PRINCIPAL SOLUTIONS OF NON-OSCILLATORY SELF-ADJOINT LINEAR DIFFERENTIAL SYSTEMS

WILLIAM T. REID

1. Introduction. In their study of real quadratic functionals

$$\int_a^b [r(x)y'^2+2q(x)yy'+p(x)y^2]dx$$

admitting a singularity at the end-point $x=a$ Morse and Leighton [11] showed that if $x=a$ is not its own first conjugate point then the corresponding Euler differential equation

$$(1.1) \qquad (r(x)y'+q(x)y)'-(q(x)y'+p(x)y)=0, \quad a<x\leqq b\ ,$$

possesses a non-trivial solution $u(x)$ such that $u(x)/y(x)\to 0$ as $x\to a^+$ for each solution $y(x)$ of (1.1) that is independent of $u(x)$. Such a solution $u(x)$ was termed a focal solution belonging to $x=a$ by Morse and Leighton [11], but in a subsequent continuation of the study by Leighton [8] the terminology was changed to principal solution.

If $f(t)$ is a real-valued continuous function on $t_0\leqq t<\infty$ and

$$(1.2) \qquad x''+f(t)x=0\ , \qquad t_0\leqq t<\infty\ ,$$

is non-oscillatory, Hartman and Wintner [4] have termed a non-trivial solution $x(t)$ a principal solution if

$$(1.3) \qquad \int_{t_0}^{\infty}|x(t)|^{-2}dt=\infty\ ,$$

for t_0 greater than the largest zero of $x(t)$, and proved that a non-oscillatory equation (1.2) has a principal solution that is unique to an arbitrary non-zero constant factor; moreover, if $x(t)\not\equiv 0$ is a solution of (1.2) which is not principal then every solution $y(t)$ of (1.2) is of the form $y(t)=Cx(t)+o(|x(t)|)$ as $t\to\infty$, where the constant C is or is not zero according as $y(t)$ is or is not principal. In view of this latter result, for a non-oscillatory equation (1.2) a solution $x(t)$ is principal in the sense of Hartman and Wintner if and only if it is principal in the sense of Morse and Leighton.

Recently Hartman [5] has considered a self-adjoint vector differential equation

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$$(1.4) \qquad (R(t)x')' + F(t)x = 0 , \qquad 0 \leqq t < \infty ,$$

where $R(t)$, $F(t)$ are $n \times n$ matrices which are continuous and hermitian, while $R(t)$ is positive definite on the interval of consideration. An $n \times n$ matrix solution of the corresponding matrix differential equation

$$(1.4') \qquad (R(t)X')' + F(t)X = 0$$

is termed "prepared" by Hartman if $X^*(t)R(t)X'(t)$ is hermitian. Under the assumption that the class Γ of solutions $X = X(t)$ of (1.4′) which are prepared and non-singular on a corresponding interval $a_X < t < \infty$ is non-empty, Hartman showed that in Γ there exists a solution which is principal in the sense that the least proper value $\lambda_X(t)$ of the positive definite hermitian matrix

$$(1.5) \qquad \int_{t_0}^{t} (X^*X)^{-1} ds , \qquad (t_0 \textit{ sufficiently large} ; \ t > t_0),$$

satisfies $\lambda_X(t) \rightarrow \infty$ as $t \rightarrow \infty$, and this principal prepared solution is unique up to multiplication on the right by an arbitrary non-singular constant matrix, while there also exist in Γ solutions that are non-principal in the sense that the greatest proper value $\mu_X(t)$ of (1.5) remains finite as $t \rightarrow \infty$; moreover, if $Y(t)$ and $X(t)$ are matrices of Γ which are principal and non-principal, respectively, then $X^{-1}(t)Y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Hartman's assumption that the above defined class Γ is non-empty is indeed an hypothesis of non-oscillation, since in view of the results of a recent paper of Reid [**13**] the class Γ is non-empty if and only if (1.4) is non-oscillatory for large t in the sense that there exists a t_0 such that if $x(t)$ is a solution of (1.4) satisfying $x(t_1) = 0 = x(t_2)$ with $t_0 < t_1 < t_2$ then $x(t) \equiv 0$.

