# ON THE GENERALIZED RADIATION PROBLEM OF A. WEINSTEIN

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**1. Introduction.** The generalized radiation problem as formulated and solved by A. Weinstein [8] requires determination of a non-singular solution of the two-dimensional Euler-Poisson-Darboux (abbreviated EPD) equation

$$u_{xx}^{[k]}=u_{yy}^{[k]}+\frac{k}{y}u_y^{[k]} \tag{1.1}$$

for $-\infty<k<1$ such that

$$\lim_{y\to 0} u^{[k]}(x, y)=f(x) \quad \text{and} \quad u^{[k]}(x, y)=0 \quad \text{for} \quad y=x \tag{1.2}$$

where $f(x)$ is a function given on some interval $0\leqq x\leqq a$, possessing a specified number of continuous derivatives there and having another specified number of zero derivatives at $x=0$. These conditions on $f(x)$ depend on the parameter $k$ as stated in [8]. The classical radiation problem, requiring an axially symmetric solution of the higher dimensional wave equation with a certain type of singularity, as given in [3], is a special case. If $k$ is an integer and $u^{[k]}$ a solution of the above generalized radiation problem, then

$$u^{(2-k)}(x, y)=\frac{u^{[k]}(x, y)}{y^{1-k}} \tag{1.3}$$

is a solution of the classical radiation problem in an $m=3-k$ dimensional space (not counting time as a dimension). Thus from a regular solution $u^{[k]}$ one generates a solution $u^{[2-k]}$ of the EPD equation with that type of singularity needed to solve the radiation problem.

The first part of this paper will be devoted to uniqueness for the generalized radiation problem. Although a more complete answer to the uniqueness question would be welcome, consideration of solutions which have two continuous derivatives on $y=x$ is natural since such solutions are the ones that correspond closely to radiation phenomena. Let $T$ be a triangle with vertices $(0, 0)$, $(a, 0)$, $(a/2, a/2)$. We define a function to be regular on $T$ if it has two continuous derivatives in some triangle $G$ the interior of which contains $T$ and its sides except for the base line, $y=0$. Only a function satisfying the EPD equation, regular on $T$, and taking on the given data will be considered a solution of the

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radiation problem. Such considerations cover an important class of the Weinstein solutions.

We are concerned for uniqueness only with the difference of two solutions $u^{[k]}(x, y)$ which take on the given data $f(x)$; that is, we show that $u^{[k]}(x, 0) \equiv u^{[k]}(x, x) \equiv 0$ implies $u^{[k]}(x, y) \equiv 0$. It will be convenient to use several properties of solutions that follow from the general solution of the EPD equation. These general solutions were known to Darboux [4], except for the case $k=-(2n-1)$, $n=1, 2, \cdots$. We use the E. K. Blum [2] representation of the general solutions.

The recursion

$$u_y^{[k]}(x, y) = y u^{[k+2]}(x, y) \tag{1.4}$$

plays a basic role in our uniqueness considerations. This relation and the relation (1.3) are still valid even where $x$ represents variables $x_1$, $x_2, \cdots, x_n$ and $u^{[k]}$ is a solution of

$$\Delta_x u = u_{yy} + \frac{k}{y} u_y \,.$$

In their $n$-dimensional form both recursions are due to A. Weinstein, but in the two-dimensional form used here the recursion (1.3) was known to Darboux. In place of (1.4) Darboux uses a relation which in our notation is

$$k u_y^{[k]}(x, y) = y u^{[k+2]}(x, y)$$

and which therefore does not admit an inversion for $k=0$. Certainly the discovery and emphasis of the very important role of these recursions in the general theory of the EPD equations is the work of A. Weinstein.

Of course, any uniqueness proof which applies to solutions of (1.1) (1.2) also applies when it is required that

$$u^{[k]}(x, x) = g(x) \,, \qquad u^{[k]}(x, 0) = f(x) \tag{1.5}$$

where $f(x)$ and $g(x) \not\equiv 0$ are given functions. A later paper will be devoted to solution of the problem (1.5), and precise conditions on $f$ and $g$ required for existence of solutions regular on $T$ will be given there.

From the Weinstein solutions it can be seen that the region of determination of $f(x)$ defined for $0 \leqq x \leqq a$ is the infinite strip bounded by the lines $y=x$ and $y=x-a$. The uniqueness question, however, can be restricted to consideration of the characteristic triangle $T$ defined above. That is, for uniqueness one considers only the problem $f(x)=0$. If it follows from this prescription of $f(x)$ that the solution is identically zero in the characteristic triangle, then it is certainly zero on the characteristic $y=-x+a$. But now as the solution has been prescribed to be zero on $y=x$, it can be shown to be zero in the infinite strip by solution of a characteristic problem. The characteristic problem for the two-dimensional EPD equation is classical. It was solved by Riemann [6] in order to obtain the Riemann function for the EPD equation.

**2. Some important properties of solutions.** In this section we shall be concerned with several properties that are derived from the general solutions of the two-dimensional EPD equation for solutions $u^{[k]}(x, y)$, $k<1$, regular on $T$, and such that

$$u^{[k]}(x, x)=u^{[k]}(x, 0)=0 .$$

The general solutions which we use are valid on a characteristic triangle in which the solution has two derivatives in a region $G$ containing that characteristic triangle except for the points of its base. Certainly then the general solutions are valid for functions which are regular on $T$ in the sense described above.

