined for W_γ, then g is also defined for each initial segment of W_γ;
(4) if $W_\sigma <_k W_\tau$ and g is defined for W_σ and W_τ, then $g(W_\sigma) < g(W_\tau)$;
(5) if g is defined for W_ξ, then $g(W_\xi) < x_\xi$ for some element x_ξ in W_ξ.

Let $W_\theta = \{x_{\theta,\upsilon} | \upsilon < \alpha(\theta)\}$ and $W_{\theta,\xi} = \{x_{\theta,\upsilon} | \upsilon < \xi\}$ for $0 < \xi \leqq \alpha(\theta)$. Let $W_{\theta,\gamma}$ be the first $W_{\theta,\xi}$ for which g is not defined: If $\gamma = 1$, that is, $W_{\theta,\gamma} = \{x_{\theta,0}\}$ let $g(W_{\theta,1})$ be some element of E which is $< \min [x_{\theta,0}, f(x_{\theta,0})]$. Such an element exists since E has no first element. Suppose that $\gamma = \beta + 1$, where $\beta > 0$. By induction, $g(W_{\theta,\beta}) < x_{\theta,\beta}$ for some element $x_{\theta,\beta}$ in $W_{\theta,\beta}$. Let $g(W_{\theta,\beta+1}) = \min [x_{\theta,\beta}, f(W_{\theta,\beta+1})]$. Since $W_{\theta,\beta} < W_{\theta,\beta+1}$, $x_{\theta,\beta}$ is not the last element in $W_{\theta,\beta+1}$. Thus $g(W_{\theta,\beta+1}) < x_{\theta,\beta+1}$ for some element $x_{\theta,\beta+1}$ in $W_{\theta,\beta+1}$. Suppose that $W_\sigma <_k W_{\theta,\beta+1}$. If $g(W_{\theta,\beta+1}) = x_{\theta,\beta}$, then $g(W_\sigma) \leqq g(W_{\theta,\beta}) < x_{\theta,\beta} = g(W_{\theta,\beta})$. If $g(W_{\theta,\beta+1}) = f(W_{\theta,\beta+1})$, then

$$g(W_\sigma) \leqq g(W_{\theta,\beta}) \leqq f(W_{\theta,\beta}) < f(W_{\theta,\beta+1}) = g(W_{\theta,\beta+1}) .$$

Suppose that γ is a limit number. Then $W_{\theta,\gamma}$ has no last element. It follows from Theorem 1 that there exists an element $x_{\theta,\gamma}$ in $W_{\theta,\gamma}$ so that $f(W_{\theta,\gamma}) < x_{\theta,\gamma}$. Let $g(W_{\theta,\gamma}) = f(W_{\theta,\gamma})$. If $W_\sigma <_k W_{\theta,\gamma}$, then

$$g(W_\sigma) \leqq f(W_\sigma) < f(W_{\theta,\gamma}) = g(W_{\theta,\gamma}) .$$

By transfinite induction g becomes defined for each $W_{\theta,\xi}$, thus for W_θ so as to satisfy (1), (3), (4), and (5). Thus g becomes defined for every W_ξ. From the manner of construction, that is (4), g is a k-function. By (5) g has the property that for each element W in ωE, $g(W) < x$ for some element x in W.

THEOREM 4. *If $\overline{A} \equiv \overline{B}$*[3] *and A is a k-set, then so is B. Equivalently. if $\overline{A} \equiv \overline{B}$ and A is a k'-set, then so is B.*

[3] E being a simply ordered set, $\overline{E}$ denotes the order type of E. $\overline{A} \equiv \overline{B}$ if there exists a similarity transformation of A into B and a similarity transformation of B into A.

Proof. Let g be a similarity transformation of A into B and h a similarity transformation of B into A. Suppose that f is a k-function of ωA into A. For each well ordered subset E of B, $h(E)$ is a well ordered subset of A which is similar to E. Let f^* be the function of ωB into B which is defined by $f^*(E)=gfh(E)$. Clearly $gfh(C)<gfh(D)$ if $C \underset{k}{<} D$. Thus f^* is a k-function, so that B is a k-set.