It is to be noted that Hartman's definition of principal solution for an equation (1.4) which is non-oscillatory for large t has the undesirable feature of limiting the considered matrix solutions of (1.4′) to the class Γ ; indeed, Hartman [**5** ; **§11**] gives an example of a non-prepared solution $X(t)$ of (1.4′) that is non-singular for large t, and such that the least proper value $\lambda_X(t)$ of the corresponding hermitian matrix (1.5) satisfies $\lambda_X(t) \rightarrow \infty$ as $t \rightarrow \infty$. Moreover, as Hartman points out, his classification of principal and non-principal solutions does not present a disjunctive alternative in the class Γ.

For a self-adjoint vector differential equation of somewhat more general type than that considered by Hartman, and which is non-oscillatory for large values of the independent variable, the present paper presents a generalized definition of principal solution that distinguishes such solutions in the class Γ_0 of all matrix solutions which are non-singular for large values of the independent variable. In addition, it is shown that principal solutions possess on Γ_0 certain properties that are extensions of properties established by Hartman for the class Γ. It is to be commented also that the presented determination of a principal solution is by variational methods and is direct in nature, in contrast to the indirect character of the proofs of the existence of a principal solution in the above-cited papers of Hartman, Hartman and Wintner, and Morse and Leighton ; in this connection it is to be remarked that although the existence of a principal solution for (1.1) was established indirectly by Morse and Leighton [**11**], the properties of principal solutions derived in their Theorem 2.2 permit a ready direct determination of such a solution.

Sections 2 - 8 of the present paper deal with a self-adjoint n-dimensional vector equation with complex coefficients that is a direct generalization of the scalar equation (1.1) ; Section 9 is devoted to a more general differential system with complex coefficients that is of the general form of the accessory differential equations for a variational problem of Bolza type.

Matrix notation is used throughout ; in particular, matrices of one column are termed vectors, and for a vector $y=(y_\alpha)$, $(\alpha=1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2+\cdots+|y_n|^2)^{1/2}$. The symbol E is used for the $n\times n$ identity matrix, while 0 is used indiscriminately for the zero matrix of any dimensions ; the conjugate transpose of a matrix M is denoted by M^*. Moreover, the notation $M\geqq N$, $(M>N)$, is used to signify that M and N are hermitian matrices of the same dimensions and $M-N$ is a nonnegative (positive) hermitian matrix.

2. **Formulation of the problem.**

For x on a given interval X: $a<x<\infty$ let $\omega(x, y, \pi)$ denote the hermitian form

$$\omega(x, y, \pi)=\pi^*R(x)\pi+\pi^*Q(x)y+y^*Q^*(x)\pi+y^*P(x)y \ , \tag{2.1}$$

in the $2n$ variables $y, \pi=(y_1, \cdots, y_n, \pi_1, \cdots, \pi_n)$. *It will be assumed throughout Sections 2 - 8 that $R(x)$, $Q(x)$, $P(x)$ are $n\times n$ matrices having complex-valued continuous elements on X, with $R(x)$, $P(x)$ hermitian, and $R(x)$ non-singular on this interval.*

If c, d are points of X the symbol $I[y\,;\,c, d]$ will denote the hermitian functional

$$I[y\,;\,c, d]=\int_c^d \omega(x, y, y')\,dx \ . \tag{2.2}$$

For the functional (2.2) the vector Euler equation is

$$L[u]\equiv(R(x)u'+Q(x)u)'-(Q^*(x)u'+P(x)u)=0 \ , \tag{2.3}$$

which may be written in terms of the canonical variables

$$u(x),\ v(x)=R(x)u'(x)+Q(x)u(x)$$

as the first order system

$$(2.4) \qquad u' = A(x)u + B(x)v\ , \quad v' = C(x)u - A^*(x)v\ ,$$

where the $n \times n$ coefficient matrices of (2.4) are continuous on X and given by $A = -R^{-1}Q$, $B = R^{-1}$, $C = P - Q^*R^{-1}Q$; in particular, the matrices $B(x)$, $C(x)$ are hermitian on X and $B(x)$ is non-singular on this interval.

Corresponding to (2.3) and (2.4) are the respective matrix equations

$$(2.3') \qquad L[U] \equiv (R(x)U' + Q(x)U)' - (Q^*(x)U' + P(x)U) = 0\ ,$$

$$(2.4') \qquad U' = A(x)U + B(x)V, \qquad V' = C(x)U - A^*(x)V\ .$$

In [13] the author has discussed various criteria of oscillation and non-oscillation for an equation (2.3) in which the coefficient matrices satisfy weaker conditions than those imposed above; although the results of the present paper hold for equations of the generality discussed in [13], for simplicity specific attention is restricted to the case described above.