The general solutions for $k$ negative are obtained [2] from repeated application of (1.3) and (1.4) (and certain considerations associated with them) to solutions $u^{[s]}(x, y)$, $0\leqq s<2$. Consider coefficients $a_{rn}$ defined by

$$a_{rn}=\left(-\frac{1}{2}\right)^{n-r}\frac{[r+2(n-r)-1]!}{(n-r)!(r-1)!}\,, \qquad a_{nn}=1 . \tag{2.1}$$

The general solutions are:

*Case* 1. $0<k<1$

$$\begin{aligned} u^{[k]}(x, y)= & -2^{k-1}y^{1-k}\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-k/2}\,d\alpha \\ & -2^{-k+1}\int_1^{-1}\psi[x+\alpha y](1-\alpha^2)^{k/2-1}\,d\alpha . \end{aligned} \tag{2.2}$$

For solutions which are regular on $T$, the arbitrary functions $\phi$ and $\psi$ have one continuous derivative on the closed interval $[0, a]$.

*Case* 2. $k<0$, $k$ non-integral,

$$\begin{aligned} u^{[k]}(x, y)= & -2^{s-1}\sum_{r=1}^{n} a_{rn}r!\sum_{j=0}^{r}\frac{(1-s)(-s)\cdots(-s-r-j)}{j!\,(r-j)!}\,y^j \\ & \times\int_1^{-1}\phi^{(j)}[x+\alpha y](1-\alpha^2)^{-s/2}\alpha^j\,d\alpha \\ & -2^{-s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\psi^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha \end{aligned} \tag{2.3}$$

where $0<s<2$, $s\neq 1$ and $n$ is an integer given by $2-k=2n+s$. Here if $u^{[k]}$ is regular on $T$, $\phi$ and $\psi$ have $(n+1)$ continuous derivatives on $[0, a]$.

*Case* 3(a). $k=0$, $u^{[0]}(x, y)=F(x+y)+G(x-y)$.

*Case* 3(b). $k=-2n$, $n=1, 2, \cdots$,

$$u^{[k]}(x, y)=\sum_{r=1}^{n+1} a_{r,n+1}y^{r-1}[F^{(r)}(x+y)+(-1)^r G^{(r)}(x-y)] \ . \tag{2.4}$$

Here if $u^{[k]}$ is regular on $T$, $F$ and $G$ have $(n+3)$ continuous derivatives on $[0, a]$.

*Case* 4. $k=-(2n+1)$, $n=0, 1, 2, \cdots$

$$u^{[k]}(x, y)=\sum_{r=1}^{n-1} a_{r,n+1}y^r \frac{\partial^r u^{[1]}}{\partial y^r} \tag{2.5}$$

where

$$\begin{aligned} u^{[1]}(x, y)=2\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-1/2}\,d\alpha \\ +2\int_1^{-1}\psi[x+\alpha y](1-\alpha^2)^{-1/2}\log[y(1-\alpha^2)(1/2)^2]\,d\alpha \ . \end{aligned} \tag{2.6}$$

Here if $u^{[k]}$ is regular on $T$, $\phi$ and $\psi$ have $n+2$ continuous derivatives on $[0, a]$.

LEMMA 1. *If* $u^{[k]}(x, 0)=0$ *and* $u^{[k]}(x, y)$ *is regular on* $T$, *then*

*Case* 1. $\psi\equiv 0$

*Case* 2. $\phi\equiv 0$

*Case* 3(a). $F\equiv -G$

(b). $F'\equiv -G'$

*Case* 4. $\psi\equiv 0$.

*Proof*. There results were known to Blum [2]. The hypothesis is, as stated above, intended in the sense that $\lim_{y\to 0} u^{[k]}(x, y)=0$. In Case 2, for example, let $y\to 0$. As $s+r-1$ is always positive, since $r\geqq 1$, we have

$$u^{[k]}(x,0)=\lim_{y\to 0} u^{[k]}(x,y)$$
$$=-2^{s-1}\sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r)\phi(x)\int_1^{-1}(1-\alpha^2)^{-s/2}\,d\alpha$$

or

$$\phi(x)=\frac{u^{[k]}(x,0)}{-2^{s-1}\sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r)\int_1^{-1}(1-\alpha^2)^{-s/2}\,d\alpha}\,. \tag{2.7}$$

But the integral cannot be zero since it is a symmetric integral of an even function—in fact, the integral is

$$\frac{-\Gamma(1/2)\Gamma(1-s/2)}{(3/2-s/2)}$$

and it can be shown that $\sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r)\neq 0$. Thus $u^{[k]}(x,0)=0$ implies $\phi(x)=0$ as stated.

Consider now Case 4. Here we have

$$\psi(x)=\frac{u^{[k]}(x,0)}{\pi\left[a_{1,n+1}+\sum_{r=2}^{n+1} a_{r,n+1}(r-1)(-1)^{r+1}\right]} \tag{2.8}$$

and $\psi\equiv 0$ if $u^{[k]}(x,0)=0$. For example, take $k=-1$,

$$u^{[-1]}(x,y)=2y\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{1/2}\alpha\,d\alpha$$
$$+2\int_1^{-1}\psi'[x+\alpha y]y\log[y(1-\alpha^2)1/2^2](1-\alpha^2)^{-1/2}\alpha\,d\alpha$$
$$+2y\int_1^{-1}\psi[x+\alpha y]\left(\frac{1}{y}\right)(1-\alpha^2)^{-1/2}\,d\alpha\,.$$

On letting $y\to 0$ we notice that $y\log c\ y\to 0$ for any constant $c$ so that

$$0=u^{(-1)}(x,0)=2\int_1^{-1}\psi[x](1-\alpha^2)^{-1/2}\,d\alpha$$

and again, since

$$\int_1^{-1}(1-\alpha^2)^{-1/2}\,d\alpha\neq 0\,,\qquad \psi[x]=0\,.$$

Case 1 and Case 3 are now entirely trivial.

