ON CERTAIN SUMS GENERATING THE DEDEKIND SUMS AND THEIR RECIPROCITY LAWS

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1. Introduction. Let $\{u\}=u-[u]$ denote the fractional part of u and let $((u))=\{u\}-\frac{1}{2}$. Dedekind sums are defined for example, by

$$s_1(h, k)=\sum_{\lambda=0}^{k-1}\left(\left(\frac{\lambda}{k}\right)\right)\left(\left(\frac{\lambda h}{k}\right)\right) \tag{1.1}$$

where h and k are relatively prime positive integers. These sums which were studied by Dedekind [7], and more recently by Rademacher and Whiteman [9], [12] in connection with the theory of the modular function $\eta(\tau)$, occur also in the theory of partitions and in a great number of special papers. (Cf. for example [1]-[13].) The most important property of $s_1(h, k)$ is the reciprocity law

$$s_1(h, k)+s_1(k, h)=(h^2+3hk+k^2+1)/(12hk)\ . \tag{1.2}$$

A few years ago, Apostol [1] (for $r=\nu$) and Carlitz [3] introduced and investigated the so-called generalized Dedekind sums

$$s_r^{(\nu)}(h, k)=\sum_{\lambda=0}^{k-1}P_{\nu+1-r}\left(\frac{\lambda}{k}\right)P_r\left(\frac{\lambda h}{k}\right) \qquad 0\leqq r\leqq \nu+1\ , \tag{1.3}$$

P_r denoting the well-known Bernoulli function defined by the expansion

$$ze^{uz}/(e^z-1)=\sum_{n=0}^{\infty}P_n(u)z^n/n! \qquad |z|<2\pi$$

for $0\leqq u<1$ and by $P_r(u)=P_r(\{u\})$ for u arbitrary real. They found the corresponding extensions of (1.2) too.

Now, we shall continue to develop these results in two directions. Next we give a systematic treatment of certain exponential sums (2.1), (2.3) generating

$$\mathfrak{S}_{m,n}\binom{a\ b}{c}=\sum_{\nu=0}^{c-1}P_m\left(\frac{\lambda a}{c}\right)P_n\left(\frac{\lambda b}{c}\right) \qquad m, n=0, 1, 2, \cdots \tag{1.4}$$

with $(a, c)=(b, c)=1$, $c>0$. We obtain (among others) a three-term relation of new type (Theorem 1) which implies (in extended form) all the above reciprocity theorems (see (5.1)-(5.10)). Let us remark that the sum function (2.5) with other notations is also used in [6]. On the other hand, we get a functional equation for

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$$\mathfrak{D}_c^{a,b}(w, z)=\sum_{\lambda=1}^{c-1}\zeta\left(w, \left\{\frac{\lambda a}{c}\right\}\right)\zeta\left(z, \left\{\frac{\lambda b}{c}\right\}\right) \tag{1.5}$$

where $\zeta(s, u)$ is the Hurwitz zeta function (Theorem 2). By

$$\zeta(1-n, u)=-P_n(u)/n \qquad 0<u\leqq 1\,;\ n=1, 2, \cdots ,$$

(1.5) can be regarded substantially as a (transcendental) generalization of (1.4).

2. Preliminaries on $\mathfrak{S}_c^{a,b}(x, y)$, $\mathfrak{s}_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix}$. In what follows, x, y, w, z denote complex variables, a, b and c are integers and $c>0$; for brevity we write, as usual, $e(z)=e^{2\pi iz}$.

Let us put

$$S_c^{a,b}(x, y)=\sum_{\lambda(\text{mod } c)} e\left(\left\{\frac{\lambda a}{c}\right\}x+\left\{\frac{\lambda b}{c}\right\}y\right) \tag{2.1}$$

with $(a, c)=(b, c)=1$, the summation extending over a complete residue system modulo c. It is obvious that (2.1) is independent of the choice of this residue system[1] and for $a=b$ or $c=1, 2$ it is independent of a, b. The function $S_c^{a,b}(x, y)$ remains unaltered if we change a, b or x, y by multiplies of c. By this periodicity, it is no restriction to suppose for example, that $0\leqq\Re(x)<c$, $-c<\Re(y)\leqq 0$.

We have $S_c^{a,b}(x, y)=S_c^{b,a}(y, x)$ and

$$S_c^{-a,b}(x, y)=e(x)S_c^{a,b}(-x, y)+1-e(x)\ , \tag{2.2}$$

since $\{-u\}=0$ or $1-\{u\}$ according as u is an integer or not.

