MONOTONE MAPPINGS OF MANIFOLDS

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1. Introduction. Mappings of the 2-sphere, and more generally of the 2-manifolds, have been studied by various authors. (See, for instance, [9] and references therein, [7].) Generally, these mappings have been subjected to certain "monotoneity" conditions on the counter-images of points. Thus, in Moore's first paper [8] on the 2-sphere, it was required not only that counter-images be connected, but that they not separate the sphere. In terms of homology, then, he required of a counter-image C that $p_a^r(C)=0$ for $r=0,1$. Later studies of Moore and others usually omitted the requirement that $p^1(C)=0$, thus increasing the possible number of topological types of images. With the condition $p^1(C)=0$ imposed, the image of the 2-sphere is a 2-sphere, and of a 2-manifold is a 2-manifold of the same type. Without this condition, the various types of "cactoids" are obtained.

In the present paper we consider some higher dimensional cases. As might be expected, we impose higher dimensional "monotoneity" conditions.

DEFINITION 1. A mapping $f: A \to B$ is called *n-monotone* if $H^r(f^{-1}(b))=0$ for all $b \in B$ and $r \leq n$. (See [10; p. 904].)

EXAMPLE. Let us consider the mapping induced by decomposing the 3-sphere into disjoint closed sets each of which is a point, except that all points on some suitable "wild" arc [5; Ex. 1.1] A are identified. This mapping is r-monotone for all r, but the image-space is no longer a 3-sphere; indeed, it is not a 3-manifold in the classical sense at all, since the point corresponding to A does not have a 3-cell neighborhood.

This example makes it at first appear that because of such "homotopy" difficulties, it may be useless to look for any well-defined class of configurations in higher dimensions. However, as we show below, the class of configurations obtained is precisely that of the generalized manifolds. Moreover, we need not restrict the mappings to the mappings of 3-manifolds in the classical sense, since the generalized manifolds turn out to form a class which is closed relative to the mappings considered. This result forms, then, a new justification for the study of generalized manifolds.

2. Preliminary theorems and lemmas. In general, spaces are

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Hausdorff, but no conditions of metrizibility or separability are assumed. Except where noted to the contrary, we use augmented Čech homology with an algebraic field as coefficient domain. We recall the following definition [**11**; p. 237].

DEFINITION 2. If S is a locally compact space, such that for every pair of open sets P, Q for which $P \supset \overline{Q}$ and $\overline{Q}$ is compact, the group $H^n(S: \overline{Q}, 0; \overline{P}, 0)$ (cf. [**11**; 166, Def. 18.28]) is of finite dimension, then S is said to have property $(P, Q)^n$.

REMARK. Since the space is assumed locally compact, the above definition can be stated in a number of different but equivalent forms. Thus, Q may be replaced in the definition by any compact set M; that is, S has property $(P, Q)^n$ if for every pair of sets P, M such that P is open, M is compact, and $P \supset M$, then $H^n(S: M, 0; \overline{P}, 0)$ is of finite dimension. Another variant, but equivalent form of the definition, is obtained if in either of the above definitions it be required only that there exist at most a finite number of n-cycles on $\overline{Q}(M)$ which are lirh on compact subsets of P (that is, *in* P).

Another variant would be to require that there exist at most a finite number of cycles on compact subsets of Q (that is, *in* Q) that are lirh on $\overline{P}$ (or, that are lirh in P). Each of the equivalent forms of the definition may be found particularly adapted to a given situation.

We express the fact that S has property $(P, Q)^r$ for $r=0, 1, \cdots, n$ by stating that S has property $(P, Q)_0^n$.

THEOREM 1. *If S is a compact space having property $(P, Q)^n$, and $f: S \to S'$ is a continuous $(n-1)$-monotone mapping of S onto a Hausdorff space S', then S' has property $(P, Q)^n$.*

Proof. Let U', V' be open subsets of S' such that $U' \supset \overline{V}'$ and $\overline{V}'$ is compact. The sets $U = f^{-1}(U')$, $V^* = f^{-1}(\overline{V}')$ are open and closed subsets, respectively, of S, such that $U \supset V^*$.