LEMMA 2. *For $k<0$, if $u^{[k]}$ is regular on $T$ and $u^{[k]}(x,0)$ exists, then*

$$u_y^{[k]}(x, 0)=\lim_{y\to 0} u_y^{[k]}(x, y)=0 .$$

That is, for $k<0$, letting $u^{[k]}(x, 0)=f(x)$, given, the function $u^{[k]}(x, y)$ is a (non-unique) solution of the singular Cauchy problem.[1] This is the main result of Blum [2]; one arbitrary function is determined as seen in Lemma 1 by specification of $f(x)$, the other is left free so that the general solutions then yield the class of all solutions of the Cauchy problem for $k<0$. For $k>0$ solutions of the singular Cauchy problem are unique. One now sees that the solution of the generalized radiation problem for $k<0$ is a solution of the Cauchy problem with one additional condition. It is this condition which must provide uniqueness. The proof of the lemma consists simply in deriving the general solutions with respect to $y$ and examining limits as $y\to 0$. It should be noted that in deriving the general solutions of the EPD equation nothing is said about the behavior of $u_y$ on the line $y=0$. Also, it should be emphasized that one cannot simply look at the term $\frac{k}{y}u_y$ of the EPD equation and conclude the above immediately; for $k\geqq 0$, $u_y^{[k]}(x, 0)$ is not necessarily zero.

Lemma 2 is true for any $u^{[k]}$ regular on $T$ such that $u^{[k]}(x, 0)$ exists, but the problem of uniqueness involves only $u^{[k]}(x, 0)\equiv 0$, and in this case a more general result, valid for $k<-1$ but used here only for $k\leqq -2$, is obtained. In [8] the existence of certain derivatives of $u^{[k]}$ on $y=0$ was (tacitly) assumed. Lemma 3 allows us, for unicity only, to avoid any such assumption.

LEMMA 3. *Let $u^{[k]}(x, y)$, $k<-1$, be any solution of the EPD equation regular on $T$. Then $u^{[k]}(x, 0)\equiv 0$ implies*

$$\lim_{y\to 0}\frac{u_y^{[k]}(x, y)}{y}\equiv 0 .$$

*For $-1<k<0$, a counterexample is $u^{[k]}(x, y)=y^{1-k}$.*

*Proof.* We must again consider separately each of the general solutions. To avoid extensive manipulations a sample case only is presented; $k$ non-integral, $-2<k<-1$.

By Lemma 1 all solutions are of the form

$$u^{[k]}(x, y)=-2^{-s+1}y^s\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\, d\alpha$$

with $1<s<2$. We have

[1] For the singular Cauchy problem, specify $f(x)$ and require

$$u^{[k]}(x, 0)=f(x),\quad u_y^{[k]}(x, 0)=0 .$$

$$\lim_{y\to 0}\frac{u_y^{[k]}(x,y)}{y}=\lim_{y\to 0}-2^{-s+1}\left\{sy^{s-2}\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\, d\alpha\right.$$

$$\left.+y^{s-1}\int_1^{-1}\psi''[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^2\, d\alpha\right\}$$

$$=-2^{-s+1}s\lim_{y\to 0}y^{s-2}\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\, d\alpha\ .$$

But as $y\to 0$, the integral factor goes to

$$\psi'[x]\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha\, d\alpha\equiv 0$$

(the integrand is odd), and the L'Hospital rule is applicable. We obtain

$$\lim_{y\to 0}\frac{u_y^{[k]}(x,y)}{y}=-2^{-s+1}\frac{s}{2-s}\lim_{y\to 0}y^{s-1}\int_1^{-1}\psi''[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^2\, d\alpha\equiv 0\ .$$

LEMMA 4. *If $u^{[k]}$ is regular on $T$, in the general solution for $u^{[k]}$ we may without loss of generality take*

*Case* 2. $\psi'(0)=\psi''(0)=\cdots=\psi^{(n)}(0)=0$

*Case* 3. $F'(0)=F''(0)=\cdots=F^{(n+1)}(0)=0$

or $G'(0)=G''(0)=\cdots=G^{(n+1)}(0)=0$

*Case* 4. $\phi'(0)=\phi''(0)=\cdots=\phi^{(n+1)}(0)=0$ .

The importance of this lemma is that it is essential in the proof of Lemma 5 where these results are used in repeated application of the rule of L'Hospital. Lemma 5 in turn is essential to an important induction used in the uniqueness proof of § 4. For solutions with two derivatives inside $T$ only, the lemma can be extended by replacing the evaluation of $\psi$, $\phi$, $F$, and $G$ at 0 by evaluation at $c>0$ and considering solutions regular on a triangle $T_c$ contained in $T$.

