T-SETS AND ABSTRACT (L)-SPACES

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1. Introduction. The theory of T-sets and of F_T-functionals was developed [4] in reference to abstract (M)-spaces for application to the characterization of Banach spaces which may be represented as Banach spaces of continuous functions. The purpose of this paper is to discuss their use in reference to abstract (L)-spaces [3] for application to the representation of certain Banach spaces as spaces of integrable functions.

A distinction of three types of abstract (L)-spaces is first made and illustrated. Next an extremely simple characterization of the Banach spaces which are susceptible of a semi-ordering under which they become abstract (L)-spaces of the second or third type is established. Then a complete analysis of the role of T-sets and of F_T-functionals in the third and most important type of abstract (L)-space is given. Finally a few remarks are appended relative to T-sets in abstract (L)-spaces of the first type.

2. Preliminary concepts. Let BL be a semi-ordered Banach space which is a linear lattice under its semi-ordering, and in which the collection P of elements $a \geqq 0$ is closed with respect to the norm. Consider, with reference to the subset P of BL, three possible additional requirements:

(I) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$.

(II) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$, and P is a subset of BL maximal with respect to this property.

(III) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$, and if $a \wedge b = 0$, then $\|a-b\| = \|a+b\|$.

A space BL wherein the subset P possesses property III is usually called an abstract (L)-space. If property III obtains in P, then property II also obtains in P with respect to BL. Thus for any $a \in BL$ with $a \notin P$, $a = a^+ - a^-$ with $a^+,\ a^- \in P$, $a^+ \wedge a^- = 0$, while $a^- \neq 0$. Then

$$\begin{aligned}\|a+a^-\| &= \|a^+\| < \|a^+\| + \|a^-\| + \|a^-\| \\ &= \|a^+ + a^-\| + \|a^-\| = \|a^+ - a^-\| + \|a^-\| = \|a\| + \|a^-\|,\end{aligned}$$

so that P is maximal in BL with respect to the stated property. Thus for the subset P of BL, we have III $\Rightarrow$ II $\Rightarrow$ I. It will presently be seen, however, that I does not imply II and that II does not imply III. Hence let BLI denote the space BL under the additional assumption

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that the subset P possesses property I but not property II, and similarly for BLII while BLIII denotes the space BL under the assumption that the subset P possesses property III. It is known [3] that a space BLI, under an easy change to an equivalent norm, becomes a space BLIII, neither the elements of the subset P nor their norms being disturbed in the process. Hence reference will be made to spaces BLI, BLII and BLIII as abstract (L)-spaces of type I, II and III.

Now let B represent an arbitrary Banach space. Let P be a subset of B maximal with respect to the property: for every finite set of elements $(b_1, \cdots, b_n)$ in P,

$$\left\| \sum_{i=1}^{n} b_i \right\| = \sum_{i=1}^{n} \|b_i\| .$$

Such subsets are called [4] T-sets. Each T-set P of B has the properties [4, Lemma 2.1]: if $a,\ b \in P$, then $a+b \in P$: if $\|a+b\| = \|a\| + \|b\|$ for all $a \in P$, then $b \in P$. In view of these properties the T-sets of B may be described as subsets P of B that are closed under addition and, as subsets of B, are maximal with respect to the property: $a,\ b \in P$ implies $\|a+b\| = \|a\| + \|b\|$.

For each such T-set P of B define an associated F_T-functional F_P with $F_P(a) = \inf_{b \in P} \{\|a+b\| - \|b\|\}$ for each element a of B. Each such F_T-functional F_P has the following pertinent properties [4, Lemma 2.2]: $F_P(b) = \|b\|$ if and only if $b \in P$; the functional F_P is linear over the linear extension of P in B.

The fact and the general form of the role played by T-sets in abstract (L)-spaces is clear from the definition of these spaces and from their representation as spaces of integrable functions. Using this guide, the possibilities when a beginning is made not with a space BL but with an arbitrary Banach space B are not difficult to discern.