In the mapping $f(V^*) = \overline{V}'$, counter-images of points are all r-acyclic for $r = 0, 1, \cdots, n-1$. Hence [3] for any cycle γ^n on $\overline{V}'$, there exists a cycle Z^n on V^* such that

$$(1) \qquad f(Z^n) \sim \gamma^n \text{ on } \overline{V}' .$$

Since S has property $(P, Q)^n$, there exist cycles Z_i^n, $i=1, \cdots, m$ of V^* such that if Z^n is any cycle of V^*, then

(2) $$Z^n \sim \sum_{i=1}^{m} a^i Z_i^n \text{ in } U .$$

Consequently, since (2) implies

(3) $$f(Z^n) \sim \sum_{i=1}^{m} a^i f(Z_i^n) \text{ in } U' ,$$

we have, combining (1) and (3), that

$$\gamma^n \sim \sum_{i=1}^{m} a^i f(Z_i^n) \text{ in } U' .$$

It follows that at most m cycles on $\overline{V}'$ are lirh in U and hence that S' has property $(P, Q)^n$.

REMARK. It is worthwhile noting that the above proof gives the following: If $f:S\to S'$ is a continuous $(n-1)$-monotone proper mapping of a locally compact space S onto a Hausdorff space S', and P', F' are open and compact subsets of S', respectively, such that $P'\supset F'$, then

$$f\,|\,P_*:H^n(S:F, P)\to H^n(S':F', P')$$

is a homomorphism onto, where $F=f^{-1}(F')$, $P=f^{-1}(P')$, and $H^n(S:F, P)$ denotes the group of n-cycles on F reduced modulo the subgroup of n-cycles that bound *in* P. A similar argument shows that $f\,|\,P_*$: $H^{n-1}(S:F, P)\to H^{n-1}(S':F', P')$ is an isomorphism onto. These are generalizations of the Vietoris mapping theorem [2], [3].

THEOREM 2. *If S is an lc^n compact space, $n>0$, and $f:S\to S'$ is a continuous $(n-1)$-monotone mapping of S onto a Hausdorff space S', then S' is lc^n.*

Proof. By [11; p. 70, Th. 1.6], S' is 0-lc. And since S' is a compact 0-lc space, it has property $(P, Q)^0$. (See [11; p. 106, 3.7]). That S' has property $(P, Q)^r$ for $r=1, 2, \cdots, n$ follows from Theorem 1. Since, for compact spaces, lc^n and $(P, Q)_0^n$ are equivalent, we conclude that S' is lc^n (see 11; p. 238, 7.17]).

LEMMA 1. *In a locally compact space S, let P and Q be open sets such that $\overline{P}$ is compact and $P\supset\overline{Q}$; and let M be a closed subset of Q such that for any open set Q_ν for which $M\subset Q_\nu\subset Q$, the dimension of $H_r(S:S, S-\overline{P}; S, S-\overline{Q}_\nu)$* [11; 166, Def. 18.29] *is the same finite number k. If $Z_r^1, \cdots, Z_r^k$ form a base for r-cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}$, then for every open set Q_ν such that $M\subset Q_\nu\subset Q$,* *the cocycles* Z_r^i *form a base for* r*-cocycles mod* $S-\overline{P}$ *relative to cohomologies mod* $S-\overline{Q}_\nu$.

Proof. Let $\gamma_r^1, \cdots, \gamma_r^k$ be a base for cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}_\nu$. Then there exist cohomologies:

$$(1) \qquad \gamma_r^j \sim \sum_{i=1}^{k} c_i^j Z_r^i \bmod S-\overline{Q}, \qquad j=1, \cdots, k.$$

Relations (1) hold a fortiori mod $S-\overline{Q}_\nu$.