*Proof*. *Case* 2. Let the function $\psi_*^{(1)}(z)$ be defined by

$$\psi_*^{(1)}(z)=\psi^{(1)}(z)-\psi^{(1)}(0)-\psi^{(2)}(0)z-\frac{\psi^{(3)}(0)}{2!}z^2-\cdots-\frac{\psi^{(n)}(0)}{(n-1)!}z^{n-1}\ . \tag{2.14}$$

Of course, $\psi_*^{(1)}(0)=\psi_*^{(2)}(0)=\cdots=\psi_*^{(n)}(0)=0$, and we show that $\psi_*^{(r)}(z)$ can replace $\psi^{(r)}(z)$, $r=1,\cdots,n$, in equation (2.3). Differentiating (2.14) $r-1$ times we obtain

$$\psi^{(r)}(z)=\psi_*^{(r)}(z)+\sum_{m=r}^{n}\frac{\psi^{(m)}(0)}{(m-r)!}z^{m-r}\ ,$$

and using binomial expansion we have

$$\psi^{(r)}(x+\alpha y)=\psi_*^{(r)}(x+\alpha y)+\sum_{m=r}^{n}\sum_{j=0}^{m-r}\frac{\psi^{(m)}(0)}{(m-r)!}\left\{\frac{(m-r)!\,x^{m-r-j}y^j\alpha^j}{j!(m-j-r)!}\right\}. \tag{2.15}$$

Then using (2.15), we may rewrite equation (2.3) as

$$\begin{aligned} u^{[k]}(x, y) = &-2^{s-1}\sum\cdots\sum\int_1^{-1}\phi\cdots \\ &-2^{s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\psi_*^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha \\ &-2^{s+1}y^s\left\{\sum_{r=1}^{n} a_{rn}\sum_{m=r}^{n}\sum_{j=0}^{m-r}\frac{\psi^{(m)}(0)}{j!(m-j-r)!}x^{m-r-j}y^{j+r-1}\right. \\ &\left.\times\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{r+j}\,d\alpha\right\} \end{aligned} \tag{2.16}$$

so that our lemma will be proved when we have shown that the last group of terms sum to zero for all $x$ and $y$. In this group for terms where $r+j$ is odd, the integral factor vanishes. We prove that the indicated brackets is zero for each $r+j$ even. Reordering terms, the brackets in (2.16) becomes

$$\begin{aligned} \sum_{m=r}^{n}\psi^{(m)}(0)&\left\{\sum_{r=1}^{n}\sum_{j=0}^{m-r} a_{rn}\frac{1}{j!(m-j-r)!}x^{m-(r+j)}y^{(r+j)-1}\right. \\ &\left.\times\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{r+j}\,d\alpha\right\} \end{aligned} \tag{2.17}$$

and it will be possible to show that the new brackets, denoted by $S(n)$ is zero for all $n$ such that $r\leqq m\leqq n$. Letting $2\nu=r+j$, for $0<j<m-r$ we have $r<2\nu<m$ or since the least value of $r$ is 1, $1\leqq\nu\leqq\left[\frac{m}{2}\right]$.[2] Then

$$S(n)=\sum_{\nu=1}^{[\frac{m}{2}]}\frac{1}{(m-2\nu)!}x^{m-2\nu}y^{2\nu-1}\left\{\sum_{r=1}^{2\nu}\frac{a_{rn}}{(2\nu-r)!}\right\}\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{2\nu}\,d\alpha \tag{2.18}$$

and we only need show that

$$\sigma=\sum_{r=1}^{2\nu}\frac{a_{rn}}{(2\nu-r)!} \tag{2.19}$$

is zero for all $n$ and $\nu$.

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[2] $\left[\frac{m}{2}\right]$ is the Legendre symbol—the greatest integer less than or equal to $\frac{m}{2}$.

From (2.19) and (2.1), then

$$\sigma=\sum_{r=1}^{2\nu}(-1/2)^{n-r}\frac{(2n-r-1)!}{(n-r)!\,(r-1)!\,(2\nu-r)!}$$

and this is the quantity which is our present concern.

Consider the polynomial

$$P(z)=\sum_{r=1}^{2\nu}\frac{(-1/2)^{2n-r-1}(1-z)^{2n-r-1}}{(r-1)!\,(2\nu-r)!}=\sum_{l=1}^{2(n-1)}b_l z^l\ . \tag{2.21}$$

Then

$$P^{(n-1)}(z)=\sum_{r=1}^{2\nu}(-1/2)^{2n-r-1}\frac{(2n-r-1)!\,(-1)^{n-1}}{(n-r)!\,(r-1)!\,(2\nu-r)!}(1-z)^{n-r}$$

and

$$P^{(n-1)}(0)=(-1)^{n-1}(-1/2)^{n-1}\sum_{r=1}^{2\nu}(-1/2)^{n-r}\frac{(2n-r-1)!}{(n-r)!\,(r-1)!\,(2\nu-r)!}$$

$$=(-1)^{n-1}(-1/2)^{n-1}\sigma\ .$$

Thus $P(z)$ has been chosen so that it will be sufficient to demonstrate that the coefficient $b_{n-1}$ is zero. Let us rewrite $P(z)$ as follows

$$P(z)=\sum_{r=1}^{2\nu}\frac{(1/2z-1/2)^{2n-r-1}}{(r-1)!\,(2\nu-1)!}=\frac{1}{(2\nu-1)!}\sum_{r=1}^{2\nu}\binom{2\nu-1}{r-1}(1/2z-1/2)^{2n-r-1}$$

