SIMPLIFIED PROOFS OF "SOME TAUBERIAN THEOREMS" OF JAKIMOVSKI

C. T. RAJAGOPAL

1. Introduction. In this note, the preceding paper (mentioned in the title) will be referred to as [J], the papers or books numbered 1, 2, $\cdots$ in the bibliography concluding [J] will be referred to as [J1], [J2], $\cdots$, while those in the numbered list of references at the end will be referred to by their numbers in square brackets.

The notation in [J] is retained with a slight simplification as follows. As in Hardy's *Divergent series* [J3], a sequence $\{t_n\}$ is called a Hausdorff transform of another sequence $\{s_n\}$ when there is a sequence $\{\mu_n\}$ such that

$$(1) \qquad \Delta^n t_0 = \mu_n \Delta^n s_0 .$$

If α is a real number, the special case of $\{t_n\}$ defined by (1) with $\mu_n = (n+1)^{-\alpha}$, called the (H, α) transform, will be denoted by $H^\alpha s$ where s denotes the sequence $\{s_n\}$. Since two Hausdorff transformations are commutable, the operator H^α is such that $H^\alpha H^\beta = H^\beta H^\alpha = H^{\alpha+\beta}$ and H^0 is the identity operator.

From the Abel or (A) transform of $\{s_n\}$, defined as the left-hand member of

$$(2) \qquad (1-x)\sum_0^\infty s_n x^n = (-1)^p (1-x)^{-p+1} \sum_0^\infty \Delta^p s_{n-p} x^n ,$$
$$0 < x < 1, \; p=1, 2, 3, \cdots,$$

we deduce the equality (2) by induction on p. It is in the form of the right-hand member of (2) that the (A) transform is used in this note.

For any sequence $\{s_n\}$, summability (H, α) to a finite value l and summability (A) to l have their usual meanings as in [J].

2. The fundamental theorem in [J]. This theorem ([J], Theorem 2) may be restated as follows with its non-trivial parts separated, so that Tauber's first theorem ([J3], Theorem 85) emerges as the case $k=1$ of the first part, with the conclusion of the convergence of $\{s_n\}$ restated as that of the (H, -1) summability of $\{s_n\}$.

THEOREM 1. (a) *If* (i) $\{s_n\}$ *is summable* (A) *to* l, (ii) *for a positive integer* k, $n^k \Delta^k s_{n-k} = o(1)$, $n \to \infty$, *then* $\{s_n\}$ *is summable* (H, $-k$) *to* l.

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(b) *Conditions* (i) *and* (ii) *are also necessary for* $\{s_n\}$ *to be summable* (H, $-k$) *to* l.

For establishing this theorem, Jakimovski's tools are (1) the Tauberian technique embodied in Lemma 2 of this note with the additional complications necessary to bring in $n^p \Delta^p s_{n-p}$ to take the place of $n\Delta^p s_{n-1}$, (2) the technique of repeated differences (or differentiation) implicit in his appeal to one particular case of a theorem proved by Parthasarathy and Rajagopal ([J6], case $k=l+r$ of Theorem C). However, the second technique, while generally useful in proving Tauberian theorems of the Hardy-Littlewood class, is not required at all for proving the original Tauberian theorems; and it is perhaps not very satisfactory to use it to prove Theorem 1 which is essentially of the latter class of theorems. The present note supplies a new proof of Theorem 1 whose merit is that it depends only on Lemma 2 as it stands and on the interpretation, in Lemma 1, of $n(n-1)\cdots(n-p+1)\Delta^p s_{n-p}$, which is asympotically equal to $n^p \Delta^p s_{n-p}$, as a Hausdorff transform of s_n. Although the content of Lemma 1 is due to Jakimovski, the proof of Lemma 1 as it appears here is a simplification of his proof, resulting from the symbolic representation (5) of the Hausdorff transformation of s_n in question, suggested to me by Mr. M. R. Parameswaran.

LEMMA 1. *If k is a positive integer and*

$$t_n \equiv \binom{n}{k} \Delta^k s_{n-k}, \tag{3}$$

then t_n is related to s_n by (1) *with*

$$\mu_n = (-1)^k \binom{n}{k}, \tag{4}$$

that is, $\{t_n\}$ in (3) *is the Hausdorff transform of $\{s_n\}$ corresponding to the $\{\mu_n\}$ defined by* (4), *and further we have symbolically*

$$\{t_n\} \equiv \frac{(-1)^k}{k!} H^{-k}(H^0-H^1)(H^0-2H^1)\cdots(H^0-kH^1)s \tag{5}$$

$$\equiv \left(\sum_{r=0}^{k} a_r^{(k)} H^{-k+r}\right)s, \quad \sum_{r=0}^{k} a_r^{(k)} = 0,$$

the order of factors in (5) *being immaterial.*

Here I must record may indebtedness to Dr. Jakimovski who has pointed out an implication of the first part of (5), namely, that

$$\binom{n}{k}\Delta^k s_{n-k} = \frac{(-1)^k}{k!}\sum_{r=0}^{k} S_{k+1}^{k-r+1} H^{-k+r} s ,$$

where S_n^m are Stirling's numbers of the first kind ([1], p. 142, (3)).