Throughout the subsequent discussion of Sections 2 - 8 we shall deal consistently with the cononical system (2.4) and associated matrix system (2.4′), instead of the equivalent respective equations (2.3) and (2.3′), since in Section 9 there is considered a vector differential system more general than (2.3), but with associated canonical system still of the form (2.4).

If $U(x) \equiv \| U_{\alpha j}(x) \|$, $V(x) = \| V_{\alpha j}(x) \|$, $(\alpha = 1, \cdots, n\ ;\ j = 1, \cdots, r)$ are $n \times r$ matrices, for typographical simplicity the symbol $(U(x)\ ;\ V(x))$ will be used to denote the $2n \times r$ matrix whose j-th column has elements $U_{1j}(x), \cdots, U_{nj}(x), V_{1j}(x), \cdots, V_{nj}(x)$. In the major portion of the following discussion we shall be concerned with matrices $(U(x)\ ;\ V(x))$ which are solutions of the matrix differential system (2.4′).

If $(U_1(x)\ ;\ V_1(x))$ and $(U_2(x)\ ;\ V_2(x))$ are individually solutions of (2.4′) then, as noted in Lemma 2.1 of [13], the matrix $U_1^*(x)V_2(x) - V_1^*(x)U_2(x)$ is a constant. This matrix will be denoted by $\{U_1, U_2\}$; it is to be remarked that for the problem formulated above there is no ambiguity in this notation, since the $V(x)$ belonging to a solution $(U(x)\ ;\ V(x))$ of (2.4′) is uniquely determined as $V(x) = R(x)U'(x) + Q(x)U(x)$. As in [13], two solutions $(u_1(x)\ ;\ v_1(x))$ and $(u_2(x)\ ;\ v_2(x))$ of (2.4) are said to be (mutually) *conjoined* if $\{u_1, u_2\} = 0$. If $(U(x)\ ;\ V(x))$ is a solution of (2.4′) whose column vectors are conjoined solutions of (2.4), then $(U(x);\ V(x))$ will be termed a matrix of conjoined solutions. In particular, if $U(x)$, $V(x)$ are $n \times n$ matrices such that $(U(x)\ ;\ V(x))$ is a matrix of conjoined solutions of (2.4), then $U(x)$ is a prepared solution of (2.3′) in the sense of Hartman [5]. If the coefficients of (2.1) are real-valued, then two real-valued solutions of (2.4) are conjoined if and only if they are conjugate in the sense introduced originally by von Escherich. The reader is referred to [13; pp. 737, 743] for comments on the use of the synonym "conjoined" for the case of (2.1) with complex-valued coefficients.

Two points s, t of X are said to be (mutually) *conjugate*, (with respect to (2.3) or (2.4)), if there exists a solution $(u(x)\,;\,v(x))$ with $u(x)\not\equiv 0$ on $[s, t]$ and satisfying $u(s)=0=u(t)$. The system (2.4) will be termed *non-oscillatory on a given interval* provided no two distinct points of this interval are conjugate; moreover, (2.4) will be called *non-oscillatory for large* x if there exists a subinterval $a_0 < x < \infty$ of X on which this system is non-oscillatory.

3. **Related matrix solutions of** (2.4′). Suppose that $(U(x)\,;\,V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on a given subinterval X_0 of X, and denote by K the $n\times n$ constant matrix such that $\{U, U\}\equiv K$. If $(U_0(x)\,;\,V_0(x))$ is a $2n\times r$ matrix solution of (2.4′) on X_0, and K_0 is the $n\times r$ constant matrix such that $\{U, U_0\}\equiv K_0$, then from this latter relation it follows that the $n\times r$ matrix $H(x)=U^{-1}(x)U_0(x)$ is such that

$$U_0(x)=U(x)H(x),\quad V_0(x)=V(x)H(x)+U^{*-1}(x)[K_0-KH(x)]\,, \tag{3.1}$$

and in view of the relation $K=-K^*$ it may be verified readily that

$$\{U_0, U_0\}\equiv -H^*(x)KH(x)+H^*(x)K_0-K_0{}^*H(x)\equiv K_1\,, \tag{3.2}$$

where K_1 is a constant $r\times r$ matrix. Moreover, from the differential equations $U_0'=AU_0+BV_0$, $U'=AU+BV$ it follows that

$$H'(x)=U^{-1}(x)B(x)U^{*-1}(x)[K_0-KH(x)],\quad x\in X_0\,. \tag{3.3}$$

Conversely, if K_0 is an arbitrary $n\times r$ constant matrix, and $H(x)$ is an $n\times r$ matrix satisfying the corresponding matrix differential equation (3.3), then it follows readily that the $2n\times r$ matrix $(U_0(x)\,;\,V_0(x))$ defined by (3.1) is a solution of (2.4′) with $\{U, U_0\}\equiv K_0$, and $\{U_0, U_0\}$ given by (3.2).

