A THREE POINT CONVEXITY PROPERTY

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There exist an interesting variety of set properties determined by placing restrictions on each triple of points of the set. It is the purpose here to study those closed sets in the n-dimensional Euclidean space E_n (in particular the plane E_2) which satisfy the following condition.

DEFINITION 1. A set S in E_n is said to possess the *three-point convexity property* P_3 if for each triple of points x, y, z in S at least one of the closed segments xy, yz, xz is in S.

The principal result obtained in this paper appears in Theorem 2. In order to achieve this result a series of lemmas and theorems is first established. Most of these are also of independent interest.

1. Closed connected sets in E_n, $n \geqq 1$.

In this section we assume that S is a closed connected set in E_n, $n \geqq 1$. The concept of *local convexity* is a useful one for our purpose, so we restate the well-known definition.

DEFINITION 2. A set S is said to be *locally convex at a point* $q \in S$ if there exists an open sphere N with center at q such that $S \cdot N$ is convex. If a set is locally convex at each of its points, it is said to be *locally convex*.

NOTATION 1. The open segment determined by points x and y is denoted by (xy), whereas xy denotes the closed segment. The line determined by x and y is denoted by $L(x, y)$. The boundary of a set S is $B(S)$, and $H(S)$ denotes the closed convex hull of S. The symbol $+$ stands for set union, and the symbol $\cdot$ stands for set product.

THEOREM 1. *Let S be a closed connected set in E_n $(n \geqq 1)$ which has property P_3. Then either S is convex or S is starlike with respect to each of its points of local nonconvexity (It may be starlike elsewhere).*

Proof. If S is locally convex, then by a theorem of Tietze [4, pp. 697–707], [2, pp. 448–449], the set S is convex, in which case it is starlike with respect to each of its points. Hence, suppose S is not locally convex, and let $q \in S$ be a point of local nonconvexity. This implies that in each spherical neighborhood N_i of q, there exist points

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x_i and y_i of S such that $(x_iy_i)\cdot S=0$ (see Notation 1). Choose *any* point $x \in S$. Property P_3 implies that either xy_i or xx_i is in S. If the radius of N_i is $1/i$, then as $i\to\infty$ the set xx_i+xy_i converges to qx, which then must belong to S. This completes the proof.

REMARK 1. *The set of all starlike points of a set S is called the convex kernel of S. The convex kernel of a set $S\subset E_n$ is convex. See Brunn* [1].

COROLLARY 1. *Each point of local nonconvexity of the set S in Theorem* 1 *is contained in the boundary of the convex kernel of S.*

COROLLARY 2. *For the set S above, let H be any r-dimensional plane section of S, where* $(1\leqq r\leqq n-1)$. *Then either $H\cdot S$ is starlike or $H\cdot S$ consists of two convex components.*

Proof of Corollary 2. If $H\cdot S$ is connected, then since $H\cdot S$ has property P_3, Theorem 1 implies $H\cdot S$ is starlike. If $H\cdot S$ is not connected, property P_3 implies trivially that $H\cdot S$ consists of two and only two components, each of which must be convex.

COROLLARY 3. *Each component of the complement of S is unbounded.* This is an immediate consequence of the starlikeness of S.

2. Closed connected sets in E_2. We restrict ourselves to closed connected sets in E_2 in this section, and the following definitions are useful.

DEFINITION 3. A component of the complement of a closed connected set S is called a *residual domain of* S. A *cross-cut xy of a residual domain K of S* is a closed segment such that $x\in S$, $y\in S$ and $(xy)\subset K$ (See Notation 1).

DEFINITION 4. An isolated point of local nonconvexity of S is called a p-point. A point of S which is a p-point or a limit point of p-points is called a *q-point.*

LEMMA 1. *Each open segment (uv) of the convex kernel of S contains no q-points of S.* (see Corollary 1).

Proof. Suppose w is a q-point contained in (uv). Clearly $S\not\subset L(u, v)$ (see Notation 1). Choose $z\in S-L(u, v)$. Since uv belongs to the convex kernel of S, we have triangle $uzv\subset S$. But this implies that each sufficiently small neighborhood of w contains no cross-cuts of the complement of S, since such a cross cut xy would have to have its interior (xy) in one of the open half-planes bounded by $L(u, v)$.