The function

$$\mathfrak{S}_c^{a,b}(x, y)=[e(x)-1]^{-1}[e(y)-1]^{-1}S_c^{a,b}(x, y) \qquad x, y\neq 0, \pm 1, \cdots \tag{2.3}$$

has corresponding trivial properties; in particular, (2.2) implies

$$\mathfrak{S}_c^{-a,b}(x, y)=-\mathfrak{S}_c^{a,b}(-x, y)-[e(y)-1]^{-1}\ . \tag{2.4}$$

By the definition of Bernoulli functions and (1.4) we obtain

$$xy\mathfrak{S}_c^{a,b}(x/2\pi i, y/2\pi i)=\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!n!}\mathfrak{s}_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix} \qquad |x|, |y|<2\pi\ . \tag{2.5}$$

Here

[1] Hence we see that $S_c^{a,b}(x, y)=S_c^{1,b'}(x, y)$ for a suitable integer b'; however, the above symmetric notation seems the most convenient.

$$\mathfrak{S}_{0,n}\binom{a\ b}{c}=\sum_{l=0}^{c-1}P_n\left(\frac{l}{c}\right)=c^{1-n}B_n \qquad n=0,1,\cdots, \tag{2.6}$$

$B_n=P_n(0)$ denoting the Bernoullian numbers.

Note that $\mathfrak{S}_{m,n}\binom{a\ b}{c}=\mathfrak{S}_{n,m}\binom{b\ a}{c}$ and $\mathfrak{S}_{m,n}\binom{a\ a}{c}$ does not depend on a; especially we have $\mathfrak{S}_{m,n}\binom{1\ b}{c}=\mathfrak{S}_n^{(m+n-1)}(b,c)$, furthermore

$$\mathfrak{S}_{m,n}\binom{a\ b}{1}=B_mB_n\,, \qquad \mathfrak{S}_{m,n}\binom{a\ b}{2}=B_mB_n[1+(1-2^{1-m})(1-2^{1-n})] \tag{2.7}$$

$$m,n=0,1,\cdots.$$

3. Representation by cotangents and Eulerian numbers respectively. Let $c>1$. The identity

$$\sum_{\mu=0}^{c-1}e\left(\frac{\mu x}{c}\right)e\left(\frac{\mu\nu}{c}\right)=[e(x)-1]\left[e\left(\frac{x+\nu}{c}\right)-1\right]^{-1} \tag{3.1}$$

yields after multiplication by $e\left(-\frac{\mu\nu}{c}\right)$ ($\nu=0,1,\cdots,c-1$) and summation

$$e\left(\frac{\mu x}{c}\right)=\frac{1}{c}[e(x)-1]\sum_{\nu=0}^{c-1}\left[e\left(\frac{x+\nu}{c}\right)-1\right]^{-1}e\left(-\frac{\mu\nu}{c}\right) \tag{3.2}$$

$$\mu=0,1,\cdots,\nu-1;$$

(3.1) and (3.2) hold clearly provided that $(x+\nu)/c$ is not an integer ($\nu=0,1,\cdots,c-1$). Hence by putting $\mu=c\{a\lambda/c\}$, a and c being coprime we get

$$e\left(x\left\{\frac{a\lambda}{c}\right\}\right)=\frac{1}{c}[e(x)-1]\sum_{\nu=0}^{c-1}\left[e\left(\frac{x+\nu}{c}\right)-1\right]^{-1}e\left(-\nu\frac{a\lambda}{c}\right). \tag{3.3}$$

Furthermore, by using the corresponding expression for $e(y\{b\lambda/c\})$, $(b,c)=1$,

$$S_c^{a,b}(x,y)=\frac{1}{c^2}[e(x)-1][e(y)-1]\sum_{p,q(\mathrm{mod}\,c)}\left[e\left(\frac{x+p}{c}\right)-1\right]^{-1}\left[e\left(\frac{y+q}{c}\right)-1\right]^{-1}$$

$$\times\sum_{\lambda=0}^{c-1}e\left(-\frac{\lambda(ap+bq)}{c}\right).$$

If we consider the complete residue systems (mod c): $p=-br$, $q=a\rho$ ($r,\rho=0,1,\cdots,c-1$) and take into account that $\sum_{\lambda=0}^{c-1}e\left(-\lambda\frac{ab(\rho-r)}{c}\right)$

vanishes except for $\rho=r$ when it has the value c, it follows simply that

$$(3.4)\qquad \mathfrak{S}_c^{a,b}(x, y)=\frac{1}{c}\sum_{r(\mathrm{mod}\,c)}\left[e\left(\frac{x-br}{c}\right)-1\right]^{-1}\left[e\left(\frac{y+ar}{c}\right)-1\right]^{-1},$$

holds for all $x, y\neq 0, \pm 1, \cdots$ and, because of the definition (2.3), in the case $c=1$ too. By $[1-e(z)]^{-1}=\frac{1}{2}(1+i\operatorname{ctg}\pi z)$ and

$$\sum_{\mu=0}^{c-1}\operatorname{ctg}\pi\left(z+\frac{\mu}{c}\right)=c\cdot\operatorname{ctg} c\pi z\,,$$

we have the equivalent formula:

$$(3.5)\qquad \mathfrak{S}_c^{a,b}(x,y)=\frac{1}{4}[1+i(\operatorname{ctg}\pi x+\operatorname{ctg}\pi y)]$$
$$-\frac{1}{4c}\sum_{r(\mathrm{mod}\,c)}\operatorname{ctg}\pi\frac{x-br}{c}\operatorname{ctg}\pi\frac{y+ar}{c}\,;$$

(3.4) or (3.5) expresses the sum (2.3) by means of periodic elementary functions, *without* using the arithmetical function $\{u\}$.