The matrix $\|c_i^j\|$ is of rank k, since otherwise there would exist a cohomology relation between the γ_r^j's, mod $S-\overline{Q}_\nu$.

Suppose the Z_r^i's are not lircoh mod $S-\overline{Q}_\nu$. Then there exists a relation

$$\sum a_i Z_r^i \sim 0 \qquad \bmod S-\overline{Q}_\nu .$$

But the system of equations

$$c_i^1 x_1 + \cdots + c_i^j x_j + \cdots + c_i^k x_k = a_i , \qquad i=1, \cdots, k$$

has a non-trivial solution in the x_j's. Hence, multiplying the relations (1) by $x_1, \cdots, x_k$, respectively, we get

$$\sum x_j \gamma_r^j \sim \sum a_i Z_r^i \sim 0 \qquad \bmod S-\overline{Q}_\nu .$$

Thus, the assumption that the Z_r^i are not liroch mod $S-\overline{Q}_\nu$ leads to contradiction; and since the dimension of $H_r(S: S, S-\overline{P}; S, S-\overline{Q}_\nu)=k$, we conclude that the Z_r^i's form a base for cohomologies mod $S-\overline{Q}_\nu$.

LEMMA 2. *In a locally compact space* S, *let* M *be a compact set such that* $H^r(M)=0$; *and suppose that there exist open sets* P, Q *such that* $M \subset Q \subset P$ *and such that* $H^r(S: \overline{Q}, 0; \overline{P}, 0)$ *has finite dimension. Then there exists an open set* Q_ν *such that* $M \subset Q_\nu \subset Q$ *and* $H^r(S: \overline{Q}_\nu, 0; \overline{P}, 0)=0$.

Proof. Suppose, on the contrary, that for all such Q_ν, $H^r(S: \overline{Q}_\nu, 0; \overline{P}, 0) \neq 0$. Since $H^r(S: \overline{Q}, 0; \overline{P}, 0)$ is of finite dimension, we may assume Q shrunk so that all dimensions of groups $H^r(S: \overline{Q}_\nu, 0; \overline{P}, 0)$ are equal to the same positive integer k for all Q_ν such that $M \subset Q_\nu \subset Q$.

Since

$$H_r(S: S, S-\overline{P}; S, S-\overline{Q}_\nu) \simeq H^r(S: \overline{Q}_\nu, 0; \overline{P}, 0)$$

[11; 166, 18.30], there exist, by Lemma 2, cocycles Z_r^i, $i=1, \cdots, k$, mod $S-\overline{P}$, that form a base for cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}_\nu$ for all Q_ν such that $M\subset Q_\nu\subset Q$. Consider Z_r^1, and $\mathfrak{U}$ a fcos of $\overline{P}$ such that $Z_r^1(\mathfrak{U})$ exists. Let $\mathfrak{B}>\mathfrak{U}$ be a normal refinement of $\mathfrak{U}$ rel. M [11; 140], and let Q_ν be such that if a simplex of $\mathfrak{B}$ meets Q_ν, then it meets M. Since $Z_r^1\not\sim 0 \bmod S-\overline{Q}_\nu$, there exists on $\overline{Q}_\nu$ a cycle Z^r such that $Z_r^1\cdot Z^r=1$. And by the choice of $\mathfrak{B}$, the coordinate $Z^r(\mathfrak{B})$ is on M. Hence $\pi_{\mathfrak{U}\mathfrak{B}}Z^r(\mathfrak{B})$ is the coordinate on M of a Čech cycle γ^r.