LEMMA 2. *Let S be a closed connected set in E_2 having property P_3. Then if S is not convex, it contains at least one isolated point of local nonconvexity.*

Proof. Let xy be a cross-cut of a residual domain K of S. Since S is closed and connected, the set $K-(xy)$ is the union of two mutually exclusive open sets, denoted by K_1 and K_2 [3, p. 118]. Since S is star-like, Corollary 3 implies that one and only one of these two sets is bounded. Let it be K_1, and denote the boundary $B(K_1)-(xy)\equiv C(K_1)$. The set $B(K_1)$ is a continuum [3, p. 124]. Since K_1 is a bounded domain, and since $C(K_1)\cdot(xy)=0$, it follows that $C(K_1)$ is a continuum. Define B_1 to be the set of points $z_1\in C(K_1)$ such that $xz_1\subset S$, and define B_2 to be the points $z_2\in C(K_1)$ such that $yz_2\subset S$. Since $xy\not\subset S$, property P_3 implies $C(K_1)=B_1+B_2$. Moreover, $x\in B_1$, $y\in B_2$, and moreover B_1 and B_2 are each closed since S and $C(K_1)$ are closed. Hence, since $C(K_1)$ is a continuum, it is well-known that $B_1\cdot B_2\neq 0$. Hence, let $p\in B_1\cdot B_2$, and we must have $xp\subset S$, $yp\subset S$, so that K_1 is interior to triangle xpy. Since $p\in B(K_1)$, it is clear that each neighborhood of p contains a cross-cut of K_1, so that p is a point of local nonconvexity of S.

To prove that p is an *isolated* point of local nonconvexity, observe that the lines $L(x, p)$ and $L(y, p)$ determine four V-shaped domains, each bounded by two rays. Order these V_1, V_2, V_3, V_4 so that $xyp\subset \overline{V}_1$, and so that the sets V_i are arranged consecutively in a clockwise direction about the point p. Suppose a p-point $p_1\in \overline{V}_1-p$ exists. Then since $p_1x+p_1y\subset S$, we would have $K_1\subset xyp_1$, which would violate the fact $p\in B(K_1)$. Suppose a p-point, say p_1 exists in V_2. But this implies that $xpp_1\subset S$, $ypp_1\subset S$. But this again would violate the fact $p\in B(K_1)$. In exactly the same way V_4 contains no p-point of S. Now consider V_3. If p_1 is a p-point of S in V_3, then $ypp_1+xpp_1\subset S$, which implies that p is an isolated p-point since V_1 contains no p-point of S. Finally, Lemma 1 implies that no sequence of p-points of S can exist on $L(x, p)\cdot\overline{V}_3$ or on $L(y, p)\cdot\overline{V}_3$ having p as a limit point. Thus we have shown that p is an isolated p-point of S.

REMARK 2. Let xy be the cross-cut in the above proof, and let p be the associated isolated p-point. Then the closed triangle xyp is such that the set $xyp\cdot S$ is the union of two convex sets having only the point p in common. One of these convex sets contains xp and is denoted by $C(xp)$, and the other denoted by $C(yp)$ contains py.

Proof. Let L_i and L be lines parallel to xy, such that L_i separates p and xy, and such that $p \in L$. Let H_i be the closed half-plane bounded by L_i and containing xy. Suppose $L_i \to L$ as $i \to \infty$ so that $H_{i+1} \supset H_i$. Since $S \cdot H_i \cdot xyp$ is locally convex, by Tietze's Theorem [4, loc. cit.] each of its components is convex. Property P_3 implies that there are at most two such components. The fact that $xy \not\subset S$, implies there are exactly two such components. Denote them by C_i and D_i. Clearly $C_{i+1} \supset C_i$, $D_{i+1} \supset D_i$, and hence C_i and D_i converge to convex sets having p in common. They have only p in common, otherwise p would not be a boundary point of K_1 as defined in the proof of Lemma 2. One of the convex sets contains xp and the other yp so that the notation in the remark is justified.

DEFINITION 5. Let Q be the set of q-points of S.

REMARK 3. Corollary 1 implies that Q is contained in the boundary of its own convex hull $H(Q)$, designated by $B(H)$.

