THE CENTER OF A COMPACT LATTICE IS TOTALLY DISCONNECTED

ALEXANDER DONIPHAN WALLACE

The purpose of this note is to prove the theorem of the title. A topological lattice is a Hausdorff space together with a pair of continuous functions $\wedge: L\times L \to L$, $\vee: L\times L \to L$ satisfying the usual conditions for lattice operations. As is customary we may write $x \wedge y$ in place of $\wedge(x, y)$. All references are to Chapter II of [1]. We assume the reader to be familiar with the elementary facts concerning topological algebras (groups, lattices, semigroups) and set-theoretic topology.

THEOREM. *The center of a compact lattice is totally disconnected.*

Proof. Let L be a compact lattice. As is wellknown L has a zero and a unit, 0 and 1. If A is the set of pairs $(x, y)\in L\times L$ such that $x\wedge y=0$ and $x\vee y=1$ then $A=\wedge^{-1}(0)\cap\vee^{-1}(1)$ so that A is closed. The projection $(x, y)\to x$ takes A onto the closed set B and B is the set of all $x\in L$ which admit a complement.

Now N, the set of neutral elements of L, is the intersection of the maximal distributive sublattices by Theorem 11. But if D is a distributive sublattice of L its closure is also a distributive sublattice. It follows that N is closed. By the corollary to Theorem 10 the center C of L is $N\cap B$ so that C is closed.

By the lemma on page 27 each element $x\in C$ has a *unique* complement $k(x)\in C$. We will show that $k: C\to C$ is continuous. If G is the subset of $C\times C$ consisting of all $(x, k(x))$ with $x\in C$ it is enough to show that G is closed since C is compact. But by the remarks above we have $G=(C\times C)\cap \wedge^{-1}(0)\cap \vee^{-1}(1)$.

Now C is a distributive lattice (Theorem 9 and Corollary p. 29) with unique complements. Thus C is a commutative *topological* group under the operations

$$x+y=(x\wedge k(y))\vee(k(x)\wedge y), \quad -x=x$$

all of whose elements are of order 2, that is, $x+x=0$ for all x. If Q is the component of C containing 0 and if $q\in Q$, $q\neq 0$, then there is a continuous homomorphism f taking Q into Z, the reals mod 1, such that $f(q)\neq f(0)$. Since $f(Q)$ is connected it contains an interval of Z and therefore contains an element not of finite order. Since the order of each element of Q is two this is a contradiction. Hence Q contains

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only 0 and therefore is totally disconnected. The proof of the Theorem is complete.

Reference

1. G. Birkhoff, *Lattice theory*, New York, 1948.

The Tulane University of Louisiana