But $H^r(M)=0$ and consequently $\gamma^r\sim 0$ on M, and a fortiori, $\gamma^r(\mathfrak{U})\sim 0$ on $\overline{Q}$; and since $Z^r(\mathfrak{U})\sim\pi_{\mathfrak{U}\mathfrak{B}}Z^r(\mathfrak{B})$ on $\overline{Q}$, it follows that $Z^r(\mathfrak{U})\sim 0$ on $\overline{Q}$. But then $Z^r(\mathfrak{U})\cdot Z_r^1(\mathfrak{U})=0$, in contradiction to the choice of $Z^r(\mathfrak{U})$. We conclude, then, that for some Q_ν, $H^r(S:\overline{Q}_\nu, 0; \overline{P}, 0)=0$.

THEOREM 3. *A necessary and sufficient condition that a locally compact space S be lc^n is that if M is any compact subset of S such that $H^r(M)=0$, for some $r\leqq n$, then for any open set P containing M there exist an open set Q such that $M\subset Q\subset\overline{Q}\subset P$ and such that $H^r(S:\overline{Q}, 0; \overline{P}, 0)=0$.*

Proof of sufficiency. Trivial. (See [11; 193, 6.14]).

Proof of necessity. With M and P as in the hypothesis, and any open set Q such that $\overline{Q}$ is compact and $M\subset Q\subset\overline{Q}\subset P$, the dimension of $H^r(S:\overline{Q}, 0; \overline{P}, 0)$ is finite [11; 193, 6.16]. Lemma 2 now gives the desired result.

LEMMA 3. *If S is an orientable n-gm and M a compact subset of S which is r- and $(n-r-1)$-acyclic for some r such that $r\leqq n-2$, then for any open set P containing M, there exists an open set Q such that $M\subset Q\subset\overline{Q}\subset P$ and such that all compact r-cycles in $Q-M$ bound in $P-M$.*

Proof. Since S is lc^n [11; 244], there exists by Theorem 3 an open set Q containing M such that $\overline{Q}\subset P$ and such that all r- and $(n-r-1)$-cycles in Q bound in P. Suppose there exists a cycle Z^r in $Q-M$ that does not bound in $P-M$.

By Lemma VIII 5.4 of [11; 255] there exists a cocycle $Z_{n-r}=\tau^*Z^r$ in $Q-M$ such that $Z_{n-r}\frown\Gamma^n\sim Z^r$ in $Q-M$, where Γ^n is the fundamental n-cycle of S. And since $Z^r\sim 0$ in P, we may assume that $Z_{n-r}\sim 0$ in P. There exists a covering $\mathfrak{U}$ and a relation.

$$\delta C_{n-r-1}(\mathfrak{U})=Z_{n-r}(\mathfrak{U}) \quad \text{in } P. \tag{1}$$

The chain $C_{n-r-1}(\mathfrak{U})$ is clearly a cocycle mod $P-M=S-[(\text{Ext } P)\cup M]$.

And if $C_{n-r-1} \not\sim 0$ mod $S-[(\text{Ext } P) \cup M]$, then by [11; 164, 18.19] there exists a cycle Z^{n-r-1} on $(\text{Ext } P) \cup M$ such that $C_{n-r-1} \cdot Z^{n-r-1} = 1$. Since $Z^{n-r-1} = Z_1 + Z_2$, where Z_1 is on Ext P and Z_2 on M, we may neglect Z_1 (as $C_{n-r-1}(\mathfrak{U})$ is in P) and write $C_{n-r-1} \cdot Z_2 = 1$. But $Z_2 \sim 0$ on M since M is $(n-r-1)$-acyclic, implying $C_{n-r-1} \cdot Z_2 = 0$. We conclude, then, that $C_{n-r-1} \sim 0$ mod $P-M$. There exists, therefore, a covering $\mathfrak{V} > \mathfrak{U}$ and a relation

$$\delta C_{n-r-2}(\mathfrak{V}) = \pi^*_{\mathfrak{U}\mathfrak{V}} C_{n-r-1}(\mathfrak{U}) - L_{n-r-1}(\mathfrak{V}) , \tag{2}$$

where $L_{n-r-1}(\mathfrak{V})$ is in $P-M$.