Let P be a T-set of Banach space B. Define a relation $\overset{P}{\leqq}$ on $B \times B$ with $a \overset{P}{\leqq} b$ exactly when $(b-a) \in P$. Since every T-set is closed under addition and under scalar multiplication by non-negative real scalars, this relation determines a linear semi-ordering for B. Since every T-set is closed under the norm and contains the zero element, the set of elements $a \overset{P}{\geqq} 0$ of B coincides with P and is closed under the norm. Reference will be made to the relation $\overset{P}{\leqq}$ as the canonical semi-ordering induced on B by P.

Of course, B is not necessarily a linear lattice with respect to this semi-ordering. However, the F_T-functional F_P associated with P provides a simple test of the semi-ordering in this respect. First apply the fact that $F_P(a) = \|a\|$ exactly when a is an element of P. This means that, for any element $a \in B$, an element $a^+ \in P$ with $(a^+ - a) \in P$ serves as the element $a \vee 0$ with respect to $\overset{P}{\leqq}$ exactly when $F_P(b-a^+) = ||b-a^+||$ for each $b \in P$ with $(b-a) \in P$. Next apply the fact that F_P is linear on the linear extension of P in B. This means that with b, $a^+ \in P$, $F_P(b-a^+) = ||b|| - ||a^+||$. Note, lastly that $a=b-c$ with b, $c \in P$ is equivalent to having both b and $(b-a)$ in P. This may be summarized.

LATTICE CRITERION *For any element $a \in B$ and T-set $P \subset B$, an element $a^+ \in P$ with $(a^+ - a) \in P$ serves as the element $a \vee 0$ under the canonical semi-ordering induced on B by P exactly when $a=b-c$, b, $c \in P$, always implies $||b-a^+|| = ||b|| - ||a^+||$.*

If B becomes a linear lattice and thus an abstract (L)-space of at least type II under the canonical semi-ordering induced on B by T-set P, the significance of the functional F_P is easily found. Thus, if $a = a^+ - a^-$ with a^+, $a^- \in P$ and defined as usual, then

$$F_P(a) = F_P(a^+) - F_P(a^-) = ||a^+|| - ||a^-|| ,$$

so that in the representation of B as a space of integrable functions, the value $F_P(a)$ equals the value of the integral over the representing space of the function representing the element a.

Finally, if a particular T-set P and the corresponding F_T-functional F_P may be thus employed in representing the space B, surely the other T-sets and F_T-functionals of B are eligible for similar usages, presumably in respect to the measurable subsets of the representing space.

With this outline of the possibilities completed, attention is turned to specific details.

3. **Preliminary examples.** Let L_2 be the set $R \times R$ of all ordered pairs of real numbers. Let L_2 be regarded as a linear lattice using the usual definitions of addition and of scalar multiplication, while $(a, b) \geqq (c, d)$ exactly when $a \geqq c$ and $b \geqq d$ as real numbers. Within L_2 distinguish subsets N_1, N_2 and N_3. In geometric terms, let N_3 be the area about the origin bounded by the pairs of lines $x+y=1$, $x+y=-1$ and $x-y=-1$, $x-y=1$. Let N_2 be the area about the origin bounded by the lines $x+y=1$ and $x+y=-1$ in the first and third quadrants, but by the circle $x^2+y^2=1$ in the second and fourth quadrants. Let N_1 be the area about the origin bounded by the lines $x+y=1$, $x+y=-1$ and by the circle $x^2+y^2=5$.