$$=\frac{(1/2z-1/2)^{2n-2}}{(2\nu-1)!}\left(1+\frac{1}{1/2z-1/2}\right)^{2\nu-1}$$

$$=\frac{1}{(2\nu-1)!}(1/2z-1/2)^{2(n-\nu)-1}(1/2z+1/2)^{2\nu-1}$$

$$=c(z-1)^{2(n-\nu)-1}(z+1)^{2\nu-1}\,,\qquad c=\frac{1}{(2\nu-1)!}(1/2)^{2(n-1)}$$

$$=cz^{n-\nu-1/2}\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^{2(n-\nu)-1}\left(\frac{1}{\sqrt{z}}+\sqrt{z}\right)^{2\nu-1}z^{\nu-1/2}$$

$$=cz^{n-1}Q(z)$$

where

$$Q(z)=\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^{2(n-\nu)-1}\left(\frac{1}{\sqrt{z}}+\sqrt{z}\right)^{2\nu-1}\ .$$

We note that $Q(z)=-Q(1/z)$, and that, therefore

$$cQ(z)=\frac{P(z)}{z^{n-1}}=-P\left(\frac{1}{z}\right)z^{n-1}=-cQ\left(\frac{1}{z}\right)$$

or $P(z)\equiv -z^{2(n-1)}P(1/z)$. Thus

$$\sum_{l=0}^{2(n-1)} b_l z^l \equiv -\sum_{l=0}^{2(n-1)} b_l z^{2(n-1)-l} = -\sum_{l=0}^{2(n-1)} b_{2(n-1)-l} z^l$$

and $b_l+b_{2(n-1)-l}=0$. Putting $l=n-1$, the required result $b_{n-1}=0$ is obtained.

It is noted that the coefficients $a_{rn}$ of the general solutions do not arise from consideration of any polynomials.

*Case* 3(b). It remains only to show that this treatment reduces after a certain point to that of Case 2.

Let

$$F_*^{(1)}(x)=F^{(1)}(x)-G^{(1)}(0)-xG^{(2)}(0)-\frac{x^2}{2!}G^{(3)}(0)-\cdots-\frac{x^n}{n!}G^{(n+1)}(0)$$

and

$$G_*^{(1)}(x)=G^{(1)}(x)-G^{(1)}(0)-xG^{(2)}(0)-\frac{x^2}{2!}G^{(3)}(0)-\cdots-\frac{x^n}{n!}G^{(n+1)}(0)\ .$$

Then $F_*^{(1)}$ and $G_*^{(1)}$ have the required number of continuous derivatives and $G_*^{(0)}(0)=G_*^{(2)}(0)=\cdots=G_*^{(n+1)}(0)=0$. (Of course, if we subtracted the "Taylor part" of $F^{(1)}(x)$ from $F^{(1)}$ and $G^{(1)}$ we would find that $F_*^{(1)}(0)=F_*^{(2)}(0)=\cdots=F_*^{(n+1)}(0)=0$.)

$$F^{(r)}(x)=F_*^{(r)}(x)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}x^{m-r}$$

and

$$G^{(r)}(x)=G_*^{(r)}(x)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}x^{m-r}\ .$$

From (2.4)

$$u^{[-2n]}=\sum_{r=1}^{n+1}a_{r,n+1}y^{r-1}\Bigg[F_*^{(r)}(x+y)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}(x+y)^{m-r}$$

$$+(-1)^r G_*^{(r)}(x-y)+(-1)^r\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}(x-y)^{m-r}$$

$$=\sum_{r=1}^{n+1}a_{r,n+1}y^{r-1}[F_*^{(r)}(x+y)+(-1)^r G_*^{(r)}(x-y)]-\sum$$

where

$$\sum = \sum_{r=1}^{n+1} a_{r,n+1} y^{r-1} \Big[ \sum_{m=1}^{n+1} \frac{G^{(m)}(0)}{(m-r)!} \sum_{j=1}^{m-r} \Big( \frac{(m-r)!}{j!\,(m-j-r)!} + \frac{(-1)^r (m-r)!\,(-1)^j}{j!\,(m-j-r)!} \Big) \times x^{m-r-j} y^j \Big]$$

and, of course, we must show that $\sum$ is zero. We have

$$\sum = \sum_{m=r}^{n+1} G^{(m)}(0) \Big\{ \sum_{r=1}^{n+1} \sum_{j=0}^{m-r} a_{r,n+1} \frac{1}{j!\,(m-j-r)!} x^{m-(r+j)} y^{(r+j)-1} \Big\} \times (1+(-1)^{r+j}) . \tag{2.22}$$

But the expression in brackets in (2.22) is exactly that of (2.17) with $n$ replaced by $(n+1)$, and the factor $1+(-1)^{r+j}$ plays exactly the role of

$$\int_1^{-1} (1-\alpha^2)^{\frac{s}{2}-1} \alpha^{r+j}\, d\alpha ,$$

each being zero for $r+j$ odd.

LEMMA 5. *If $k<0$ and $u^{[k]}(x, y)$ is a solution of the* EPD *equation regular on $T$ and such that*

$$u^{[k]}(x, x)=0 ,$$

*then*

(i) $u_y^{[k]}(x, x)=Bx^{-k/2}$ , $\quad$ *$B$ constant*

(ii) $u^{[k]}(x, 0)\equiv 0 \Longrightarrow B=0$ .