Applying δ to (2) and utilizing (1), we get

$$\delta L_{n-r-1}(\mathfrak{V}) = \pi^*_{\mathfrak{U}\mathfrak{V}} Z_{n-r}(\mathfrak{U}) .$$

That is, $Z_{n-r} \sim 0$ in $P-M$. But this implies $Z^r \sim 0$ in $P-M$, contrary to supposition.

REMARK. In the hypothesis of Lemma 3 it was assumed that $r \leqq n-2$, that is, $n-r-2 \geqq 0$. The necessity for this is shown by the following example: Let S be the 2-sphere, S^2, and in S let M be a circular disk, and U and V open circular disks concentric with M and such that $M \subset V \subset \bar{V} \subset U$. Then in $V-M$ an S^1 which encloses M carries a Z^1 which fails to bound in $U-M$.

Note also that if M is an S^1 in S^4, then M is 2-acyclic but in any open set P containing M there exist 2-dimensional cycles linking M. This shows the necessity for the assumption that M be $(n-r-1)$-acyclic in the hypothesis.

LEMMA 4. *Let Z^{n-1} be a cycle carried by a closed subset K of an orientable n-gcm S, and M a connected subset of $S-K$. If $Z^{n-1} \sim 0$ on S, then must $Z^{n-1} \sim 0$ on a compact subset of $S-M$.*

Proof. This is analogous to that of Lemma XII 3.12, p. 375 of [11].

For the purposes of the proof of the next theorem, let us recall the following form of the definition of an orientable *n-gcm*: An n-dimensional compact space S such that (1) $p^n(S) = 1$ and all n-cycles on closed proper subsets of S bound on S; (2) S is semi-r-connected for all r such that $1 \leqq r \leqq n-1$; (3) S is completely r-avoidable at all points for all $r \leqq n-2$; (4) S is n-extendible at all points. (This is IX 3.6, p. 273, of [11]). (By Lemmas VII 5.2, 5.3, p. 224 of [11], condition (4) may be replaced by the requirement that S is locally $(n-1)$-avoidable at all points; this fact will be utilized in the proof of Main Theorem A below.)

3. Main theorems.

MAIN THEOREM A. *Let S be an orientable n-gcm and $f: S \rightarrow S'$ an $(n-1)$-monotone continuous mapping of S onto an at most n-dimensional nondegenerate Hausdorff space S'. Then S' is an orientable n-gcm of the same homology type as S.*

Proof. Since S' is nondegenerate, f is n-monotone and therefore by the Vietoris-Begle Theorem [2], $p^n(S')=p^n(S)=1$. And since $p^n(S')>0$, S' is at least n-dimensional, and therefore, by the dimensionality assumption of the hypothesis, is exactly n-dimensional. And if F' is a proper closed subset of S', and Z^n a cycle on F', there exists on the set $F=f^{-1}(F')$ a cycle γ^n such that $f(\gamma^n)\sim Z^n$ on F' (see [2; § 5]). As F is a proper closed subset of $S, \gamma^n \sim 0$ on S and therefore $f(\gamma^n)\sim 0$ on S'—implying that $Z^n \sim 0$ on S'. Thus S' satisfies condition (1) above.

That condition (2) is satisfied, follows from the fact that S' is $\bar{l}c^n$ by Theorem 2.