For each element (x, y) of L_2 define

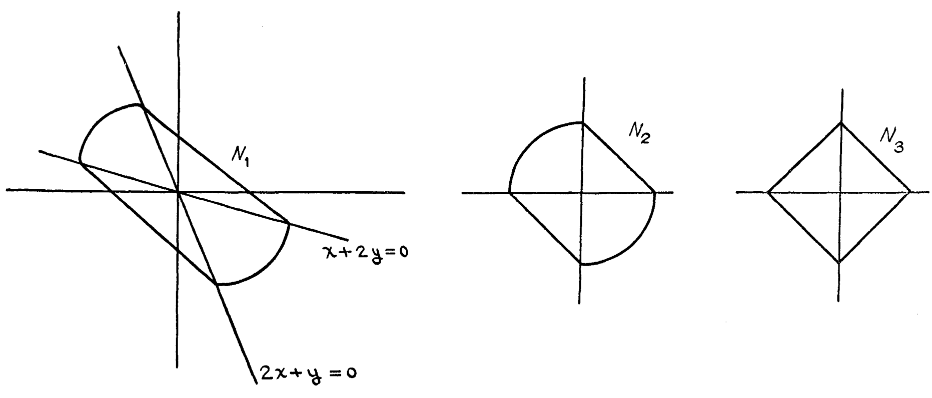
$$||(x, y)||_i = \inf \{a | (x/a, y/a) \in N_i, a > 0\} , \qquad i=1, 2, 3.$$

The third of these norms is familiar: $||(x, y)||_3 = |x| + |y|$ for each element

(x, y) of L_2. The second of these norms was discussed in [3]: $\|(x, y)\|_2 = |x+y| = |x|+|y|$ for elements (x, y) in the first and third quadrants, while $\|(x, y)\|_2 = \sqrt{x^2+y^2}$ for elements (x, y) of L_2 in the second and fourth quadrants. The first of these norms is presumably new: $\|(x, y)\|_1 = (1/\sqrt{5})\cdot\sqrt{x^2+y^2}$ for elements (x, y) on or within the cones formed in the second and fourth quadrants by the intersecting lines $x+2y=0$ and $2x+y=0$, while $\|(x, y)\|_1 = |x+y|$ for all other elements (x, y) of L_2.

Now let BL_2I, BL_2II and BL_2III denote respectively the linear lattice L_2 as under the distinct norms based on the subsets N_1, N_2 and N_3. Then BL_2I is an example of a space BLI wherein the subset P possesses property I but not property II. Specifically, P consists of all points in the first quadrant, while the unique T-set of BL_2I containing P consists of all points on or within the angle determined by the line $x+2y=0$ for $x\geqq 0$ and the line $2x+y=0$ for $y\geqq 0$. Similarly BL_2II is an example of a space BLII while BL_2III is an example of a space BLIII.

The fact that in abstract (L)-spaces of type I the set P is not a T-set complicates the following discussion. The basic relation between T-sets and abstract (L)-spaces of type II and III is treated first. Then, because of its superior importance and because of the perfect application of the T-set theory, the type III situation is discussed in full detail. Last of all, some remarks pertinent to the type I situation will be made.



4. Canonical semi-orderings. This section is devoted to a single theorem.

THEOREM 4.1. *A Banach space B is susceptible of a semi-ordering in respect to which it becomes an abstract (L)-space of type* II *or* III *exactly when it contains a T-set P such that for each $a \in B$ there exist a^+, $a^- \in P$ with the double property that $a = a^+ - a^-$ while $a = b - c$, b, $c \in P$,* *always implies* $\|b-a^+\|=\|b\|-\|a^+\|$, *the semi-ordering then being identical with the canonical semi-ordering induced on* B *by* P. *With this condition satisfied, a space* BLIII *rather than a space* BLII *results exactly when the additional relation* $\|a\|=\|a^+\|+\|a^-\|$ *is satisfied for each* $a\in B$.

Proof. Assume first that B has been endowed with a semi-ordering in respect to which it may be regarded as a space BLII or BLIII. Let P be the subset of B consisting of all elements $a\geqq 0$ under the given semi-ordering. In either case P is a T-set in B: for the case BLII by explicit assumption, and for the case BLIII by assumption and easy conclusion as explained earlier. The canonical semi-ordering induced on B by P obviously duplicates the semi-ordering assumed on B as a space BLII or BLIII.