Now if $x=s$ is a point of X and $T(x)=T(x, s\,;\,U)$ is the solution of the matrix differential system

$$T'=-U^{-1}(x)B(x)U^{*-1}(x)KT,\quad T(s)=E\,, \tag{3.4}$$

then by the method of variation of parameters it follows immediately that $H(x)$ is a solution of (3.3) for a given $n\times r$ matrix K_0 if and only if there is an $n\times r$ constant matrix $H_0=H(s)$ such that

$$H(x)=T(x, s\,;\,U)[H_0+S(x, s\,;\,U)K_0]\,, \tag{3.5}$$

where

$$(3.6)\qquad S(x,\ s\ ;\ U)=\int_s^x T^{-1}(t,\ s\ ;\ U)U^{-1}(t)B(t)U^{*-1}(t)\,dt,\quad x,\ s\in X_0\ .$$

The corresponding solution $(U_0(x)\ ;\ V_0(x))$ of (2.4′) determined by (3.1) is such that

$$(3.7)\qquad U_0(x)=U(x)T(x,\ s\ ;\ U)[U^{-1}(s)U_0(s)+S(x,\ s\ ;\ U)\{U,\ U_0\}]\ .$$

In general, if $F(x)$ is a continuous $n\times n$ matrix and $Y(x)$ is the fundamental matrix of $Y'=F(x)Y$ satisfying $Y(s)=E$, then $Z=Y^{*-1}(x)$ is the fundamental matrix solution of $Z'=-F^*(x)Z$ satisfying $Z(s)=E$. As $K=\{U,\ U\}$ satisfies $K=-K^*$ it follows that $T^{*-1}(x)=T^{*-1}(x,\ s\ ;\ U)$ is the solution of $(T^{*-1})'=-KU^{-1}(x)B(x)U^{*-1}(x)T^{*-1}$ satisfying $T^{*-1}(s)=E$. Now if $H(x)$ is a solution of (3.3) then

$$[K_0-KH(x)]'=-KU^{-1}(x)B(x)U^{*-1}(x)[K_0-KH(x)]\ ,$$

and hence $K_0-KH(x)=T^{*-1}(x,\ s\ ;\ U)[K_0-KH_0]$. Since $K=\{U,\ U\}$ and $K_0=\{U,\ U_0\}$, this latter relation may be written as the following identity for solutions $(U_0(x)\ ;\ V_0(x))$ and $(U(x)\ ;\ V(x))$ of (2.4′), with $U(x)$ non-singular on the interval of consideration X_0 and x, s arbitrary values on this interval,

$$(3.8)\qquad \begin{aligned}\{U,\ U_0\}-\{U,\ U\}U^{-1}(x)U_0(x)\equiv T^{*-1}(x,\ s\ ;\ U)[\{U,\ U_0\}\\ -\{U,\ U\}U^{-1}(s)U_0(s)]\ .\end{aligned}$$

In particular, if $\{U,\ U\}=0$ then

$$(3.9)\qquad K=0,\ T(x,\ s\ ;\ U)\equiv E,\ H(x)=H_0+\int_s^x U^{-1}(t)B(t)U^{*-1}(t)\,dt\ ,$$

and the $U_0(x)$, $V_0(x)$ given by (3.1) satisfy $\{U_0,\ U_0\}=0$ if and only if the $r\times r$ constant matrix $H_0^*K_0$ is hermitian. In case $\{U,\ U\}=0$ the formula (3.7) reduces to a relation that may be found in various recent papers, (see Sternberg and Kaufman [14]; Barrett [1 and 2]; Hartman [5]). For future reference the above results are collected in the following theorem.