That the solution be regular on $T$ implies that all second derivatives exist on the line $y=x$ and that the EPD equation be satisfied there.

*Proof.*

(i) On $y=x$ the EPD equation may be written, using $x$ as a parameter,

$$\frac{d}{dx}(u_x^{[k]}(x, x)-u_y^{[k]}(x, x))=\frac{k}{x}u_y^{[k]}(x, x) . \tag{2.23}$$

Differentiating $u^{[k]}(x, y)$ on the line $y=x$, we have

$$0=u_x^{[k]}(x, x)+u_y^{[k]}(x, x) \tag{2.24}$$

so that (2.23) may be rewritten

$$\frac{d}{dx}(u_y^{[k]}(x, x)) = -\frac{k}{2x}u_y^{[k]}(x, x)$$

and the first part of the lemma follows. This elementary procedure is basic in our problem and similar techniques will be used often.
(ii) To demonstrate the second part of the lemma we note that since $u^{[k]}(x, y)$ has been assumed to be regular on $T$ the general solutions apply on the line $y=x$, and from Lemma 1 the condition $u^{[k]}(x, 0)$ gives the general solutions a simplified form. Thus for Case 2, $k$ non-integral, noting that $k/2=-s/2+1-n$, we have

$$(2.25)\qquad \begin{aligned} B&=x^{k/2}u_y^{[k]}(x, x)\\ &=-2^{-s+1}\sum_{r=1}^{n}a_{rn}\left\{(s+r+1)x^{s/2+r-n-1}\int_1^{-1}\psi^{(r)}[(1+\alpha)x](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha\right.\\ &\quad\left.+x^{s/2+r-n}\int_1^{-1}\psi^{(r+1)}[(1+\alpha)x](1-\alpha^2)^{s/2-1}\alpha^{r+1}\,d\alpha\right\}. \end{aligned}$$

We can now conclude that $B=0$ by taking the limit of (2.25) as $x\to 0$. To do this we apply the rule of L'Hospital $(n+1-r)$ times to the $r^{\text{th}}$ term in the first set of terms and $(n-r)$ times to the $r^{\text{th}}$ term in the second set of terms. The purpose in presenting Lemma 4 was to justify this procedure.

The Cases 1 and 3(a) are irrelevant to this lemma as we require $k$ to be negative. Treatment of Case 4 is precisely analogous to Case 2 except that here, by Lemma 1, $\psi(x)\equiv 0$, and (2.25) appears in terms of integrals of $\phi$ instead of $\psi$ and with a slightly different kernel.

Consider Case 3(b), $k=-2n$, $n=1, 2, \cdots$. The analogue of (2.25) is

$$(2.26)\qquad \begin{aligned} B=x^{k/2}u_y^{[k]}(x, x)&=\sum_{r=1}^{n+1}a_{r,n+1}x^{r-n-1}[F^{(r+1)}(2x)+(-1)^{r+1}F^{(r)}(0)]\\ &\quad+\sum_{r=1}^{n+1}a_{r,n+1}(r-1)x^{r-n-2}[F^{(r)}(2x)+(-1)^{r}F^{(r)}(0)]. \end{aligned}$$

We again conclude that $B=0$ by taking the limit of (2.26) as $x\to 0$, applying the rule of L'Hospital $(n+1-r)$ times to the $r^{\text{th}}$ term of the first set of terms and $(n+2-r)$ times to the $r^{\text{th}}$ term of the second set of terms. For this purpose an immediate extension of Lemma 4 is used; that is, without loss of generality, in the expression from the general solutions for $u_y^{[k]}$, we may assume that

$$F^{(1)}(0)=F^{(2)}(0)=\cdots=F^{(n+2)}(0)=0\ ;$$

it is only $u_y^{[k]}$, not $u^{[k]}$ itself, which enters into (2.26).

Since the coefficients of the EPD equation do not depend on $x$, it is evident that if a solution $u^{[k]}(x, y)$ has three continuous derivatives in a region, then $u_x^{[k]}(x, y)$ is a solution with at least two continuous derivatives in that region. This is the motivation of the following lemma which is essential to the induction of § 4. A solution which has *three* continuous derivatives in a triangle $G$ the interior of which contains the triangle $T$ and its sides except for the base line, will be said to be *regular plus one* on $T$.

LEMMA 6. *Let $U^{[k]}(x, y)$ be any solution regular on $T$ such that $U^{[k]}(x, 0)=0$. There exists a solution $u^{[k]}(x, y)$ regular plus one on $T$ such that*

$$U^{[k]}(x, y)=u_x^{[k]}(x, y)$$

*and such that*

$$u^{[k]}(x, 0)=0 .$$

*Proof.* This lemma is obtained in a trivial manner from the general solutions using Lemma 1. For Case 2,

$$\begin{aligned} U^{[k]}(x, y) &= -2^{s+1}\sum_{r=1}^{n} a_{rn} y^{s+r-1}\int_1^{-1}\psi^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\, d\alpha \\ &= \frac{\partial}{\partial x}\left[-2^{s+1}\sum_{r=1}^{n} a_{rn} y^{s+r-1}\int_1^{-1}\psi^{(r-1)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\, d\alpha\right], \end{aligned}$$

or for Case 3(b)

$$\begin{aligned} U^{[k]}(x, y) &= \sum_{r=1}^{n+1} a_{r,n+1} y^{r-1}\{F^{(r)}(x+y)+(-1)^{r+1}F^{(r)}(x-y)\} \\ &= \frac{\partial}{\partial x}\sum_{r=1}^{n+1}\left[a_{r,n+1} y^{r-1}\{F^{(r-1)}(x+y)+(-1)^{r+1}F^{(r)}(x-y)\}\right]. \end{aligned}$$

In both cases the quantity in square brackets is a solution of the EPD equation which is regular plus one on $T$ since the arbitrary functions $\psi$ and $F$ have $(n+1)$ and $(n+3)$ continuous derivatives respectively. Of course, $u^{[k]}(x, 0)=0$ as required. Again the treatment of Case 4 is analogous to Case 2.