For any element $a\in B$, let $a^+=a\vee 0$ and $a^-=-(a\wedge 0)$ be as defined under the assumed lattice ordering of B. Then $a=a^+-a^-$ with a^+, $a^-\in P$. Next, if $a=b-c$, b, $c\in P$, then $b\geqq 0$ and $(b-a)\geqq 0$ by definition of P. Hence $b\geqq a$ and $b\geqq a^+$ so that $(b-a^+)\in P$. Then

$$\|b\|=\|(b-a^+)+a^+\|=\|(b-a^+)\|+\|a^+\| \text{ or } \|b-a^+\|=\|b\|-\|a^+\|.$$

Finally, if the assumed ordering is of type III, then for each $a\in B$,

$$\|a\|=\|a^+-a^-\|=\|a^++a^-\|=\|a^+\|+\|a^-\|,$$

since $a^+\wedge a^-=0$.

Conversely, assume that B contains a T-set P as described in the theorem. Let $\overset{P}{\leqq}$ be the canonical semi-ordering induced on B by P. Then, as explained in the Lattice Criterion, the space B with semi-ordering $\overset{P}{\leqq}$ is a space BLII if not a space BLIII, noting that the existence of $a\vee 0$ in the usual sense is the single additional requirement needed in order that $\overset{P}{\leqq}$ be a linear lattice ordering. Finally, if the condition that $\|a\|=\|a^+\|+\|a^-\|$ for each $a\in B$ is satisfied, then, for $a=b-c$ with $b\wedge c=0$,

$$\begin{aligned}\|b-c\|&=\|(b-c)^+\|+\|(b-c)^-\|=\|(b-c)^++(b-c)^-\|\\&=\|b\vee c-c+c\vee b-b\|=\|(b-b\wedge c)+(c-c\wedge b)\|\\&=\|b\|+\|c\|-2\|b\wedge c\|=\|b\|+\|c\|=\|b+c\|.\end{aligned}$$

5. T-sets and F_T-functionals in Type III Spaces. Assume now that Banach space B contains a T-set P_0 such that B is a space BLIII with respect to the canonical semi-ordering $\overset{P_0}{\leqq}$ induced on B by P_0. With P_0 fixed, write $\leqq$ instead of $\overset{P_0}{\leqq}$ and let all lattice notation refer to this fixed lattice ordering of B as BLIII. Certain concepts and results found in [3] will be needed:

(A) For $a, b \in P_0$, $||a-b||=||a||+||b||-2||a\wedge b||$.

(B) An element 1 of P_0, $||1||=1$, is said to be a weak unit in BLIII if $a\wedge 1>0$ or, equivalently, $||a-1||<[||a||+||1||]$, for each $a\in P_0$, $a\neq 0$. It is assumed for the present that BLIII contains a weak unit, the adjustments necessary in the contrary case being indicated later.

(C) Associated with each $a\in P_0$ is a projection function P_a. It is defined by the relation $P_a(b)=\lim_n\{[na]\wedge b\}$ for each $b\in P_0$. If $a\wedge b=0$, then $P_a(c)\wedge P_b(c)=0$ and $P_c(a)\wedge P_c(b)=0$ for each $c\in P_0$. If $a\in P_0$ with $P_a(1)=0$, then $a=0$.

(D) An element e of P_0 is said to be a characteristic element of BLIII if $e\wedge(1-e)=0$. For each $a\in P_0$, $P_a(1)$ is a characteristic element of BLIII. For any $a\in P_0$ and any characteristic element e of BLIII, $a=P_1(a)=P_e(a)+P_{1-e}(a)$ with $P_e(a)\wedge P_{1-e}(a)=0$.

(E) The characteristic elements of BLIII with weak unit form a Boolean algebra, and if $\{e_n\}$ be a sequence of such elements with $e_n \leqq e_{n+1}$, then there is a characteristic element e of BLIII with $e_n\leqq e$ for all n and $\lim_n\{e_n\}=e$ in terms of the norm.

With this information, and with B, P_0, $\leqq$, BLIII, 1 as explained above, two lemmas are in order.