(3.4) leads immediately to corresponding representations of $\mathfrak{s}_{m,n}\begin{pmatrix} a & b\\ & c\end{pmatrix}$ by means of the so-called Eulerian numbers $H_n(\eta^k)$, defined for a root of unity $\eta^k=e\left(\frac{k}{c}\right)$, $c>1$, $c\nmid k$ by

$$(3.6)\qquad (1-\eta^k)/(e^z-\eta^k)=\sum_{n=0}^{\infty}H_n(\eta^k)z^n/n! \qquad |z|<2\pi\{k/c\}\,.$$

In fact, after expanding the right-hand members of

$$xy\mathfrak{S}_c^{a,b}(x/2\pi i,\ y/2\pi i)=(xy/c)(e^{x/c}-1)^{-1}(e^{y/c}-1)^{-1}$$
$$+(xy/c)\sum_{r=1}^{c-1}(e^{x/c}\eta^{-br}-1)^{-1}(e^{y/c}\eta^{ar}-1)^{-1}\,,$$

we find

$$(3.7)\qquad xy\mathfrak{S}_c^{a,b}(x/2\pi i,\ y/2\pi i)=c+\sum_{n=1}^{\infty}\frac{B_n}{n!c^{n-1}}(x^n+y^n)$$
$$+\sum_{m,n=1}^{\infty}\frac{x^my^n}{m!n!c^{m+n-1}}\left[B_mB_n+mn\sum_{r=1}^{c-1}\frac{H_{m-1}(\eta^{br})H_{n-1}(\eta^{-ar})}{(\eta^{ar}-1)(\eta^{-br}-1)}\right]\quad |x|,\ |y|<\frac{2\pi}{c}\,,$$

so that comparison with (2.5) gives in addition to (2.6)

$$(3.8)\qquad \mathfrak{s}_{m,n}\begin{pmatrix} a & b\\ & c\end{pmatrix}=\frac{1}{c^{m+n-1}}\left[B_mB_n+mn\sum_{r=1}^{c-1}\frac{H_{m-1}(\eta^{br})H_{n-1}(\eta^{-ar})}{(\eta^{ar}-1)(\eta^{-br}-1)}\right]$$
$$m,\ n=1,\ 2,\ \cdots\,,$$

a formula implying a result of Carlitz [3, (6.5)]. In particular, for $m=n=1$ (3.8) becomes

$$\mathfrak{S}_{11}\begin{pmatrix} a & b \\ & c \end{pmatrix}=\frac{1}{4c}+\frac{1}{c}\sum_{r=1}^{c-1}(\eta^{ar}-1)^{-1}(\eta^{-br}-1)^{-1} \tag{3.9}$$
$$=\frac{1}{4}+\frac{1}{4c}\sum_{r=1}^{c-1}\operatorname{ctg}\frac{\pi ar}{c}\operatorname{ctg}\frac{\pi br}{c}\,,$$

which contains two equivalent representations due to Rademacher and Rédei (for $a=1$; cf. for example, [4], (2.2) and [2], (5) respectively).

4. The main property of $\mathfrak{S}_c^{a,b}(x, y)$. Our next purpose is to deduce a peculiar symmetry relation relating to the sums in question, by applying the calculus of residues.

THEOREM 1. *We have for a, b, c positive, mutually coprime, and for $0\leqq\Re(x)<1$, $-1<\Re(y)\leqq 0$ the relation*

$$\mathfrak{S}_b^{c,a}(ax+by,\ -cx)+\mathfrak{S}_c^{a,b}(cx,\ cy)+\mathfrak{S}_a^{b,c}(-cy,\ ax+by) \tag{4.1}$$
$$=[1-e(ax+by)]^{-1}\,,$$

provided that $ax+by$, cx and cy are not integers.

Proof. We consider the integral

$$\mathfrak{J}=\frac{1}{2\pi i}\int_Q [e(z)-1]^{-1}\left[e\left(x-\frac{b}{c}z\right)-1\right]^{-1}\left[e\left(y+\frac{a}{c}z\right)-1\right]^{-1}dz \tag{4.2}$$

the path of integration being a rectangle whose vertices are the points $-\varepsilon\pm ti$, $c-\varepsilon\pm ti$ with

$$t>\max\left\{\frac{c}{b}|\Im(x)|,\ \frac{c}{a}|\Im(y)|\right\}$$

and

$$0<\varepsilon<\min\left\{\frac{c}{b}(1-\Re(x)),\ \frac{c}{a}(1+\Re(y))\right\}\,,$$

taken in positive direction. A straight-forward calculation shows that only singularities of the integrand inside Q are at the points:

$$z=\lambda \qquad \lambda=0,\ 1,\ \cdots,\ c-1\,;$$

$$z=\frac{c}{b}(\mu+x) \qquad \mu=0,\ 1,\ \cdots,\ b-1\,;$$

$$z=\frac{c}{a}(\nu-y) \qquad \nu=0,\ 1,\ \cdots,\ a-1\,;$$

by our assumptions, these are all distinct and poles of order 1 only of the first, second, and third factor respectively. Since

$$\operatorname*{res}_{z=\lambda}\,[e(z)-1]^{-1}=1/2\pi i$$

$$\operatorname*{res}_{z=(c/b)(\mu+x)}\,[e(x-bz/c)-1]^{-1}=-c/2\pi ib\ ,$$

$$\operatorname*{res}_{z=(c/a)(\nu-y)}\,[e(y+az/c)-1]^{-1}=c/2\pi ia\ ,$$

the residue theorem yields

$$\begin{aligned}2\pi i\cdot\mathfrak{F}=&\sum_{\lambda=0}^{c-1}\left[e\left(x-\frac{\lambda b}{c}\right)-1\right]^{-1}\left[e\left(y+\frac{\lambda a}{c}\right)-1\right]^{-1}\\&-\frac{c}{b}\sum_{\mu=0}^{b-1}\left[e\left(\frac{a}{b}x+y+\frac{\mu a}{b}\right)-1\right]^{-1}\left[e\left(\frac{c}{b}x+\frac{\mu c}{b}\right)-1\right]^{-1}\\&+\frac{c}{a}\sum_{\nu=0}^{a-1}\left[e\left(x+\frac{b}{a}y+\frac{\nu b}{a}\right)-1\right]^{-1}\left[e\left(-\frac{c}{a}y+\frac{\nu c}{a}\right)-1\right]^{-1}\end{aligned}$$

and therefore, by (3.4), we obtain

$$\mathfrak{S}_c^{a,b}(cx,\,cy)-\mathfrak{S}_b^{c,-a}(ax+by,\,cx)+\mathfrak{S}_a^{c,b}(ax+by,\,-cy)=(2\pi i/c)\mathfrak{F}\ . \tag{4.3}$$

Now, if we write

$$\int_Q=\int_{c-\varepsilon-ti}^{c-\varepsilon+ti}+\int_{c-\varepsilon+ti}^{-\varepsilon+ti}+\int_{-\varepsilon+ti}^{-\varepsilon-ti}+\int_{-\varepsilon-ti}^{c-\varepsilon-ti}$$

with the integrand of (4.2) and straight-line paths, the sum of the first and third member on the right vanishes because of the periodicity (with period c) of

$$[e(z)-1]^{-1}[e(x-bz/c)-1]^{-1}[e(y+az/c)-1]^{-1}\ .$$

On the other hand, using the estimate $|e(u+iv)-1|\geqq|e^{-2\pi v}-1|$ (u, v arbitrary real), we find at once that the integrals along the horizontal segments tend to zero as $t\to\infty$. Hence (4.3) implies for $t\to\infty$

$$\mathfrak{S}_a^{c,b}(ax+by,\,-cy)-\mathfrak{S}_b^{c,-a}(ax+by,\,cx)+\mathfrak{S}_c^{a,b}(cx,\,cy)=0 \tag{4.4}$$

which is, by (2.4), equivalent to (4.1).

5. Applications; extension of the well-known reciprocity theorems. (1) If we write

$$\mathfrak{T}_c^{a,b}(x,\,y)=\frac{1}{c}\sum_{r(\mathrm{mod}\ c)}\operatorname{ctg}\pi\frac{x-br}{c}\operatorname{ctg}\pi\frac{y+ar}{c} \tag{5.1}$$

and use (3.5), then (4.1) becomes

$$\mathfrak{T}_b^{c,a}(ax+by,\ -cx)+\mathfrak{T}_c^{a,b}(cx,\ cy)+\mathfrak{T}_a^{b,c}(-cy,\ ax+by)=1\ . \tag{5.2}$$

By (3.9), this may be regarded as a generalization of the reciprocity theorem of Dedekind sums. For, by putting $y=-x$ in (5.2) and making $x\to 0$, we obtain on the basis of the Laurent expansion $\operatorname{ctg} z=z^{-1}-\frac{1}{3}z-\cdots$

$$\mathfrak{s}_{11}\begin{pmatrix} b & c \\ & a \end{pmatrix}+\mathfrak{s}_{11}\begin{pmatrix} c & a \\ & b \end{pmatrix}+\mathfrak{s}_{11}\begin{pmatrix} a & b \\ & c \end{pmatrix}=\frac{1}{2}+\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right), \tag{5.3}$$

a remarkably symmetric three-term relation which for $a=1$ reduces to (1.2) with $h=b$, $k=c$. (Cf. also a result of Rademacher in [11].)