Let $p' \in S'$, and U' an open set containing p'. Then $U=f^{-1}(U')$ is an open set containing the set $M=f^{-1}(p')$. Let r be any integer such that $1 \leqq r \leqq n-2$. Since $H^r(M)=H^{n-r-1}(M)=0$, there exists by Lemma 3 an open set P such that $M \subset P \subset \bar{P} \subset U$ and such that all r-cycles in $P-M$ bound in $U-M$. Let W' be an open set such that $p' \in W' \subset \overline{W'} \subset U'$, and such that $f^{-1}(\overline{W'}) \subset P$. Let Q' be an open set such that $p' \in Q' \subset \overline{Q'} \subset W'$. As S' is $\bar{l}c^n$, there exists a finite base $Z_1^r, \cdots, Z_k^r$ of r-cycles of $F(W')$ relative to homologies in $U'-\overline{Q'}$. Let $W=f^{-1}(W')$, $Q=f^{-1}(Q)$, and consider any cycle Z_i^r. There exists a cycle γ_i^r on $f^{-1}(F(W'))$ such that $f(\gamma_i^r)\sim Z_i^r$ on $F(W')$. And as $\gamma_i^r \sim 0$ in $U-M$, Z_i^r must bound in $U'-P'$. Finally, since there are only a finite number of the r-cycles Z_i^r, there must exist an open set R' such that $p' \in R' \subset \overline{R'} \subset Q'$ and such that all r-cycles on $F(W')$ bound in $U'-\overline{R'}$. Thus S' satisfies condition (3).

To show that S' satisfies condition (4), let p', U', U and M be as before. Since by hypothesis $p^{n-1}(M)=0$, there exists by Theorem 3 an open set V such that $M \subset V \subset \bar{V} \subset U$ such that all $(n-1)$-cycles of $\bar{V}$ bound on $\bar{U}$. Let P' be an open set such that $p' \in P' \subset \overline{P'} \subset U'$ and such that if $F'=F(P')$, then the set $F=f^{-1}(F')$ lies in V. Let Q' be an open set such that $p' \in Q' \subset \overline{Q'} \subset P'$. As above, there exist cycles Z_i^{n-1}, $i=1, \cdots, k$, of F' forming a base for $(n-1)$-cycles of F' relative to homologies in $S'-\overline{Q'}$. And for each Z_i^{n-1} there exists a cycle γ_i^{n-1} on F such that $f(\gamma_i^{n-1})\sim Z_i^{n-1}$ on F'. But since $\gamma_i^{n-1}\sim 0$ on $\bar{U}$, hence on S, it follows from Lemma 4 that $\gamma_i^{n-1}\sim 0$ in $S-M$. Therefore each $Z_i^{n-1}\sim 0$

in $S'-p'$, and it follows that, as above, an open set R' exists such that $p'\in R'\subset Q'$ and all Z_i^{n-1} bound in $S'-\overline{R}'$. Thus S' is locally $(n-1)$-avoidable.

The necessity for assuming that S' is at most n-dimensional above may be avoided if the monotoneity condition on f is strengthened. We recall that for the Vietoris Mapping Theorem to hold when the coefficient group is not a field or an elementary compact topological group, it is necessary to phrase the monotoneity condition in terms of the individual coordinates of cycles (just as, for example, may be done with the r-*lc* condition; compare [**11**; 176, Defs. 1.1, 1.2]). In terms of the generalized Vietoris cycles such as Begle employed [**2**], the condition is defined as follows:

DEFINITION 3. A mapping f of a space X onto a space Y is a *Vietoris mapping of order* n if for each covering $\mathfrak{U}$ of X and $y\in Y$ there exists a refinement $\mathfrak{B}=\mathfrak{B}(\mathfrak{U}, y)$ of $\mathfrak{U}$ such that every r-cycle of $X(\mathfrak{B})\wedge f^{-1}(y)$ [**11**; 131], $r\leqq n$, bounds on $X(\mathfrak{U})\wedge f^{-1}(y)$. (By $X(\mathfrak{U})$ is denoted the complex consisting of all simplexes σ such that the vertices of σ are points of X and diameter of $\sigma<\mathfrak{U}$.)

