A REAL INVERSION FORMULA FOR A CLASS OF BILATERAL LAPLACE TRANSFORMS

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1. Introduction. The Post-Widder inversion formula for unilateral Laplace transformations [1] states that, under certain weak restrictions on $\phi(u)$,

$$\lim_{k\to\infty}\left(\frac{k}{c}\right)^{k+1}\frac{1}{k!}\int_0^\infty \phi(u)u^k \exp\left(-k\frac{u}{c}\right)du=\phi(c) ,$$

for any continuity point c of $\phi(u)$.

This formula applies when $\phi(u)$ is defined only for $u\geqq 0$. A similar formula may be deduced if $\phi(u)$ is defined for $u\geqq -a$, for some positive a. In such a case, we may let $\phi^*(u)=\phi(u-a)$, and we may then use the Post-Widder formula to determine $\phi^*(u)$ at the point $u=c+a$. The inversion formula then becomes

$$\lim_{k\to\infty}\left(\frac{k}{c+a}\right)^{k+1}\frac{1}{k!}\int_0^\infty \phi(u-a)u^k \exp\left(-k\frac{u}{c+a}\right)du=\phi(c) ,$$

or, if we make the transformation $z=u/(c+a)$,

$$(1)\qquad \lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_0^\infty \phi[(c+a)z-a]z^k \exp(-kz)dz=\phi(c) .$$

This suggests that, if $\phi(u)$ is defined for $-\infty<u<\infty$, some sort of limiting form of (1) applies. We shall prove that under suitable restrictions on ε and on the behavior of $\phi(u)$,

$$(2)\qquad \lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_{-\infty}^\infty \phi[(c+k^\varepsilon)z-k^\varepsilon]z^k \exp(-kz)dz=\phi(c) .$$

2. Remarks. In the following sections $\phi(u)$ will be assumed to be integrable over the interval from $-\infty$ to ∞, and c will be assumed to be a continuity point of $\phi(u)$. All limits should be understood to be for increasing values of k.

The expression $\delta/(c+k^\varepsilon)$, where δ and ε are positive numbers, occurs frequently. It will be denoted by $\delta(k, \varepsilon)$.

Finally, it may be noted that in terms of the Laplace transform of $\phi(u)$ for real t,

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$$f(t)=\int_{-\infty}^{\infty}\phi(u)\exp(-tu)du\ ,$$

the inversion formula (2) may be written in the form

$$\lim\frac{(-1)^k}{k!}\left(\frac{k}{c+k^\varepsilon}\right)^{k+1}\frac{d^k}{dt^k}[f(t)\exp(-tk^\varepsilon)]_{t=k/(c+k^\varepsilon)}=\phi(c)\ .$$

3. Preliminary proofs. The results of the following four lemmas will be needed below. Proofs are given for the first two. The second two are proved in a similar way.

LEMMA 1. *If n is any fixed number and $1/3<\varepsilon<1/2$, then*

$$\lim k^n[1+\delta(k,\varepsilon)]^k\exp[-k\delta(k,\varepsilon)]=0\ .$$

Proof. If the logarithm of the expression under the limit sign is expanded in powers of $\delta(k,\varepsilon)$, the sum of two of the terms in the expansion approaches $-\infty$ as $k\to\infty$, while the sum of the rest of the terms is bounded.

LEMMA 2. *If $1/3<\varepsilon<1/2$, then*

$$\lim\frac{k^{k+1}}{k!}\int_1^{1+\delta(k,\varepsilon)}z^k\exp(-kz)dz=\frac{1}{2}\ .$$

Proof. It is well known [1] that

$$\lim\frac{k^{k+1}}{k!}\int_1^{\infty}z^k\exp(-kz)dz=\frac{1}{2}\ .$$

Therefore, it is sufficient to show that

$$\lim\frac{k^{k+1}}{k!}\int_{1+\delta(k,\varepsilon)}^{\infty}z^k\exp(-kz)dz=0\ .$$

Since $z\exp(-z)$ is a decreasing function of z for $z>1$, the above expression is, for fixed k, no larger than

$$\frac{k^{k+1}}{k!}[1+\delta(k,\varepsilon)]^{k-1}\exp[-(k-1)(1+\delta(k,\varepsilon))]\int_{1+\delta(k,\varepsilon)}^{\infty}z\exp(-z)dz\ .$$

By applying Stirling's formula and Lemma 1, we see that the upper bound approaches zero as k increases.

LEMMA 3. *If n is any fixed number and $0<\varepsilon<1/2$, then*

$$\lim k^n[1-\delta(k, \varepsilon)]^k \exp [k\delta(k, \varepsilon)]=0 ,$$

LEMMA 4. *If* $0<\varepsilon<1/2$, *then*

$$\lim \frac{k^{k+1}}{k!}\int_{1-\delta(k,\varepsilon)}^{1} z^k \exp(-kz)dz=\frac{1}{2} .$$

4. The inversion formula.

THEOREM. *If*

$$\text{(a)} \qquad \left|\int_{-\infty}^{-x} \phi(z)\,dz\right| \leqq A \exp(-dx^{2+\alpha})$$

for some positive quantities A, d, *and* α, *and if*

$$\text{(b)} \qquad \max(1/3,\ 1/(2+\alpha))<\varepsilon<1/2,$$

then

$$\lim I_k=\lim \frac{k^{k+1}}{k!}\int_{-\infty}^{\infty} \phi[(c+k^\varepsilon)z-k^\varepsilon]z^k \exp(-kz)dz=\phi(c) .$$