Turning to the construction of k-sets we have

THEOREM 5. *If $\{E_v | v \in V\}$ is a family of pairwise disjoint k-sets, and V is the dual*[4] *of a well ordered set, then the ordered sum ΣE_v is a k-set.*

Proof. Let f_v be a k-function from ωE_v to E_v. Now let A be a nonempty well ordered subset of ΣE_v. Denote by w the largest element v in V such that $A \cap E_v$ is nonempty. Since V is the dual of a well ordered set, w exists. Let h be the function which is defined by $h(A) = f_w(A \cap E_w)$. There is no trouble verifying that h is a k-function from $\omega \Sigma E_v$ to ΣE_v.

COROLLARY. *The dual of a well ordered set is a k-set. One particular k-function is the mapping which takes a well ordered subset into its largest element.*

Another method of obtaining k-sets is to use the next result.

THEOREM 6. *Let $\{A_v | v \in V\}$ be a family of pairwise disjoint simply ordered sets where V is the dual of a well ordered set of order type α, α being a limit number. Furthermore suppose that for each element w in V, there exists a simply ordered extension f_w of $A^w = \omega \sum_{v>w} A_v$ into A_w*[5]. *Then $A = \sum_{v \in V} A_v$ is a k-set.*

Proof. Let X be any nonempty well ordered subset of A. Let x_0 be the first element in X. x_0 is in one of the sets A_v, say A_r. Since α is a limit number, r has an immediate predecessor in V, say r^-. By hypothesis there exists a simply ordered extension f_{r^-} of $\omega A^{r^-} = \omega \sum_{v>r^-} A_v$ into A_{r^-}. Let $f(X) = f_{r^-}(X)$. Thus f is a well defined function from ωA into A.

Suppose that $Y \underset{k}{<} Z$ in ωA. The first element in Y, say y_0, is also the first element in Z. If y_0 is in A_s, then $f(Y) = f_{s^-}(Y) < f_{s^-}(Z) = f(Z)$. Thus f is a k-function and A is a k-set.

Now let E_0 be any simply ordered set. It is known that each

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[4] $(\rho, <')$ is the dual of $(\rho, <)$ if $x<'y$ if and only if $x>y$, for every x and y in ρ.

[5] f is a simply ordered extension of the partially ordered set B into the simply ordered set A if f maps B into A in such a manner that whenever $x<y$ in B, $f(x)<f(y)$ in A.

partially ordered set has a simply ordered extension [3]. Let f_0 be a simply ordered extension of ωE_0 into some set, say F_0. Let E_1 be a simply ordered set such that $\overline{E}_1 = \overline{F}_0 + \overline{E}_0$. Continuing by induction we obtain for each ordinal number υ, a simply ordered extension f_υ of ωG_υ, where $\overline{G}_\upsilon = \cdots + \overline{E}_\xi + \cdots + \overline{E}_1 + \overline{E}_0$ $(\xi < \upsilon)$, into a simply ordered set F_υ. Let E_υ be a simply ordered set such that $\overline{E}_\upsilon = \overline{F}_\upsilon + \overline{G}_\upsilon$. In particular, by Theorem 6, G_ω is a k-set. Thus we have

THEOREM 7. *Each simply ordered set E is a terminal segment*[1] *of some k-set $F(E)$.*

REMARK. Theorem 2 shows that there exist simply ordered sets E such that for no k-set $F(E)$ is E similar to an initial segment of $F(E)$.

We now consider products of simply ordered sets, ordered by last differences.