THEOREM 3.1. *If $(U(x)\ ;\ V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on a subinterval X_0 of X, and K is the constant $n\times n$ matrix such that $\{U,\ U\}\equiv K$, then an $n\times r$ matrix $U_0(x)$ belongs to a solution $(U_0(x)\ ;\ V_0(x))$ of (2.4′) on X_0 if and only if $U_0(x)=U(x)H(x)$, where $H(x)$ is of the form (3.5) with $T(x,\ s\ ;\ U)$ and $S(x,\ s\ ;\ U)$ determined by (3.4) and (3.6), respectively, and H_0, K_0 are $n\times r$ constant matrices. Moreover, for such a $U_0(x)$ the corresponding $V_0(x)$ is given by (3.1), $\{U,\ U_0\}=K_0$, $\{U_0,\ U_0\}$ has the value (3.2), and the identities (3.7), (3.8) hold for x, $s\in X_0$; in particular, if $K=0$ then $T(x,\ s\ ;\ U)\equiv E$ and $\{U_0,\ U_0\}\equiv 0$ if and only if the constant $r\times r$ matrix $H_0^*K_0$ is hermitian.*

It is to be emphasized that the above theorem is quite independent of any non-oscillatory character of (2.4). For example, the scalar equation $u''+u=0$ has solution $u(x)=\exp(ix)$ which satisfies $u(x)\neq 0$ on $(-\infty, \infty)$, and with $\{u, u\}\equiv 2i$, $T(x, s; u)=\exp(-2i(x-s))$, $S(x, s; u)=\sin(x-s)\exp(i(x-s))$; moreover, $u_0(x)=\sin x$ is a second solution of this equation for which $\{u, u_0\}\equiv 1$, and one may verify readily the identities (3.7) and (3.8).

THEOREM 3.2. *Suppose that $(U(x); V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on a subinterval X_0 of X. If $s\in X_0$ then for $t\in X_0$, $t\neq s$, the matrix $S(t, s; U)$ is singular if and only if t is conjugate to s. In particular, if (2.4) is non-oscillatory on a subinterval $X_0: a_0<x<\infty$, and $(U(x); V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on X_0, then for $s\in X_0$ the matrix $S(t, s; U)$ is non-singular for $t\in X_0$, $t\neq s$; moreover, if there exists an $s\in X_0$ such that $S^{-1}(x, s; U)\to 0$ as $x\to\infty$ then $S^{-1}(x, r; U)\to 0$ as $x\to\infty$ for arbitrary $r\in X_0$.*

As $B(x)$ is non-singular, if $u(x)\equiv 0$, $v(x)$ is a solution of (2.4) on a given subinterval of X then $v(x)\equiv 0$ on this subinterval. In view of this condition, which is a property of "normality" of (2.4), it follows that if $(U_0(x); V_0(x))$ is a solution of (2.4′) with $U_0(s)=0$ and $V_0(s)$ non-singular then t is conjugate to s if and only if $U_0(t)$ is singular. Now if $(U(x); V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on X_0, then for $s\in X_0$ the above-defined $(U_0(x); V_0(x))$ is such that $\{U, U_0\}$ is the non-singular matrix $U^*(s)V_0(s)$, and from (3.7) it follows that $U_0(x)=U(x)T(x, s; U)S(x, s; U)U^*(s)V_0(s)$ for $x\in X_0$, and thus $S(t, s; U)$ is singular for a value $t\in X_0$, $t\neq s$, if and only if t is conjugate to s. Consequently, if (2.4) is non-oscillatory on a subinterval X_0, and $(U(x); V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on X_0, then $S(t, s; U)$ is non-singular for $t\in X_0$, $t\neq s$. Now the fundamental matrix $T(x, s; U)$ of (3.4) satisfies the well-known relation $T(x, s; U)=T(x, r; U)T(r, s; U)$ for $r, s\in X_0$, and by direct computation it follows that

$$S(x, s; U)=T(s, r; U)[S(x, r; U)-S(s, r; U)] \tag{3.10}$$

for $r, s, x\in X_0$. If for a general non-singular matrix M the supremum and infimum of $|My|$ on the sphere $|y|=1$ are denoted by $\mu(M)$ and $\lambda(M)$, respectively, then the relation

$$\mu(M^{-1})|My|\geqq|M^{-1}(My)|=|y|=|M(M^{-1}y)|\geqq\lambda(M)|M^{-1}y|,$$

implies that $1=\lambda(M)\mu(M^{-1})$. As the condition that $S^{-1}(x, s; U)\to 0$ as $x\to\infty$ is equivalent to $\mu(S^{-1}(x, s; U))\to 0$ as $x\to\infty$, this condition holds if and only if $\lambda(S(x, s; U))\to\infty$ as $x\to\infty$. Now in view of the non-singularity of $T(s, r; U)$ it follows from (3.10) that for $r, s\in X_0$ we have $\lambda(S(x, s; U))\to\infty$ as $x\to\infty$ if and only if $\lambda(S(x, r; U))\to\infty$ as $x\to\infty$.