## 3. Uniqueness for $-2<k<1$.

In this section we show that when $-2<k<1$

$$\left.\begin{cases} u^{[k]}(x, y) \text{ regular on } T \\ \lim\limits_{y\to 0} u^{[k]}(x, y)\equiv 0 \\ u^{[k]}(x, x)\equiv 0 \end{cases}\right\} \Longrightarrow u^{[k]}(x, y)\equiv 0 .$$

The argument is divided into the cases $0<k<1$, $k=0$, and $-2<k<0$. In § 4 it will be shown that uniqueness for all $k\leqq 0$ follows from the uniqueness for $-2<k\leqq 0$.

The $k=0$ case is entirely trivial. We have

$$u^{[0]}(x, y)=F(x+y)+G(x-y) .$$

The boundary conditions yield

$$0=u^{[0]}(x, x)=F(2x)+G(0) \quad \text{or} \quad F(x)=-G(0)$$

$$0=u^{[0]}(x, 0)=F(x)+G(x) \quad \text{or} \quad G(x)=-F(x)=G(0)$$

so that

$$u^{[0]}(x, y)=-G(0)+G(0)\equiv 0 .$$

Consider now $0<k<1$. We have from (2.2) by Lemma 1

$$u^{[k]}(x, y)=-2^{k-1}y^{1-k}\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-k/2}\,d\alpha \tag{3.1}$$

and

$$0=u^{[k]}(x, x)=-2^{k-1}x^{1-k}\int_1^{-1}\phi[x(1+\alpha)](1-\alpha^2)^{-k/2}\,d\alpha .$$

Let $\sigma=x(1+\alpha)$. Then

$$0=2^{k-1}\int_0^{2x}\sigma^{-k/2}\phi[\sigma](\sigma-2x)^{-k/2}\,d\sigma$$

or

$$0=I^{1-k/2}[(2x)^{-k/2}\phi[2x]]$$

where $I^{\alpha}f[x]$ is the Riemann-Liouville integral of $f$ to the order $\alpha$ (see e.g. [8]). Then $(2x)^{-k/2}\phi[2x]=0$ and $\phi[2x]\equiv 0$. Of course, then, from (3.1)

$$u^{[k]}(x, y)\equiv 0 .$$

The case $-2<k<0$ is similar. We treat only the case $k\neq -1$ because using Lemma 1, the treatments of $k=-1$ and $k$ fractional become entirely analogous. We have

$$u^{[k]}(x, y)=-2^{s+1}y^{s}\int_1^{-1}\psi'[x-\alpha y](1-\alpha^2)^{s/2-1}\alpha\,d\alpha \tag{3.2}$$

where $0<s<2$ and $\psi$ has two continuous derivatives on $[0, a]$. Then

$$0=u^{[k]}(x, x)=-2^{s+1}x^{s}\int_1^{-1}\psi'[x(1+\alpha)](1-\alpha^2)^{s/2-1}\alpha\,d\alpha$$

or, integrating once by parts,

$$0=\frac{x^{s+1}}{s}\int_1^{-1}\psi''[x(1+\alpha)](1-\alpha^2)^{s/2}\,d\alpha$$

for all $x$. As above let $\sigma=x(1+\alpha)$ and obtain

$$0=\frac{1}{s}\int_0^{2x}\sigma^{s/2}\psi''[\sigma](2x-\sigma)^{s/2}\,d\sigma$$

or

$$0=I^{s/2+1}[(2x)^{s/2}\psi''[2x]]\ .$$

Again

$$(2x)^{s/2}\psi''[2x]=0$$

or *for* $x\neq 0$, $\psi''[2x]=0$ and $\psi'[2x]\equiv\text{constant}=K$. But with $\psi'=K$, (3.2) becomes

$$u^{[k]}(x,y)=-2^{s+1}y^sK\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha\,d\alpha\equiv 0$$

since the integrand is odd.

**4. An induction, uniqueness for all $k<1$.** Uniqueness for $-2<k\leqq 0$ as proven in the last section together with the lemmas of § 2 are used here to establish uniqueness for all $k\leqq 0$, the case $0<k<1$ having already been considered in § 3.

Define (negative) numbers $k_n$ recursively by the relation $k_{n+1}=k_n-2$, $n=1,2,\cdots$ where $-2<k_1<0$; that is, such that $-2n<k_n<-2(n-1)$. We apply a complete induction. In § 3 it was shown that for $n=1$ (that is, for any $k$ which is a $k_1$) $u^{[k]}(x,0)\equiv u^{[k]}(x,x)\equiv 0$ implies $u^{[k]}(x,y)\equiv 0$ provided $u^{[k]}$ is regular on $T$. It remains only to show that if this statement is true for $k=k_n$, then it is true for $k=k_{n+1}=k_n-2$.