Proof. The relation between s_n and t_n is proved directly, starting from

$$\Delta^n t_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} t_r ,$$

and showing that substitution for t_r from (3) leads to (1) with the $\{\mu_n\}$ in (4).

Equation (5) follows from the fact that (4) can be written:

$$\mu_n = \frac{(-1)^k}{k!}(n+1)^k\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k}{n+1}\right) .$$

Now the factors $(n+1)^k$, $1-(n+1)^{-1}$, $1-2(n+1)^{-1}$, $\cdots$, $1-k(n+1)^{-1}$, taken successively instead of μ_n in (1), make the $\{t_n\}$ of $\{1\}$ the Hausdorff transforms of $\{s_n\}$ corresponding to the operators H^{-k}, H^0-H^1, H^0-2H^1, $\cdots$, H^0-kH^1 respectively. Hence the $\{t_n\}$ of (3) is the product of the several Hausdorff transforms last mentioned multiplied by $(-1)^k/k!$. We thus have the representation in (5) of the $\{t_n\}$ in (3), and we can take the factors in this representation in any order since Hausdorff transformations are commutable.

LEMMA 2. *If $\{s_n\}$ is such that $n\Delta^1 s_{n-1}=O(1)$, then, for $x=1-n^{-1}$,*

$$\limsup_{n\to\infty}\left|\sum_{r=0}^{n}\Delta^1 s_{r-1} - \sum_{r=0}^{\infty}\Delta^1 s_{r-1}x^r\right| \leqq \tau \limsup_{n\to\infty} |n\Delta^1 s_{n-1}|$$

where τ is the 'Tauberian constant':

$$\tau = C + 2\int_1^\infty e^{-x}x^{-1}\,dx , \qquad C=\textit{Euler's constant}.$$

This result, due to Hadwiger ([J2], inequality (15)), is a particular case of a more general result (e.g. [2], case $\alpha=1$, $\lambda_n=n$, $\phi^*(u)=e^{-u}$ of Theorem 2(b)).

Proof of Theorem 1. (a) We may suppose without loss of generality that $l=0$. For, we have only to consider, instead of $\{s_n\}$, the new sequence $\{s_n-l\}$ which is clearly subject to hypothesis (i) with $l=0$ and also hypothesis (ii).

First, we take $\Delta^k s_{n-k}$ instead of $\Delta^1 s_{n-1}$ in Lemma 1 and obtain, for $x=1-n^{-1}$,

$$(6)\ \limsup_{n\to\infty}(1-x)^{-k+1}\left|\sum_{r=0}^{n}\Delta^k s_{r-k}-\sum_{r=0}^{\infty}\Delta^k s_{r-k}x^r\right|\leqq\tau\limsup_{n\to\infty}(1-x)^{-k+1}|n\Delta^k s_{n-k}|$$
$$=\tau\limsup_{n\leftarrow\infty}|n^k\Delta^k s_{n-k}|\,.$$

Next, we take $p=k$ in (2) and get

$$(7)\qquad (-1)^k(1-x)^{-k+1}\sum_{r=0}^{\infty}\Delta^k s_{r-k}x^r=(1-x)\sum_{r=0}^{\infty}s_r x^r=o(1)\,,\qquad x\to 1-0,$$

as a result of hypothesis (i) where $l=0$ according to our supposition. Using in (6) hypothesis (ii) and (7) with $x=1-n^{-1}$, we obtain

$$n^{k-1}\Delta^{k-1}s_{n-k+1}=-(1-x)^{-k+1}\sum_{r=0}^{n}\Delta^k s_{r-k}=o(1)\,,\qquad n\to\infty\,.$$

If $k=1$, we infer at once that s_n converges to 0. If $k\geqq 2$, we repeat the foregoing argument with $k-1$, $k-2$, $\cdots$, 1 successively in place of k and find that $n^p\Delta^p s_{n-p}=o(1)$ for $p=k-2$, $k-3$, $\cdots$, 0, thus finally drawing the same inference as before. After this we use the fact, following from $n^p\Delta^p s_{n-p}=o(1)$, $1\leqq p\leqq k$, taken along with (5), that

$$(8)\qquad \frac{(-1)^p}{p!}H^{-p}(H^0-H^1)(H^0-2H^1)\cdots(H^0-pH^1)s\equiv\binom{n}{p}\Delta^p s_{n-p}=o(1)$$

as $n\to\infty$ for $p=1, 2, \cdots, k$, and prove successively that $H^{-1}s$, $H^{-2}s$, $\cdots$, $H^{-k}s$ all converge to $0=l$.