When the coefficient group is a field or elementary compact group, this definition is equivalent to that of n-monotone. It will be convenient, then, to retain the term "n-monotone" with, however, a qualification regarding the coefficient group employed. Also, for working with Čech cycles, the definition is more suitable in the following form:

DEFINITION 3′. A mapping f of a space X onto a space Y is n-monotone over (an abelian group) G if for each covering $\mathfrak{U}$ of X, $y\in Y$ and $M=f^{-1}(y)$, there exists a refinement $\mathfrak{B}$ of $\mathfrak{U}$ such that for every r-cycle $Z^r(\mathfrak{B})$ over G, $r\leqq n$, on $\mathfrak{B}\wedge M$ the projection $\pi_{\mathfrak{U}\mathfrak{B}}Z^r(\mathfrak{B})$ bounds on $\mathfrak{U}\wedge M$.

A routine argument shows that the two Definitions 3 and 3′ are equivalent.

LEMMA 4. *If f is an n-monotone mapping over the additive group I of integers of a compact space S onto a Hausdorff space S', then f is n-monotone over every abelian group G.*

(Remark. As will be seen from the proof below, it is sufficient to assume the condition of the Definition 3′ only for $r=n$ and $n-1$.)

Proof. For $n=0$ the lemma follows at once since, as is easily shown, 0-monotone over any group G is equivalent to the connectedness of $f^{-1}(x)$ for all $x\in S'$.

For $n>0$ we proceed as follows (see Čech [4; 11-13], where a similar type of argument is employed for quite different purposes): Given a covering $\mathfrak{U}_1$ of S and $x \in S'$, $M-f^{-1}(x)$, we choose $\mathfrak{U}_2>\mathfrak{U}_1$ such that for every n-cycle $Z^n(\mathfrak{U}_2)$ over I on $\mathfrak{U}_2 \wedge M$, the projection $\pi_{12}Z^n(\mathfrak{U}_2)$ thereof from $\mathfrak{U}_2$ to $\mathfrak{U}_1$ bounds on $\mathfrak{U}_1 \wedge M$; and $\mathfrak{U}_3>\mathfrak{U}_2$ such that for every $(n-1)$-cycle $Z^{n-1}(\mathfrak{U}_3)$ over I on $\mathfrak{U}_3 \wedge M$, the projection $\pi_{23}Z^{n-1}(\mathfrak{U}_3)$ thereof bounds on $\mathfrak{U}_2 \wedge M$.

There exists a base for n-chains over I for the complex $\mathfrak{U}_3 \wedge M$ consisting of chains $C_i^n(\mathfrak{U}_3)$, $i=1, \cdots, \alpha_n$, such that

$$\begin{aligned} &\partial C_i^n(\mathfrak{U}_3)=\eta_i^n C_i^{n-1}(\mathfrak{U}_3)\ , && i=1, \cdots, \beta_n\ , \\ &\partial C_i^n(\mathfrak{U}_3)=0\ , && i=\beta_n+1, \cdots, \alpha_n\ , \end{aligned}$$

where $0 \leqq \beta_n \leqq \min(\alpha_n, \alpha_{n-1})$.

Consider any cycle $Z^n(\mathfrak{U}_3)$ over G of $\mathfrak{U}_3 \wedge M$. Then

$$Z^n(\mathfrak{U}_3)=\sum_{i=1}^{\alpha_n} g_i C_i^n(\mathfrak{U}_3)\ , \qquad g_i \in G\ .$$

And since $Z^n(\mathfrak{U}_3)$ is a cycle,

$$\sum_{i=1}^{\beta_n} g_i \eta_i^n C_i^{n-1}(\mathfrak{U}_3)=0\ ,$$

implying that

$$(1) \qquad g_i \eta_i^n=0 \qquad \text{for } 1 \leqq i \leqq \beta_n\ .$$

Also, since for $\beta_n+1 \leqq i \leqq \alpha_n$ the chain $C_i^n(\mathfrak{U}_3)$ is a cycle, there exist chains $H_i^{n+1}(\mathfrak{U}_1)$ over I of $\mathfrak{U}_1 \wedge M$ such that

$$(2) \qquad \partial H_i^{n+1}(\mathfrak{U}_1)=\pi_{12}\pi_{23}C_i^n(\mathfrak{U}_3)\ , \qquad \beta_n+1 \leqq i \leqq \alpha_n\ .$$