(2) Let us replace in (4.1) x, y by $x/2\pi i$ and $y/2\pi i$ respectively, multiply both sides by $c^2xy(ax+by)$ and expand every member by applying (2.5), (2.6) and the power series of $z/(e^z-1)$. We obtain

$$\begin{aligned} &cy\sum_{m,n=1}^{\infty}\frac{(ax+by)^m(-cx)^n}{m!n!}\mathfrak{s}_{m,n}\begin{pmatrix} c & a \\ & b \end{pmatrix}-(ax+by)\sum_{m,n=1}^{\infty}\frac{(cx)^m(cy)^n}{m!n!}\mathfrak{s}_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix} \\ &+cx\sum_{m,n=1}^{\infty}\frac{(-cy)^m(ax+by)^n}{m!n!}\mathfrak{s}_{m,n}\begin{pmatrix} b & c \\ & a \end{pmatrix}=c^2xy\left[1+\sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!}(ax+by)^\nu\right] \\ &-cy\sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!b^{\nu-1}}[(ax+by)^\nu+(-cx)^\nu]+c(ax+by)\sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!}(x^\nu+y^\nu) \\ &-cx\sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!\,a^{\nu-1}}[(-cy)^\nu+(ax+by)^\nu]\ , \end{aligned}$$

this holding identically for $|x|$, $|y|<2\pi$. If one uses still the binomial theorem and arranges our absolutely convergent series in terms of x^ν, y^ν ($\nu=1, 2, \cdots$), then comparison of the corresponding coefficients leads without difficulty to the following system of relations:

$$\begin{aligned} &a^\nu\cdot(\nu+1)b^\nu c\,\mathfrak{s}_{1,\nu}\begin{pmatrix} b & c \\ & a \end{pmatrix}+b^\nu\sum_{\mu=1}^{\nu}(-1)^{\mu+1}\binom{\nu+1}{\mu}c^\mu a^{\nu+1-\mu}\mathfrak{s}_{\nu+1-\mu,\mu}\begin{pmatrix} c & a \\ & b \end{pmatrix} \\ &+c^\nu\cdot(\nu+1)ab^\nu\,\mathfrak{s}_{\nu,1}\begin{pmatrix} a & b \\ & c \end{pmatrix}=B_{\nu+1}(a^{\nu+1}+\nu b^{\nu+1}+(-c)^{\nu+1})-(\nu+1)B_\nu(ab)^\nu c \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu=1,\ 2,\ \cdots\ , \end{aligned} \tag{5.4}$$

furthermore, by $\binom{\alpha}{\beta}\binom{\gamma}{\alpha}=\binom{\gamma}{\beta}\binom{\gamma-\beta}{\gamma-\alpha}$,

$$\begin{aligned} &a^\nu\cdot\binom{\nu+1}{p+1}\sum_{\mu=1}^{p}(-1)^{\mu+1}\binom{p+1}{\mu}b^{\nu+1-\mu}c^\mu\mathfrak{s}_{\mu,\nu+1-\mu}\begin{pmatrix} b & c \\ & a \end{pmatrix} \\ &+b^\nu\cdot\binom{\nu+1}{p}\sum_{\mu=1}^{\nu+1-p}(-1)^{\mu+1}\binom{\nu+1-p}{\mu}c^\mu a^{\nu+1-\mu}\mathfrak{s}_{\nu+1-\mu,\mu}\begin{pmatrix} c & a \\ & b \end{pmatrix} \end{aligned} \tag{5.5}$$

$$+c^{\nu}\cdot\Bigg[\binom{\nu+1}{p+1}a^{p+1}b^{\nu-p}\mathfrak{S}_{\nu-p,p+1}\binom{a\;\;b}{c}+\binom{\nu+1}{p}a^{p}b^{\nu+1-p}\mathfrak{S}_{\nu+1-p,p}\binom{a\;\;b}{c}$$

$$=B_{\nu+1}\Bigg[\binom{\nu+1}{p}a^{\nu+1}+\binom{\nu+1}{p+1}b^{\nu+1}\Bigg]-(\nu+1)B_{\nu}\binom{\nu}{p}(ab)^{\nu}c$$

$$1\leqq p\leqq \nu-1\ .$$

The results can be written briefly in symbolic form as follows

$$(5.6)\qquad (\nu+1)\Bigg[ca^{\nu}\mathfrak{S}_{1,\nu}\binom{b\;\;c}{a}+c^{\nu}a\mathfrak{S}_{\nu,1}\binom{a\;\;b}{c}\Bigg]-(a\mathfrak{S}-c\bar{\mathfrak{S}})^{\nu+1}\binom{c\;\;a}{b}$$

$$=\nu B_{\nu+1}b-(\nu+1)B_{\nu}a^{\nu}c\qquad \nu=1,\,2,\,\cdots\,,$$

$$(5.7)\qquad a^{\nu}\cdot\binom{\nu+1}{p+1}(b\mathfrak{S}-c\bar{\mathfrak{S}})^{p+1}(b\mathfrak{S})^{\nu-p}\cdot\binom{c\;\;b}{a}$$

$$+b^{\nu}\cdot\binom{\nu+1}{p}(a\mathfrak{S}-c\bar{\mathfrak{S}})^{\nu+1-p}(a\mathfrak{S})^{p}\binom{c\;\;a}{b}$$