LEMMA 5.1. *For arbitrary T-set P of B the following statements are true*:

(a) *If $a, b\in P$, then $(a^++b^+)\wedge(a^-+b^-)=0$.*

(b) *If $a\in P$, then $a^+\in P$ and $a^-\in -P$.*

(c) *If $0\leqq b\leqq a$ with $a\in P$, then $b\in P$.*

(d) *There exists a unique characteristic element e such that $e\in P$ and $(1-e)\in -P$.*

(e) *For this e and for arbitrary $a\in B$, there exist elements $a_e^+ =[P_e(a^+)-P_{1-e}(a^-)]$ and $a_e^-=[P_e(a^-)-P_{1-e}(a^+)]$ in P with the double property that $a=a_e^+-a_e^-$ with $||a||=||a_e^+||+||a_e^-||$ while $a=b-c$ with b, $c\in P$ implies $||b-a_e^+||=||b||-||a_e^+||$.*

(f) *For arbitrary $a\in B$, $\frac{1}{2}\{F_{P_0}(a)+F_P(a)\}=||P_e(a^+)||-||P_e(a^-)||$.*

LEMMA 5.2. *For arbitrary characteristic element e of BLIII, the subset of all elements $P_e(a^+)-P_{1-e}(a^-)$, $a=a^+-a^-\in BL$III, of B constitute a T-set P of B with $e\in P$ and $(1-e)\in -P$.*

The truth of Lemma 5.2 is easily established in terms of the representation of BLIII as a concrete (L)-space. Because of the routine nature of the proofs for the various parts of Lemma 5.1, attention is restricted to two comments on parts (e) and (f).

First, suppose $a \in P$ with $a \geqq 0$. Then $na \in P$ and thus $[na] \wedge 1 \in P$. Then $e = P_a(1) = \lim_n \{[na] \wedge 1\}$ is in P since every T-set is closed under the norm, and $\|e\| \leqq 1$. Let

$$s = \sup \{\|e\| \,\Big|\, e = P_a(1),\ a \in P,\ a \geqq 0\}\ .$$

Form $\{a_n\}$, $a_n \in P$, $a_n \geqq 0$, and then $\{e_n\}$ with $e_n = P_{a_n}(1)$ such that $\lim_n \{\|e_n\|\} = s$. Then let $a_n^* = a_1 + \cdots + a_n$ and $e_n^* = P_{a_n^*}(1)$ so that $a_n^* \in P$, $e_n^* \in P$ with $e_n \leqq e_n^* \leqq e_{n+1}^*$ and $\lim_n \{\|e_n^*\|\} = s$. Now use (E) to select characteristic element e with $e_n^* \leqq e$ and $\lim_n \{e_n^*\} = e$ under the norm. Since each $e_n^* \in P$, also $e \in P$. Also $\|e\| = s$. But if P is a T-set in B, so also is $-P$. Repeating the above process for $-P$, a second characteristic element is obtained which is disjoint from the e obtained above since P and $-P$ have only the zero element in common. It is then but a small matter to show that this second element is $(1-e)$ and that this e and $(1-e)$ are unique with respect to the stated property.

To prove (f), use is again made of the fact that a F_T-functional is linear on the linear extension in B of the T-set used. Thus for arbitrary $a = a^+ - a^- \in B$, with P_0, P and e as above:

$$F_{P_0}(a) = [\|P_e(a^+)\| - \|P_{1-e}(a^-)\|] - [\|P_e(a^-)\| - \|P_{1-e}(a^+)\|]$$

$$F_P(a) = [\|P_e(a^+)\| + \|-P_{1-e}(a^-)\|] - [\|P_e(a^-)\| + \|-P_{1-e}(a^+)\|]\ .$$