THEOREM 8. *If E and F are k-sets, then* so *is $E \times F$.*

Proof. Let f and g be k-functions for E and F respectively, and z a definite element of E. Let A be any well ordered subset of $E \times F$. Define A_τ to be the set $\{v|\text{for some } u, (u, v) \text{ is in } A\}$. Obviously A_τ is a well ordered subset of F. If A_τ has a last element, say w, let $A_\sigma = \{u|(u, w) \text{ is in } A\}$ and let $h(A) = (f(A_\sigma), g(A_\tau))$. If A_τ has no last element, let $h(A) = (z, g(A_\tau))$. To see that h is a k-function let $A <_k B$ in $\omega E \times F$. Since A is a proper initial segment of B, either A_τ is a proper initial segment of B_τ, or else $A_\tau = B_\tau$. If the former holds, then since $g(A_\tau) < g(B_\tau)$, $h(A) < h(B)$. Suppose that the latter holds. Since $A <_k B$, there exists an element (x, y) in B which is not in A. Thus $A \subseteq \{(u, v)| (u,v) < (x, y), (u, v) \text{ in } B\}$. Since $A_\tau = B_\tau$, it follows that y must be the last element of B_τ, thus also of A_τ. Therefore A_σ and B_σ exist. Since A is a proper initial segment of B, $A_\sigma <_k B_\sigma$. As f is a k-function, $f(A_\sigma) < f(B_\sigma)$. Hence

$$h(A) = [f(A_\sigma), g(A_\tau)] < [f(B_\sigma), g(A_\tau)] = h(B).$$

REMARKS. (1) Theorem 8 is no longer true if one of the sets, either A or B is a k'-set. This is seen by two examples.

(a) Let E be a set of one element and F a set order type ω. Then $E \times F$ is of order type ω, thus a k'-set.

(b) Interchange E and F in (a).

(2) The conclusion of Theorem 8 may be true if one of the sets is a k-set and the other is not. For example

(a) Let $\overline{E} = \omega^{\omega^*}$ and $\overline{F} = \omega$. Then $\overline{E} \times \overline{F} = \overline{E}$, and as easily seen, E

is a k-set. It is also easy to show that for each ordinal number α and each limit number δ, $A_\alpha \times B_\delta$ is a k-set, where $\overline{A_\alpha}=\alpha$ and $\overline{B_\delta}=\delta^*$. If $\alpha \geqq \omega$, then $B_\delta \times A_\alpha$ is a k'-set.

(b) Let $A_0=R$, f_1 be a simply ordered extension of wA_0 into B_1, and $A_{-1}=(A_0 \times B_1)$. In general, let f_n be a simply ordered extension of $\omega(\sum_{i<n} A_{-i})$ into B_n, and $A_{-n}=(A_0 \times B_n)$. Let $F=\sum_{n<\omega} A_{-n}$. By Theorem 6, F is a k-set. Then $\overline{A_0 \times F}=\sum \overline{(A_0 \times A_{-n})}=\sum \overline{A_{-n}}=\overline{F}$. Thus $A_0 \times F$ is a k-set. It is known [1;2] that A_0 is a k'-set.

(3) Theorem 8 is no longer true if we have a product of an infinite number of k-sets. For example, for each negative integer υ let $E_\upsilon=\{0, 1\}$. Then ΠE_υ is the set of all zero-one sequences of order type ω^*, ordered by last differences. But $\overline{\Pi_\upsilon E_\upsilon} \equiv \lambda$, where $\lambda=\overline{R^+}$. R^+ is a k'-set [2]. By Theorem 4, ΠE_υ is a k'-set.

Question. Do there exist two k'-sets E and F such that $E \times F$ is a k-set?

THEOREM 9. *If E is a k'-set and F is a simply ordered set with a first element, then $E \times F$ is a k'-set.*

Proof. Let x_0 be the first element of F and $G=F-\{x_0\}$. Then $E \times F=E \times [\{x_0\}+G]=E \times \{x_0\}+E \times G$. Since $E \times \{x_0\}$ is a k'-set, by Theorem 2 so is $E \times \{x_0\}+E \times G$. Hence the result.

Since $\lambda \equiv 1+\lambda$ and $\eta \equiv 1+\eta$, where $\eta=\overline{R}$, it follows from Theorem 4 and Theorem 9 that for any k'-set A, $A \times R$ and $A \times R^+$ are k'-sets. In particular, Euclidean n-space, ordered by last differences of the coordinates of the points, is a k'-set.