$$-c^{\nu}\cdot\Bigg[\binom{\nu+1}{p+1}a\mathfrak{S}+\binom{\nu+1}{p}b\bar{\mathfrak{S}}\Bigg](a\mathfrak{S})^{p}(b\bar{\mathfrak{S}})^{\nu+p}\binom{a\;\;b}{c}$$

$$=(p+1)\binom{\nu+1}{p+1}B_{\nu}a^{\nu}b^{\nu}c\qquad p=1,\,2,\,\cdots\,;\ \nu=p+1,\,p+2,\,\cdots\,,$$

where for example

$$(b\mathfrak{S}-c\bar{\mathfrak{S}})^{p+1}(b\mathfrak{S})^{\nu-p}\binom{c\;\;b}{a}$$

means that, after formal application of the binomial theorem to the first factor and formal multiplication by $b^{\nu-p}\cdot\mathfrak{S}^{\nu-p}\cdot\binom{c\;\;b}{a}$, every product $\mathfrak{S}^{m}\bar{\mathfrak{S}}^{n}\binom{c\;\;b}{a}$ is replaced by $\mathfrak{S}_{m,n}\binom{c\;\;b}{a}$.

(3) We remark at once that (5.4), (5.6) go over for $\nu=1$ to the reciprocity relation (5.3) and for $\nu>1$ odd, $b=1$ to the formula (cf. (1.3), (2.7))

$$(5.8)\qquad (\nu+1)[ca^{\nu}\cdot s_{\nu}^{(\nu)}(c,\,a)+c^{\nu}a,\ s_{\nu}^{(\nu)}(a,\,c)]=(Bc-Ba)^{\nu+1}+\nu B_{\nu+1}$$

with 2

$$(Bc-Ba)^{\nu+1}=\sum_{\mu=0}^{\nu+1}(-1)^{\mu}\binom{\nu+1}{\mu}c^{\mu}a^{\nu+1-\mu}B_{\mu}B_{\nu+1-\mu}\ ;$$

[2] The factor $(-1)^{\mu}$ may plainly be suppressed in the last summand, that is,

$$(Bc-Ba)^{\nu+1}=(Bc+Ba)^{\nu+1}\ .$$

therefore (5.4), (5.6) generalize (5.3) and Apostol's reciprocity theorem [1, Theorem 1].

On the other hand, putting $\nu=3, 5, 7, \cdots$ in (5.7), we get for $c=1$

$$(5.9)\qquad \binom{\nu+1}{p+1}a^{\nu-p}(s^{(\nu)}-b)^{p+1}(b,\ a)-\binom{\nu+1}{p}b^{p}(s^{(\nu)}-a)^{\nu+1-p}(a,\ b)$$
$$=\binom{\nu+1}{p+1}aB_{\nu-p}B_{p+1}-\binom{\nu+1}{p}bB_{\nu+1-p}B_{p}\ ,$$

while the case $b=1$ yields

$$(5.10)\qquad c^{\nu}\left[\binom{\nu+1}{p+1}as^{(\nu)}_{\nu-p}(a,\ c)+\binom{\nu+1}{p}s^{(\nu)}_{\nu+1-p}(a,\ c)\right]$$
$$=\binom{\nu+1}{p+1}(s^{(\nu)}-c)^{p+1}(as^{(\nu)})^{\nu-p}(c,\ a)+\binom{\nu+1}{p}(aB-c\bar{B})^{\nu+1-p}B^{p}\ ,$$

the symbolic notations being understood in similar sense as above. (5.9) and (5.10) express the first and second reciprocity law of Carlitz respectively [3, Theorems 1, 2][3], so that we have in (5.5), (5.7) a common extension of them.

6. The sum $\mathfrak{D}_c^{a,b}(w,\ z)$. We now use the generalized zeta function, defined by

$$\zeta(z,\ u)=\sum_{n=0}^{\infty}(u+n)^{-z}$$

for $\Re(z)>1$ and by analytic continuation for other values $\neq 1$ of z, u denoting a fixed number with $0<u\leq 1$. There holds the well-known formula of Hurwitz:

$$(6.1)\qquad \zeta(z,\ u)=2(2\pi)^{z-1}\Gamma(1-z)$$
$$\times\left(\sin\frac{\pi z}{2}\sum_{n=1}^{\infty}n^{z-1}\cos 2n\pi u+\cos\frac{\pi z}{2}\sum_{n=1}^{\infty}n^{z-1}\sin 2n\pi u\right)\qquad \Re(z)<0\ .$$

Next we establish a functional equation for the sum

$$(6.2)\qquad \mathfrak{D}_c^{a,b}(w,\ z)=\sum_{\lambda=1}^{c-1}\zeta\left(w,\ \left\{\frac{\lambda a}{c}\right\}\right)\zeta\left(z,\ \left\{\frac{\lambda b}{c}\right\}\right)$$

with $(a,\ c)=(b,\ c)=1$, $c>1$, in observing that [cf. (1.4)]

$$(6.3)\qquad \mathfrak{D}_c^{a,b}(1-m,\ 1-n)=\frac{1}{mn}\left[\mathfrak{S}_{m,n}\binom{a\ \ b}{c}-B_mB_n\right]\qquad m,\ n=1,\ 2,\ \cdots$$