(b) If $\{s_n\}$ is summable (H, $-k$) to l, then $H^{-p}s$, $p=k, k-1,\cdots, 0$, are obviously each convergent to l and (8) necessarily holds for $p=k$; also $\{s_n\}$, being convergent to l, is necessarily summable (A) to l.

3. Remarks on other theorems in [J]. It may be pointed out how (5) in conjunction with the notation of this note simplifies the presentation of Jakimovski's main theorems ([J], Theorems 3,5) restated in this notation as Theorems 2,3. The simplified presentation, like the one given by Jakimovski, depends only on the results of the preceding section, O. Szász's theorem for the product of a regular Hausdorff method of summability and (A) summability ([J], Theorem D, generalized by Rajagopal in [3], Theorem I), and finally an idea whose simplest expression is the lemma which follows.

LEMMA 3. *If $\{s_n\}$ is summable* (A) *to l and the sequence denoted by $H^\alpha s$, where α is any real number, is bounded on one side, then $H^{\alpha+1}s$ is convergent to l.*

The case $\alpha=0$ of Lemma 4 is classical. The case $\alpha\neq 0$ is included in one of Jakimovski's theorems ([J4], Theorem (9.6)). However, it is best to deduce it from the case $\alpha=0$ by means of the following observation. If $\alpha>0$, then $H^{\alpha}s$ is summable (A) to l by Szász's product-theorem referred to above; while, if $\alpha<0$, $H^{\alpha}s$ is again summable (A) to l since it is summable (H, $-\alpha+1$) to l as a result of $s\equiv H^{-\alpha}(H^{\alpha}s)$ being bounded on one side and summable (A) to l.

In Lemma 3 we extend a Tauberian theorem for sequences s summable (A) by replacing s by $H^{\alpha}s$ in the Tauberian hypothesis and the conclusion. The method of extension shows that, in Theorem 1 (a), we may replace s by $H^{\alpha}s$, or, α being any real number, replace s by $H^{\alpha+k}s$, in hypothesis (ii) and the conclusion. The result of the replacement of s by $H^{\alpha+k}s$ is stated below.

THEOREM 2. (a) *If* (i) *the sequence* $\{s_n\}$ *is summable* (A) *to* l, (ii) *for a real number* α *and a positive integer* k, *the sequence* $H^{\alpha+k}t$ *is null, where* $t\equiv\{t_n\}$ *is defined by* (3) *or* (5), *then* $\{s_n\}$ *is summable* (H, α) *to* l.

(b) *Conditions* (i) *and* (ii) *are also clearly necessary for* $\{s_n\}$ *to be summable* (H, α) *to* l.

An immediate deduction from Theorem 2 is the next.

THEOREM 3. *If, in Theorem* 2(a), *condition* (ii) *is replaced by the condition that* $H^{\alpha+k}t$ *is bounded on one side, the conclusion will be that* $\{s_n\}$ *is summable* (H, $\alpha+1$) *to* l.

Proof. By Szász's product-theorem, $H^{k}t$ is summable (A) to 0. Hence, by Lemma 3 with $H^{k}t$ instead of s, $H^{\alpha+1+k}t$ is a null sequence, and the conclusion follows from Theorem 2(a) with $\alpha+1$ instead of α.

4. Addition. (November 23, 1956.) Szász ([4], p. 1019, Lemma 5) has proved the following theorem.

THEOREM X. *Let* $\{s_n\}$ *be a sequence which is* (i) *summable* (A) *to* l, (ii) *bounded below and quasi-monotonic-decreasing in the sense that there is a constant* $c>0$ *such that*

$$s_{n+1}\leqq(1+c/n)s_n\,, \qquad n>n_0(c).$$

Then $\{s_n\}$ *is convergent to* l.

Appealing to Lemma 3, we can replace $s\equiv\{s_n\}$ by $H^{\alpha}s$ in the hypothesis (ii) and the conclusion of Theorem X, and obtain the following theorem.

THEOREM Y. *Let* s *be a sequence such that* (i) *it is summable* (A) *to*

l, (ii) *its transform* $H^{\alpha}s$ *is bounded below and quasi-monotonic-decreasing according to the definition in Theorem* X. *Then* s *is summable* (H, α) *to* l.

The cases $\alpha=0$, $\alpha=-1$ of Theorem Y have applications to trigonometric series ([4]: p. 1020, Theorem 3 and p. 1031, Theorem 8).

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RAMANUJAN INSTITUTE OF MATHEMATICS
MADRAS, INDIA.