LEMMA 3. *The boundary of $H(Q)$ is connected, and it can contain at most one ray.*

Proof. Since $H \equiv H(Q)$ is convex, if $B(H)$ were not connected, it would have to consist of two parallel lines (this is known). However, Lemma 1 would then imply that each of these parallel lines would contain at most two q-points. But this would imply that Q is bounded in which case $B(H)$ would be connected. If $B(H)$ contained two rays, then Lemma 1 would again imply that Q is bounded, which would again be contradictory.

DEFINITION 6. An *edge* of the boundary $B(H)$ is a closed segment xy or a closed ray $x\infty$ whose endpoints are q-points. An open half-plane whose boundary contains xy(*or* $x\infty$), and which does not intersect $H(Q)$ is called an *open half-plane of support*, and it is denoted by W.

LEMMA 4. *Let W be an open half-plane of support to $H(Q)$, which abuts on an edge xy(or $x\infty$). Then $H(Q)+W \cdot S$ is a convex subset of S.*

Proof. If $u \in H(Q)$ and if $v \in S \cdot W$, then $uv \subset S$, since S is starlike with respect to u. This, together with the facts $x \in B(S)$, $y \in B(S)$, and property P_3 imply that $uv \cdot xy \neq 0$ (or $uv \cdot x\infty \neq 0$), so that $uv \subset H(Q)+W \cdot S$. Suppose $u \in S \cdot W$, $v \in S \cdot W$. Let $z \in (xy)$ or $(x\infty)$. If $(uv) \not\subset S$, since $uz \subset S$, $vz \subset S$, then triangle uvz would contain a p-point of S (See the first paragraph of the proof of Lemma 2). But this is impossible, since W contains no p-points of S, and since by Lemma 1 the open segment (xy) or $(x\infty)$ contains no p-points of S. Hence $H(Q)+$

$W \cdot S$ is convex. It should be observed that if $H(Q) \equiv xy$, then $H(Q) + W \cdot S$ may or may not be closed.

LEMMA 5. *Let $x_i y_i$ be a countable number of pairwise disjoint edges in $B(H) \equiv B(H(Q))$. Assume that $B(H)$ contains at least three edges, and let W_i be the open half-plane of support to $H(Q)$ whose boundary contains $(x_i y_i)$ ($x_i y_i$ may be $x_i \infty$).*

Then the set $H(Q) + S \cdot \Sigma_i W_i$ is a closed convex set.

Proof. Without loss of generality establish an order on the boundary $B(H)$, and assume that in terms of this order, x_i is the beginning of the edge $x_i y_i$ and that y_i is the endpoint of $x_i y_i$. Select any two disjoint edges $x_i y_i$ and $x_j y_j$, and without loss of generality assume that x_i, y_i, x_j, y_j fall in an order so that an arc of $B(H)$ has x_i and y_j as its endpoints, and so that all four points lie on this arc in the order given above ($B(H)$ may be unbounded). Let the convex set which is bounded by the two lines $L(x_i, y_j)$ and $L(x_j, y_i)$, and which contains the quadrilateral $x_i y_i x_j y_j$ be denoted by V. The segments $x_i y_i$ and $x_j y_j$ divide V into three parts; one is the closed quadrilateral $x_i y_i x_j y_j$; the second is a three sided closed polygonal set adjacent to $x_i y_i$ and denoted by $\Delta(x_i, y_i)$; the third is a three sided closed polygonal set adjacent to $x_j y_j$ and denoted by $\Delta(x_j, y_j)$. The last two sets may or may not be bounded. If the edge $x_j y_j$ is a ray $x_j \infty$ instead, then the same type of division occurs, in which $L(x_i, \infty)$ is a line parallel to the ray $x_j \infty$ so that $\Delta(x_j, \infty)$ has two bounding sides instead of three. *We must have* $S \cdot W_i \subset \Delta(x_i, y_i)$, for if this were not so, it is easily seen that either x_i or y_i would be an interior point of a triangle which would belong to S. But this would contradict the fact that $x_i \in B(S)$, $y_i \in B(S)$. Similarly $S \cdot W_j \subset \Delta(x_j, y_j)$. This is true whether $x_j y_j$ is a finite segment or a ray $x_j \infty$.