THEOREM 10. *Each infinite simply ordered group is a k'-set. If E is an ordered field, then there is no k-function from the bounded elements of ωE to E.*

Proof. First suppose that E is an ordered field. Let 1 be the multiplicative identity. For $1<x$ let $h(x)=2-1/x$ where $2=1+1$. For $0 \leqq x \leqq 1$ let $h(x)=x$. For $x<0$ let $h(x)=-h(-x)$. Then h is a similarity transformation of E onto $(-2, 2)$.

Suppose that f is a k-function from the bounded elements of ωE to E. Let $x_0=z_0=0$, $z_1=1$, $x_1=h(1)$, and $A_j=\{x_i | i<j\}$ for $j=1, 2$. Let $y_1=f(A_1)$ and $y_2=f(A_2)$. Clearly $y_1<y_2$. Let $z_2=z_1+(y_2-y_1)$. Thus $z_2-z_1=y_2-y_1$. Let $x_2=h(z_2)$. In general suppose that for $1<\xi<\alpha$, z_ξ, $x_\xi=h(z_\xi)$, $A_\xi=\{x_\upsilon | \upsilon<\xi\}$, and $y_\xi=f(A_\xi)$ are defined. Furthermore, suppose that $\{z_\xi\}$ and $\{y_\xi\}$ are strictly increasing and that $z_\xi-z_1=y_\xi-y_1$ for

$1 < \xi$. Since E is a group, z_ξ and x_ξ are elements of E. Observe that $-2 < x_\xi < 2$, that is $\{x_\xi\}$ is a bounded sequence.

(1) Suppose that $\alpha = \beta + 1$. Let $A_\alpha = \{x_\xi | \xi < \alpha\}$, $y_\alpha = f(A_\alpha)$, $z_\alpha = z_\beta + (y_\alpha - y_\beta)$, and $x_\alpha = h(z_\alpha)$. Since $A_\beta <_k A_\alpha$, $y_\beta < y_\alpha$. Thus $z_\beta < z_\alpha$ and $x_\beta < x_\alpha$. Since $z_\alpha - z_\beta = y_\alpha - y_\beta$ and $z_\beta - z_1 = y_\beta - y_1$, we get $z_\alpha - z_1 = y_\alpha - y_1$.

(2) Suppose that α is a limit number. Let $A_\alpha = \{x_\xi | \xi < \alpha\}$ and $y_\alpha = f(A_\alpha)$. Since $A_\xi <_k A_\alpha$, for $\xi < \alpha$, $y_\xi < y_\alpha$. Let $z_\alpha = z_1 + (y_\alpha - y_1)$ and $x_\alpha = h(z_\alpha)$. Since $A_\xi <_k A_\alpha$ for $\xi < \alpha$, $y_\xi < y_\alpha$ and thus $z_\xi < z_\alpha$ and $x_\xi < x_\alpha$. Note that $z_\alpha - z_1 = y_\alpha - y_1$.

In this way, for each ξ we get an x_ξ. Let δ be the smallest ordinal number such that E contains no subset of order type δ. The elements of the set $\{x_\xi | \xi < \delta\}$ form a strictly increasing sequence of order type δ. From this contradiction we see that no such function f exists.

Now suppose that E is an infinite simply ordered group. Let $z_0 = 0$ and $z_1 > 0$. Let $A_j = \{z_i | i < j\}$ for $j = 1, 2$. Let $y_1 = f(A_1)$ and $y_2 = f(A_2)$. Repeat the procedure given above, defining y_ξ and z_ξ for each ξ, with $A_\upsilon = \{z_\xi | \xi < \upsilon\}$. We obtain a strictly increasing sequence of elements $\{z_\xi\}$, $\xi < \delta$, where δ has the same significance as above. Again we arrive at a contradiction.

REMARK. The second statement in Theorem 10 cannot be extended to hold for a group. For example, let E be the group consisting of all the integers, positive, negative, and zero. The bounded, well ordered subsets of E consist of the finite subsets of E. For this family there does exist a k-function, namely the function which maps each set into its maximal element.

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