[3] In formula (3.2) of [**3**], the lack of the corresponding binomial coefficients before the Bernoullian numbers appears to be a typographical error.

and, by $\zeta(z, \frac{1}{2})=(2^z-1)\zeta(z)$ where $\zeta(z)=\zeta(z, 1)$ is Riemann's zeta function,

$$\mathfrak{D}_2^{a,b}(w, z)=(2^w-1)(2^z-1)\cdot\zeta(w)\zeta(z) \; . \tag{6.4}$$

THEOREM 2. *For* $(a, c)=(b, c)=1$, $c>2$ *and for any* w, z *distinct from* 0 *and* 1 *we have the relation*

$$\mathfrak{D}_c^{a,b}(w, z)=(c^{w+z}-1)\zeta(w)\zeta(z)+\pi^{-1}(2c\pi)^{w+z-1}\Gamma(1-w)\Gamma(1-z) \tag{6.5}$$

$$\times\left\{\cos\frac{\pi}{2}(w-z)\mathfrak{D}_c^{b,a}(1-w, 1-z)-\cos\frac{\pi}{2}(w+z)\mathfrak{D}_c^{b,-a}(1-w, 1-z)\right\} .$$

Proof. 1° First let $\Re(w)<0$, $\Re(z)<0$. We transform

$$\overline{\mathfrak{D}}_c^{a,b}(w, z)=\sum_{\lambda=1}^{c}\zeta\left(w, \left\{\frac{\lambda a}{c}\right\}\right)\zeta\left(z, \left\{\frac{\lambda b}{c}\right\}\right) \tag{6.6}$$

by means of (6.1).

Since the series involved in Hurwitz's formula are absolutely convergent, one obtains after substitution into (6.6)

$$\overline{\mathfrak{D}}_c^{a,b}(w, z)=4(2\pi)^{w+z-2}\Gamma(1-w)\Gamma(1-z) \tag{6.7}$$

$$\times\sum_{m,n=1}^{\infty}m^{w-1}n^{z-1}\left(\phi_{m,n}\cdot\sin\frac{\pi w}{2}\sin\frac{\pi z}{2}+\psi_{m,n}\cdot\cos\frac{\pi w}{2}\cos\frac{\pi z}{2}\right) ,$$

where

$$\phi_{m,n}=\sum_{\mu=1}^{c}\cos 2m\pi\frac{\mu a}{c}\cos 2n\pi\frac{\mu b}{c}=\begin{cases} c, \text{ if } \; c \mid am\pm bn \; , \\ 0 \;\text{ for } c\nmid am\pm bn \; , \\ c/2 \text{ otherwise} \; , \end{cases} \tag{6.8}$$

$$\psi_{m,n}=\sum_{\mu=1}^{c}\sin 2m\pi\frac{\mu a}{c}\sin 2n\pi\frac{\mu b}{c}=\begin{cases} c/2, \text{ if } c\mid am-bn \text{ but } \\ \qquad c\nmid am+bn \; , \\ -c/2, \text{ if } c\mid am+bn \text{ and } \\ \qquad c\nmid am-bn \; , \\ 0 \text{ otherwise} \; . \end{cases} \tag{6.9}$$

Hence it follows easily that

$$\overline{\mathfrak{D}}_c^{a,b}(w, z)=2c(2\pi)^{w+z-2}\Gamma(1-w)\Gamma(1-z)\cdot\left\{2\sin\frac{\pi w}{2}\sin\frac{\pi z}{2}\sum_{c\mid m,\, c\mid n}m^{w-1}n^{z-1}\right. \tag{6.10}$$

$$\left.+\cos\frac{\pi}{2}(w-z)\sum_{\substack{am\equiv bn \pmod c \\ c\nmid m,\, c\nmid n}}m^{w-1}n^{z-1}-\cos\frac{\pi}{2}(w+z)\sum_{\substack{am\equiv -bn \pmod c \\ c\nmid m,\, b\nmid n}}m^{w-1}n^{z-1}\right\} .$$

Now, by the functional equation of $\zeta(s)$ we have

$$4c(2\pi)^{w+z-2}\Gamma(1-w)\Gamma(1-z)\sin\frac{\pi w}{2}\sin\frac{\pi z}{2}\sum_{c|m,\,c|n} m^{w-1}n^{z-1} = c^{w+z-1}\zeta(w)\zeta(z)\,. \tag{6.11}$$