*Induction assumption.* $u^{[k_n]}(x,0)\equiv u^{[k_n]}(x,x)\equiv 0$ implies $u^{[k_n]}(x,y)\equiv 0$ provided $u^{[k_n]}$ is *regular* on $T$.

(a) Given $u^{[k_{n+1}]}(x,y)$ *regular plus one*[3] on $T$ and such that

$$u^{[k_{n+1}]}(x,0)=u^{[k_{n+1}]}(x,x)\equiv 0$$

we generate a solution $u^{[k_n]}(x,y)$ of the EPD equation which is *regular* on $T$ by the recursion

$$yu^{[k_n]}(x,y)=u_y^{[k_{n+1}]}(x,y)\ . \tag{4.1}$$

[3] See Lemma 6.

Now by Lemma 5, $u_y^{[k_{n+1}]}(x, x)=0$ so that

$$u^{[k_n]}(x, x)\equiv 0 \ . \tag{4.2}$$

Further from (4.1) by Lemma 3

$$u^{[k_n]}(x, 0)=\lim_{y\to 0} u^{[k_n]}(x, y)=\lim_{y\to 0}\frac{u_y^{[k_{n+1}]}(x, y)}{y}\equiv 0 \ . \tag{4.3}$$

(b) Now the induction assumption together with (4.2) and (4.3) imply that $u^{[k_n]}(x, y)\equiv 0$. But then by (4.1)

$$u_y^{[k_{n+1}]}(x, y)\equiv 0$$

or

$$u^{[k_{n+1}]}(x, y)=F(x) \qquad \text{for all y.}$$

However, $F(x)$ may be evaluated by setting $y$ equal either to zero or $x$ so that

$$F(x)=u^{[k_{n+1}]}(x, 0)=u^{[k_{n+1}]}(x, x)\equiv 0$$

and

$$u^{[k_{n+1}]}(x, y)\equiv 0 \ . \tag{4.4}$$

(c) Consider now $U^{[k_{n+1}]}(x, y)$ *regular* on $T$ and such that

$$U^{[k_{n+1}]}(x, 0)=U^{[k_{n+1}]}(x, x)\equiv 0 \ .$$

By Lemma 6 we can write

$$U^{[k_{n+1}]}(x, y)=u_x^{[k_{n+1}]}(x, y) \tag{4.5}$$

where $u^{[k_{n+1}]}$ is regular plus one on $T$ and

$$u^{[k_{n+1}]}(x, 0)\equiv 0 \ . \tag{4.6}$$

Let us examine the condition $U^{[k_{n+1}]}(x, x)\equiv 0$ or, equivalently, the condition $u_x^{[k_{n+1}]}(x, x)\equiv 0$. On the line $y=x$, the EPD equation may be written

$$\frac{d}{dx}\left(u_x^{[k_{n+1}]}(x, x)-u_y^{[k_{n+1}]}(x, x)\right)=\frac{k_{n+1}}{x}u_y^{[k_{n+1}]}(x, x)$$

and the condition $u_x^{[k_{n+1}]}(x, x)=0$ yields

$$\frac{d}{dx}u_y^{[k_{n+1}]}(x, x)=-\frac{k_{n+1}}{x}u_y^{[k_{n+1}]}(x, x)$$

or

$$u_y^{[k_{n+1}]}(x, x) = Ax^{-k_{n+1}}\,, \qquad A \text{ arbitrary.} \tag{4.7}$$

Differentiating $u^{[k_{n+1}]}(x, y)$ on the line $y=x$ we have

$$u_x^{[k_{n+1}]}(x, x) + u_y^{[k_{n+1}]}(x, x) = \frac{d}{dx}\,u^{[k_{n+1}]}(x, x)$$

and, again since $u_x^{[k_{n+1}]}(x, x)=0$, using (4.7) we have

$$Ax^{-k_{n+1}} = u_y^{[k_{n+1}]}(x, x) = \frac{d}{dx}\,u^{[k_{n+1}]}(x, x)$$

so that

$$u^{[k_{n+1}]}(x, x) = Bx^{1-k_{n+1}} + C\,. \tag{4.8}$$

Here $B$ is arbitrary but $C$ becomes zero since $u^{[k_{n+1}]}(x, 0)\equiv 0$. From parts (a) and (b) above in which the uniqueness of a solution $u^{[k_{n+1}]}$ which is regular plus one on $T$ was established, the unique solution of the boundary value problem (4.6) (4.8) is

$$u^{[k_{n+1}]}(x, y) = By^{1-k_{n+1}}\,.$$

Then by (4.5)

$$U^{[k_{n+1}]}(x, y) \equiv 0$$

and this completes the induction.

The following theorem summarizes the results obtained in §§ 3 and 4.

THEOREM. *For* $-\infty < k < 1$ *there is at most one solution of the* EPD *equation which is regular on* $T$ *and is such that for given functions* $f(x)$ *and* $g(x)$

$$\lim_{y\to 0} u^{[k]}(x, y) = f(x)\,, \qquad u^{[k]}(x, x) = g(x)\,.$$

It should be noted that the uniqueness theorem given in [1] does not apply here for the cases $k<0$ since the EPD equation does not satisfy the relation (A) (5″) of that paper unless $0 \leqq k \leqq 2$

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