Now, choose two points u and v in $U \equiv H(Q) + S \cdot \Sigma_i W_i$. If u and v are in $H(Q) + S \cdot W_i$, then Lemma 4 implies $uv \subset U$. If $u \in S \cdot W_i$ and $v \in S \cdot W_j$, then by the preceding paragraph $u \in \Delta(x_i, y_i)$, $v \in \Delta(x_j, y_j)$ (or $v \in \Delta(x_j, \infty)$). Since $V \equiv \Delta(x_i, y_i) + x_i y_i x_j y_j + \Delta(x_j, y_j)$ is convex, and since $\Delta(x_i, y_i) \cdot x_i y_i x_j y_j = x_i y_i$, we have $uv \cdot x_i y_i \neq 0$, whence $uv \subset U$.

To prove that U is closed, observe first that if there are a *finite* number of disjoint sets $x_i y_i$ (there are at least three edges) then U is closed, since $\overline{W}_i \cdot S \subset \Delta(x_i, y_i)$ implies $\overline{W_i \cdot S} \subset W_i \cdot S + B(H)$. If there are an infinite number of sets W_i, then let s be a limit point of an infinite sequence of sets $W_{i_n} \cdot S$. Since $W_{i_n} \subset \Delta(x_{i_n}, y_{i_n})$ by fixing (x_j, y_j) of the preceding paragraph, it follows that $(x_{i_n}, y_{i_n}) \to q$, a fixed point of $B(H)$, as $i_n \leftarrow \infty$. However, since in this situation, we must have $\Delta(x_{i_n}, y_{i_n}) \to q$ as $i_n \to \infty$, it follows that $s = q \in H(Q)$. Hence, it is clear that U is closed, since $H(Q)$ is closed.

THEOREM 2. *Suppose S is a closed connected set in E_2 such that for each triple of points x, y, z in S at least one of the segments xy, yz, xz is in S.*

Then S is expressible as the union of three or fewer closed convex sets having a nonempty intersection. The number three is best.

DEFINITION 7. Let N denote the cardinality of the set of p-points of S *in Theorem* 2.

THEOREM 3. *If N is not an odd integer greater than* 1, *then S can be expressed as the union of two or fewer closed convex sets having a nonempty intersection.*

Proofs of Theorems 2 *and* 3. We recall that Q is the closure of the set of p-points of S. The proof is divided into 5 cases, depending upon the value of N. The five cases are: $N=1$; $N=2$; $N=2m>1$; $N=2m+1>1$; $N=\infty$.

Case 1. $N=1$. Let $Q=p$, and let C be a circle with center at p and having radius r. The set $S\cdot C$ is a closed connected set having property P_3 and having p as its only p-point. If $S\cdot C$ satisfies the conclusions of either Theorem 2 or Theorem 3, it is quite clear that $S=\lim S\cdot C$ as $r\to\infty$ will satisfy the same conclusions. Let the boundary of the convex hull $H(S\cdot C)$ be $D(H)$, since $B(H)$ stands for the boundary of $H(Q)$. The rest of the proof will show incidentally that $D(H)\cdot S$ has one, two or four components.

Suppose $D(H)\cdot S$ has exactly three components and designate these by B_i $(i=1, 2, 3)$. It is easy to show that $B_i\neq\{p\}$ $(i=1, 2, 3)$. Choose points $x_i\in B_i$ with $x_i\neq p$ $(i=1, 2, 3)$. Property P_3 implies that at least one of the intervals x_1x_2, x_2x_3, x_3x_1 is in S. Suppose $x_1x_2\subset S$. Since $x_i\in D(H)$ $(i=1, 2)$, and since $B_1\cdot B_2=0$, we must have $L(x_1, x_2)\cdot S=x_1x_2$. If $p\notin(x_1x_2)$, then let H_{12} be the closed half-plane bounded by $L(x_1, x_2)$ and not containing p. Since $x_1\in B_1$, $x_2\in B_2$ with $B_1\cdot B_2=0$, there must exist a cross cut of the complement of S in H_{12}. However, by the proof of Lemma 2, there would exist a p-point in $H_{12}\cdot S$ which contradicts the fact that $Q=p$. Hence $p\in(x_1x_2)$. Since $p\in(x_1x_2)$, if $x_1x_3\subset S$, the proof of Lemma 2 would again imply the existence of a p-point in that closed half-space bounded by $L(x_1, x_3)$ which does not contain p. However this contradicts $Q=p$. Hence, $x_1x_3\not\subset S$. Similarly $x_2x_3\not\subset S$. Property P_3 and the closure of S implies that for points $x\in B_1$ sufficiently near x_1, we have $xx_2\subset S$, $xx_3\not\subset S$, $x_2x_3\not\subset S$. Applying the same reasoning to x, x_2, x_3 that we applied to x_1, x_2, x_3, we get $p\in(xx_2)$ for all x near