Furthermore there exist chains $D_i^n(\mathfrak{U}_2)$ over I of $\mathfrak{U}_2 \wedge M$ such that

$$\partial D_i^n(\mathfrak{U}_2)=\pi_{23}C_i^{n-1}(\mathfrak{U}_3)\ , \qquad 1 \leqq i \leqq \beta_n\ .$$

And since the chains $\pi_{23}C_i^n(\mathfrak{U}_3)-\eta_i^n D_i^n(\mathfrak{U}_2)$ are cycles over I, we also have relations

$$(3) \qquad \partial H_i^{n+1}(\mathfrak{U}_1)=\pi_{12}\pi_{23}C_i^n(\mathfrak{U}_3)-\pi_{12}\eta_i^n D_i^n(\mathfrak{U}_2)\ , \qquad 1 \leqq i \leqq \beta_n$$

on $\mathfrak{U}_1 \wedge M$. From (1), (2), and (3) we get

$$\begin{aligned} \partial \sum_{i=1}^{\alpha_n} g_i H_i^{n+1}(\mathfrak{U}_1) &= \sum_{i=1}^{\alpha_n} \pi_{12}\pi_{23} g_i C_i^n(\mathfrak{U}_3) \\ &=\pi_{12}\pi_{23}Z^n(\mathfrak{U}_3) \\ &\sim \pi_{13}Z^n(\mathfrak{U}_3)\ . \end{aligned}$$

on $\mathfrak{U}_1 \wedge M$.

MAIN THEOREM B. *Let S be an orientable n-gcm and $f: S \to S'$ a continuous mapping of S, $(n-1)$-monotone over the integers, onto a finite-dimensional nondegenerate Hausdorff space S'. Then S' is an orientable n-gcm of the some homology type as S.*

Proof. The defining properties of an orientable n-*gcm* S utilize an algebraic field $\mathscr{F}$ as coefficient domain, and in particular specify that if F is a proper closed subset of S, then $H^n(F; \mathscr{F})=0$. It follows that since S' is nondegenerate, f is n-monotone as defined in Definition 1, and consequently [2; 542-3] is n-monotone over $\mathscr{F}$ as defined in Definition 3′. Furthermore, f is n-monotone over I. For it is trivial that n-monotoneity over a cofinal system of coverings of a space is sufficient for n-monotoneity, and S has a cofinal system Σ of coverings of dimension n; and since a cycle $Z^n(\mathfrak{B})$, $\mathfrak{B} \in \Sigma$, over I is a fortiori a cycle over $\mathscr{F}$, for a projection $\pi_{\mathfrak{U}\mathfrak{B}} Z^n(\mathfrak{B})$, $\mathfrak{U} \in \Sigma$, to bound implies $\pi_{\mathfrak{U}\mathfrak{B}} Z^n(\mathfrak{B})=0$. We conclude then that f is n-monotone over I.

Now suppose the dimension, dim $S', > n$. Then ([6]; [1]) there exists a closed set $C \subset S'$ and cycle Z^n over R_1 (the additive group of the reals mod 1) such that $Z^n \sim 0$ on S' but $Z^n \nsim 0$ on C. As f is n-monotone over R_1 by Lemma 4, there exists [2; § 5] a cycle γ^n on $f^{-1}(C)$ such that $f(\gamma^n) \sim Z^n$ on C. But since $Z^n \sim 0$ on S', it follows [2; 542] that $\gamma^n \sim 0$ on S. As S is n-dimensional, this implies $\gamma^n = 0$ and a fortiori that $\gamma^n \sim 0$ on C and consequently $f(\gamma^n) \sim 0$ on C, implying $Z_n \sim 0$ on C, contrary to the choice of Z^n.

The theorem now follows from Main Theorem A, since by Lemma 4, γ is $(n-1)$-monotone over $\mathscr{F}$.

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