Finally, consider the case wherein B, P_0, $\leqq$, and BLIII are as before, but in which the existence of a weak unit is not assumed. Then, following [3], it may be shown that there exists a collection 1_α, $\alpha \in \mathscr{A}$. $\mathscr{A}$ an index set, of elements 1_α, $\|1_\alpha\| = 1$, of elements of P_0, maximal in BLIII with respect to the property that $1_\alpha \wedge 1_\beta, = 0$ $\alpha \neq \beta$ in $\mathscr{A}$. A characteristic element of BLIII with respect to a particular 1_α, $\alpha \in \mathscr{A}$. is then taken as an element e_α of BLIII such that $e_\alpha \wedge (1_\alpha - e_\alpha) = 0$ in BLIII. Finally, if $\mathscr{E} = \{e_\alpha,\ \alpha \in \mathscr{A}\}$ indicates any definite choice of characteristic elements, one for each 1_α, then Lemmas 5.1 and 5.2 may be restated with each reference to a particular element e replaced by a reference to a particular choice $\mathscr{E}$, and each reference to an element $[P_e(a^\pm) - P_{1-e}(a^\mp)]$ of B replaced by a reference to an element

$$\sum_{\alpha \in \mathscr{A}} [P_{e_\alpha}(a^\pm) - P_{1_\alpha - e_\alpha}(a^\mp)]$$

of B.

These observations are now summarized.

THEOREM 5.3. *Let Banach space B be a space $BL\text{III}_0$ under the canonical semi-ordering induced by a particular T-set P_0 of B. Let* $\{1_\alpha,$

$\alpha \in \mathscr{A}\}$ be a complete set of weak units in $BL\mathrm{III}_0$ and let $\mathscr{E}=\{e_\alpha, \alpha \in \mathscr{A}\}$ denote any chosen family of characteristic elements e_α, one for each 1_α. Then each such family $\mathscr{E}$ determines a unique T-set P of B with $e_\alpha \in P$, $(1_\alpha - e_\alpha) \in -P$. Also every T-set of B is determined in this fashion. Moreover the space B is a space $BL\mathrm{III}$ under the canonical semi-ordering induced by each T-set. Finally, in the concrete representation of $BL\mathrm{III}_0$, for any T-set P of B the function $\frac{1}{2}\{F_{P_0}+F_P\}$ may be interpreted as the result of the associated integration process when restricted to the measurable subsets corresponding to the choice $\{e_\alpha, \alpha \in \mathscr{A}\}$ determining P.

6. Concerning BLI spaces. Let $BL\mathrm{I}$ denote an abstract (L)-space of type I and let B denote the same space regarded simply as a Banach space. Let P be the subset of elements $a \in B$ with $a \geqq 0$ as in $BL\mathrm{I}$. By definition of P as in $BL\mathrm{I}$ and by Zorn's lemma, there is at least one T-set T of B containing the set P. For elements $a \in P \subset T$, $F_T(a) = \|a\|$. But F_T is linear on the linear extension of T in B. Thus for any element $a \in B$, with $a = a^+ - a^-$ with respect to $BL\mathrm{I}$, $F_T(a) = \|a^+\| - \|a^-\|$. However, $F_{T_1}(a) = F_{T_2}(a)$ for each $a \in B$ implies $T_1 \equiv T_2$. Thus n B the T-set T containing P is uniquely determined.

Next let $a \in B$ be any element of T and let $a = a^+ - a^-$ with respect to $BL\mathrm{I}$. Then a, $a^- \in T$ imply $\|a\| + \|a^-\| = \|a + a^-\| = \|a^+\|$ and so $\|a\| = \|a^+\| - \|a^-\|$. Conversely, let $a \in B$ be such that $\|a\| = \|a^+\| - \|a^-\|$. Then $\|a\| = F_T(a)$, so that $a \in T$. Thus T consists exactly of the elements $a \in B$ for which $\|a\| = \|a^+\| - \|a^-\|$.

It has already been seen that for any space B the type II and type III orderings are mutually exclusive, in the sense that all orderings of either type are canonical semi-orderings based on T-sets, and if one such ordering is of type III so is every other. No success has been had thus far in demonstrating a similar exclusiveness between type I orderings on the one hand, and type II or type III orderings on the other.

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