x_1. This can only be true if $B_1=x_1$. Similarly, $B_2=x_2$. However, since B_3 is contained in only one of the open half planes bounded by $L(x_1, x_2)$, the facts $B_i=x_i$ $(i=1, 2)$ simply that $x_1x_2 \subset D(H)\cdot S$, which contradicts the fact $B_1\cdot B_2=0$. Hence *$D(H)\cdot S$ cannot have exactly three components.* Suppose $D(H)\cdot S$ has at least four components, and designate four of these by B_i $(i=1, 2, 3, 4)$. The above argument implies that $B_i=x_i \in D(H)\cdot S$, and these can be renumbered so that $p \in (x_1x_2)$, $p \in (x_3x_4)$. Clearly any fifth component B_5 could not exist, since the above argument applied to x_1, x_2, x_3 and $x_5 \in B_5$ would yield $p \in (x_1x_5)$, $p \in (x_3x_5)$, so that $x_5=x_4$, a contradiction. *Thus if $D(H)\cdot S$ has more than two components, then $S\cdot C$ is the union of two line segments having an interior point in common.*

Now, suppose $D(H)\cdot S$ has exactly two components denoted by B_1 and B_2. Let the end points of B_i be x_i and y_i ordered so that y_1x_2 and y_2x_1 are cross-cuts of the complement of S. The points x_i and y_i need not be distinct. We will prove that each of the sets $P_i \equiv H(B_i+p)+C(x_ip)+C(y_ip)$, $(i=1, 2)$ is convex. (See Remark 2 following Lemma 2 for the definitions of $C(x_ip)$ and $C(y_ip)$). Property P_3 and the fact that $D(H)\cdot S=B_1+B_2$ implies that $B_i+x_ip+y_ip$ is the boundary of $H(B_i+p)$. We will prove that P_1 is convex. Since each of the sets $H(B_1+p)$, $C(x_1p)$, $C(y_1p)$ is convex, to show that P_1 is convex, it suffices to select points $z \in H(B_1+p)$, $u \in C(x_1p)$, $v \in C(y_1p)$, and to show that $uv+uz+vz \subset P_1$. We must have $uv\cdot x_1p \neq 0$, $uv\cdot y_1p \neq 0$, for if this were not so, the fact $D(H)\cdot S=B_1+B_2$ would imply that $uv\cdot P_2 \neq 0$. However, this would contradict property P_3. Hence, we have $uv \subset x_1y_1p+C(x_1p)+C(y_1p)$. Thus $uv \subset P_1$. In the same manner $uz \subset P_1$, $vz \subset P_1$, so that P_1 is convex. The same argument applies to P_2.

Finally if $D(H)\cdot S$ has exactly one component, and $Q=p$, it can be shown readily that there exists a line through p which divides $S\cdot C$ into two closed convex sets having p in common. This completes the proof for $N=1$, and oddly enough it appears to be the most difficult to prove.

Case 2. $N=2$. Let $Q=p_1+p_2$. The line $L(p_1, p_2)$ divides the plane into two open half-planes W_i $(i=1, 2)$. Lemma 4 implies that $W_i\cdot S$ is convex. If $W_1\cdot S=0$, then $S=\overline{W}_2\cdot S+S\cdot L(p_1, p_2)$ yields the desired conclusions of Theorem 2 and 3. Hence suppose $W_i\cdot S\neq 0$ $(i=1, 2)$. Let $U\equiv \overline{W_1\cdot S}+\overline{W_2\cdot S}$. If U is convex, then $S\equiv U+S\cdot L(p_1, p_2)$ yields the desired decomposition. Suppose U is not convex, then we can show that $S\cdot L(p_1, p_2)=U\cdot L(p_1, p_2)$, for suppose a point $u \in S\cdot L(p_1, p_2)-U\cdot L(p_1, p_2)$ exists. Since U is not convex, there exist points $x_i \in W_i\cdot S$ such that $x_1x_2 \not\subset S$. Moreover $ux_i \not\subset S$, since $u \notin \overline{W_i\cdot S}$. However, this violates property P_3. Thus if U is not convex, $S=\overline{W_1\cdot S}+\overline{W_2\cdot S}$, and this is a desired decomposition of S into two convex sets.