Furthermore, ar ($r=0, 1, \cdots, c-1$) and br ($r=0, 1, \cdots, c-1$) being complete systems of residues mod c, we can write

$$\begin{aligned}\sum_{\substack{am\equiv bn (\mathrm{mod}\ c)\\ c\nmid m,\, c\nmid n}} m^{w-1}n^{z-1} &= \sum_{r=1}^{c-1}\Big(\sum_{m\equiv rb(\mathrm{mod}\ c)} m^{w-1}\Big)\Big(\sum_{n\equiv ra(\mathrm{mod}\ c)} n^{z-1}\Big)\\ &= c^{w+z-2}\sum_{r=1}^{c-1}\Big[\sum_{M=0}^{\infty}\Big(\Big\{\frac{rb}{c}\Big\}+M\Big)^{w-1}\Big]\Big[\sum_{N=1}^{\infty}\Big(\Big\{\frac{ra}{c}\Big\}+N\Big)^{z-1}\Big]\\ &= c^{w+z-2}\sum_{r=1}^{c-1}\zeta\Big(1-w,\ \Big\{\frac{rb}{c}\Big\}\Big)\zeta\Big(1-z,\ \Big\{\frac{ra}{c}\Big\}\Big)\end{aligned} \tag{6.12}$$

and similarly

$$\begin{aligned}\sum_{\substack{am\equiv -bn (\mathrm{mod}\ c)\\ c\nmid m,\, c\nmid n}} m^{w-1}n^{z-1} &= \sum_{r=1}^{c-1}\Big(\sum_{m\equiv rb(\mathrm{mod}\ c)} m^{w-1}\Big)\Big(\sum_{n\equiv -ra(\mathrm{mod}\ c)} n^{z-1}\Big)\\ &= c^{w+z-2}\sum_{r=1}^{c-1}\zeta\Big(1-w,\ \Big\{\frac{rb}{c}\Big\}\Big)\zeta\Big(1-z,\ \Big\{\frac{ra}{c}\Big\}\Big)\,.\end{aligned} \tag{6.13}$$

(6.10)–(6.13) yield together

$$\begin{aligned}\overline{\mathfrak{D}}_c^{a,b}(w,\ z) &= c^{w+z-1}\zeta(w)\zeta(z)+\pi^{-1}(2c\pi)^{w+z-1}\Gamma(1-w)\Gamma(1-z)\\ &\times\Big\{\cos\frac{\pi}{2}(w-z)\mathfrak{D}_c^{b,a}(1-w,\ 1-z)-\cos\frac{\pi}{2}(w+z)\mathfrak{D}_c^{b,-a}(1-w,\ 1-z)\Big\}\,.\end{aligned} \tag{6.14}$$

2° Finally, (6.5) follows immediately from (6.14), in view of

$$\mathfrak{D}_c^{a,b}(w,\ z)=\overline{\mathfrak{D}}_c^{a,b}(w,\ z)-\zeta(w)\zeta(z) \qquad \Re(w)<0\,,\ \Re(z)<0$$

and by analytic continuation.

7. Some remarks. In [2], Apostol finds certain finite sum representations for $s_\nu^{(\nu)}(h, k)$, involving cotangents, $\zeta(z, u)$, $\Gamma'(z)/\Gamma(z)$ and he uses these expressions to give a short analytic proof of (5.8) [Theorems 1, 2]. It may be noted that the above Theorem 2 implies the results in question, arising as limiting cases for $w\to 0$, and $z\to 0$, $z=-1, -2, \cdots$.

The form of $\mathfrak{S}_c^{a,b}(x, y)$, $\mathfrak{D}_c^{a,b}(w, z)$ suggests applications in connection with certain Lambert series, generalizing those investigated by Rademacher, Apostol and Carlitz. I hope to return on this problem in another paper.

References

1. T. M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J., **17** (1950), 147–157.

2. ———, *Theorems on generalized Dedekind sums*, Pacific J. Math., **2** (1952), 1–9.

3. L. Carlitz, *Some theorems on generalized Dedekind sums*, Pacific J. Math., **3** (1953), 513–522.

4. ———, *The reciprocity theorem for Dedekind sums*, Pacific J. Math., **3** (1953), 523–527.

5. ———, *A note on generalized Dedekind sums*, Duke Math. J., **21** (1954), 399–403.

6. ———, *Dedekind sums and Lambert series*, Proc. Amer. Math. Soc., **5** (1954), 580–584.

7. R. Dedekind, *Erläuterungen zu Riemann's Fragmenten über die Grenzfälle der elliptischen Funktionen*, Gesammelte mathematische Werke, vol. 1, Braunschweig, 1930, 159–173.

8. L. J. Mordell, *The reciprocity formula for Dedekind sums*, Amer. J. Math. **73** (1951), 593–598.

9. H. Rademacher, *Zur Theorie der Modulfunktionen*, J. Reine Angew. Math., **167** (1932), 312–336.

10. ———, *Die Reziprozitätsformel für Dedekindsche Summen*, Acta Sci. Math. Szeged, **12B** (1950), 57–60.

11. ———, *Generalization of the reciprocity formula for Dedekind sums*, Duke Math. J., **21** (1954), 391–397.

12. H. Rademacher and A. Whiteman, *Theorems on Dedekind sums*, Amer. J. Math., **63** (1941), 377–407.

13. L. Rédei, *Elementarer Beweis und Verallgemeinerung einer Reziprozitätsformel von Dedekind*, Acta Sci. Math. Szeged, **12B** (1950), 236–239.

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