Proof. For any $\delta>0$, the infinite interval may be partitioned into the four subintervals $(-\infty, 1-\delta(k, \varepsilon))$, $(1-\delta(k, \varepsilon), 1)$, $(1, 1+\delta(k, z))$, and $(1+(k, \varepsilon), \infty)$. I_k may be considered as the sum of four integrals over these intervals, so that we may write

$$I_k=I_k^{(1)}+I_k^{(2)}+I_k^{(3)}+I_k^{(4)} .$$

$I_k^{(1)}$ is understood to represent the integral over $(-\infty, 1-\delta(k, \varepsilon))$ etc.

$$|I_k-\phi(c)| \leqq |I_k^{(1)}|+\left|I_k^{(2)}-\frac{\phi(c)}{2}\right|+\left|I_k^{(3)}-\frac{\phi(c)}{2}\right|+|I_k^{(4)}| .$$

We prove first that $I_k^{(1)}$ and $I_k^{(4)}$ approach zero as $k\to\infty$. For $I_k^{(1)}$, consider first the integral over the interval from 0 to $1-\delta(k, \varepsilon)$. The function $z\exp(-z)$ attains its maximum at the upper endpoint. Therefore an upper bound for the absolute value of this portion of the expression is

$$\frac{k^{k+1}}{k!}[1-\delta(k, \varepsilon)]^k \exp[-k+k\delta(k, \varepsilon)]\int_0^{1-\delta(k,\varepsilon)} |\phi[(c+k^\varepsilon)z-k^\varepsilon]|dz ,$$

which approaches zero by Stirling's formula and Lemma 3.

Consider now the integral over the interval from $-\infty$ to 0. Integrating by parts, we find that it is equal to

$$-\frac{1}{c+k^{\varepsilon}}\frac{k^{k+2}}{k!}\int_{-\infty}^{0}F[(c+k^{\varepsilon})z-k^{\varepsilon}]z^{k+1}(1-z)\exp(-kz)dz\ ,$$

where $F(z)=\int_{-\infty}^{z}\phi(u)du$. Note that, by the assumption on $F(z)$,

$$|F[(c+k^{\varepsilon})z-k^{\varepsilon}]|\leqq A\exp[-d\{-(c+k^{\varepsilon})z+k^{\varepsilon}\}^{2+\alpha}]\ ,$$

which is in turn equal to or less than

$$A\exp[dz(c+k^{\varepsilon})k^{\varepsilon(1+\alpha)}]\ .$$

The result of the integration by parts may be written as the difference between two integrals, the first containing z^{k-1} and the second containing z^{k}. The first integral is no greater in absolute value than

$$\frac{A}{(c+k^{\varepsilon})}\frac{k^{k+2}}{k!}\int_{-\infty}^{0}|z^{k-1}|\exp[z\{d(c+k^{\varepsilon})k^{\varepsilon(1+\alpha)}-k\}]dz\ .$$

Since $\varepsilon(2+\alpha)>1$, the coefficient of z in the exponent above is positive for sufficiently large k. Therefore, after some manipulation, this upper bound can be shown to be equal to

$$\frac{A}{(c+k^{\varepsilon})}\frac{k^{k+2}}{k!}\cdot\frac{\Gamma(k)}{[d(c+k^{\varepsilon})k^{\varepsilon(1+\alpha)}-k]^{k}}\ ,$$

which approaches zero as $k\to\infty$.

By the same argument, the second integral approaches zero, so that $\lim I_k^{(1)}=0$.

For $I_k^{(4)}$, observe that since $z\exp(-z)$ is a decreasing function of z for $z>1$, the expression has the following upper bound for its absolute value:

$$\frac{k^{k+1}}{k!}[1+\delta(k,\varepsilon)]^{k}\exp[-k-k\delta(k,\varepsilon)]\int_{1+\delta(k,\varepsilon)}^{\infty}|\phi[(c+k^{\varepsilon})z-k^{\varepsilon}]|dz\ .$$

Since the integral is bounded, the whole upper bound approaches zero by virtue of Stirling's formula and Lemma 1.

We now prove that

$$\left|\lim I_k^{(3)}-\frac{1}{2}\phi(c)\right|<\frac{1}{2}\eta$$

for any $\eta>0$. By Lemma 2, it is sufficient to show that

$$\left|\lim\frac{k^{k+1}}{k!}\int_{1}^{1+\delta(k,\varepsilon)}\{\phi[(c+k^{\varepsilon})z-k^{\varepsilon}]-\phi(c)\}z^{k}\exp(-kz)dz\right|<\frac{\eta}{2}\ .$$

Since c is a continuity point of $\phi(u)$, there is a $\delta > 0$ such that if $|(c+k^{\varepsilon})z-k^{\varepsilon}-c| < \delta$, that is, if $|z-1| < \delta(k, \varepsilon)$, then

$$|\phi[(c+k^{\varepsilon})z-k^{\varepsilon}]-\phi(c)| < \eta .$$

For such a δ, the absolute value of the expression above is equal to or less than

$$\eta \lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k,\varepsilon)} z^k \exp(-kz) dz = \frac{\eta}{2} .$$

By the use of Lemma 4, it may be shown in a similar way that

$$\left| \lim I_k^{(2)} - \frac{1}{2}\phi(c) \right| < \frac{1}{2}\eta .$$

Putting together these results, we have $|\lim I_k - \phi(c)| < \eta$ for any $\eta > 0$, which proves the theorem.

Reference

1. C. V. Widder, *Inversion of the Laplace transform and the related moment problem*, Trans. Amer. Math. Soc. **36** (1934), 107–200.

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