Case 3. $N=2m>2$. In this case the hull $H(Q)$ is a convex polygon, each segment of which is an edge having p-points as endpoints (See definition 6). Order the edges $x_i x_{i+1}$ of the boundary $B(H)$ counterclockwise so that ($i=1, 2, \cdots, 2m$; $x_1=x_{2m+1}$). The open half-plane of support to $H(Q)$ adjacent to $x_i x_{i+1}$ is denoted by W_i. By Lemma 5 each of the sets

$$S_1 \equiv H(Q)+S\cdot\sum_{i=1}^{m} W_{2i-1} \tag{1}$$

$$S_2 \equiv H(Q)+S\cdot\sum_{i=1}^{m} W_{2i}$$

is a closed convex set. Moreover, since $S\subset H(Q)+\sum_{i=1}^{2m} W_i$ we have $S_1+S_2=S$.

Case 4. $N=2m+1>1$. As in Case 3, let $e_i\equiv x_i x_{i+1}$ ($i=1, \cdots, 2m+1$; $x_1=x_{2m+2}$) denote the ordered edges of $B(H)$, and define S_1 and S_2 as in (1).

Let

$$S_3\equiv H(Q)+S\cdot W_{2m+1}\,.$$

By Lemma 5, the sets S_1, S_2 and S_3 satisfy the conclusions of Theorem 2.

Case 5. $N=\infty$. In order to prove this case, the following definition is helpful.

DEFINITION 8. A connected closed subset I of the boundary $B(H)$ is called a *polygonal element* if the following conditions hold:

(a) It is the closure of the union of edges of $B(H)$ (see Definition 6).

(b) Its endpoints (one, two or none) are limit points of p-points of S.

(c) If $I=B(H)$, then I contains at most one limit point of p-points. If $I\neq B(H)$, then only its endpoints (one or two) are limit points of p-points.

Observe that these conditions imply that a polygonal element is maximal in the sense that it is not a proper subset of a larger polygonal element.

The number of polygonal elements of $B(H)$ is countable, hence we can well-order them easily. Let $I_1, I_2, \cdots, I_n \cdots$ designate such a well-ordering.

For each polygonal element I_n, divide the edges it contains (see Definition 6) into two classes M_n^1 and M_n^2 such that no two edges of M_n^i ($i=1, 2$) are adjacent, that is, have an endpoint in common. It may happen that one of the M_n^i may be empty. For each edge $e \in M_n^i$ we let W_e^i denote the open half-plane of support to $B(H)$ whose boundary contains e. Define

$$F_n^i = \sum_{e \in M_n^i} W_e^i \cdot S \qquad (i=1, 2),$$

and let

$$S_i = H(Q) + \sum_n F_n^i \qquad (i=1, 2).$$

Since each edge in M_n^i is separated from each edge in M_m^i $(n \neq m)$, Lemma 5 implies that S_1 and S_2 are closed convex subsets of S. Moreover, since for each point $x \in S$, either $x \in H(Q)$, or x is contained in some $W_e^i \cdot S$, we have $S = S_1 + S_2$ and $S_1 \cdot S_2 \neq 0$.

To prove that the number "three" in Theorem 2 is best consider the familiar two cell formed by a five-pointed star. It is a simple matter to verify that this set has property P_3, and that it cannot be expressed as the union of two convex sets. The analogous $2m+1$ pointed star behaves the same way.

3. Concluding remarks.

(a) It should be noted that the converse of Theorem 2 is not true. For instance, the set consisting of three segments xx_i $(i=1, 2, 3)$, where each angle $\angle x_i x x_j = 120°$ $(i \neq j)$, is the union of three convex sets; yet it does not have property P_3.

(b) It would be of interest to characterize those sets in E_2 which are the union of two closed convex sets. It appears that such a characterization will follow from an investigation of the cardinality of the set $B(K) \cdot B(S)$, where K is the convex kernel of S.

(c) The theory in E_3 needs to be settled. In view of § 1, it is natural to ask the question. What are the closed connected sets in E_3 such that each of its plane sections is either starlike or the union of two disjoint convex sets?

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