In view of the result of Theorem 3.2, for an equation (2.4) that is non-oscillatory for large x a solution $(U(x)\,;\ V(x))$ of (2.4′) will be termed a *principal solution* if $U(x)$ is non-singular for x on some interval X_U: $a_U<x<\infty$ and $S^{-1}(x,\ s\,;\ U)\to 0$ as $x\to\infty$ for at least one (and consequently all) $s\in X_U$. If $(U(x)\,;\ V(x))$ is a matrix of conjoined solutions of (2.4) with $U(x)$ non-singular for large x this definiton clearly reduces to that of Hartman [5]. In the following sections it will be shown that if $R(x)$ is positive definite on X, and (2.4) is non-oscillatory for large x, then there does exist a principal solution of (2.4′), and this principal solution is unique up to multiplication on the right by a non-singular constant matrix. In general, however, one has the following theorem, which shows that if (2.4) is non-oscillatory for large x then a solution of (2.4′) which is principal in the sense defined above possesses a property corresponding to that used as a definitive property by Morse and Leighton [11] for the scalar eqution (1.1).

THEOREM 3.3. *If* (2.4) *is non-oscillatory for large x, then a solution $(U(x)\,;\ V(x))$ of* (2.4′) *is a principal solution if $U(x)$ is non-singular for large x and there exists a solution $(U_0(x)\,;\ V_0(x))$ of* (2.4′) *with $U_0(x)$ non-singular for large x and such that for some value $s\in X$,*

$$(3.11)\qquad U_0^{-1}(x)U(x)T(x,\ s\,;\ U)\to 0 \quad as\ x\to\infty\ ;$$

moreover, $\{U,\ U_0\}$ is non-singular for any such $(U_0(x)\,;\ V_0(x))$. Conversely, if (2.4) *is non-oscillatory for large x, and $(U(x)\,;\ V(x))$ is a principal solution of* (2.4′), *then any solution $(U_0(x)\,;\ V_0(x))$ of* (2.4′) *with $\{U,\ U_0\}$ non-singular is such that $U_0(x)$ is non-singular for large x and* (3.11) *holds for arbitrary $s\in X$.*

Suppose that (2.4) is non-oscillatory for large x, and that there is a solution $(U(x)\,;\ V(x))$ of (2.4′) with $U(x)$ non-singular on an interval X_0: $a_0<x<\infty$. If $(U_0(x)\,;\ V_0(x))$ is also a solution of (2.4′) then by (3.7),

$$(3.12)\qquad [U(x)T(x,\ s\,;\ U)]^{-1}U_0(x)=U^{-1}(s)U_0(s)+S(x,\ s\,;\ U)\{U,\ U_0\}\ ;$$

moreover, if $U_0(x)$ is non-singular and satisfies (3.11) for some $s\in X_0$, then $\lambda([U(x)T(x,\ s\,;\ U)]^{-1}U_0(x))\to\infty$ as $x\to\infty$ and from (3.12) it follows that $\{U,\ U_0\}$ is non-singular and $\lambda(S(x,\ s\,;\ U))\to\infty$ as $x\to\infty$, so that $(U(x)\,;\ V(x))$ is a principal solution of (2.4′).

On the other hand, if (2.4) is non-oscillatory for large x, and $(U(x);\ V(x))$ is a principal solution of (2.4′), then for s sufficiently large we have that $\lambda(S(x,\ s\,;\ U))\to\infty$ as $x\to\infty$. For such a value s, and $(U_0(s)\,;\ V_0(x))$ a solution of (2.4′) with $\{U,\ U_0\}$ non-singular, we have $\lambda(U^{-1}(s)U_0(s)+S(x,\ s\,;\ U)\{U,\ U_0\})\to\infty$ as $x\to\infty$, and hence from (3.12) it follows that $\lambda([U(x)T(x,\ s\,;\ U)]^{-1}U_0(x))\to\infty$ as $x\to\infty$, which is equivalent to the condition that $U_0(x)$ is non-singular for large x and satisfies (3.11). As

$T(x, s; U)=T(x, r; U)T(r, s; U)$, if (3.11) holds for one value s then this condition holds for arbitrary $s \varepsilon X$.

4. Certain basic results of the calculus of variations. For the functional (2.2) an n-dimensional vector function $y(x)$ will be termed *differentially admissible on a subinterval* of X if on this subinterval $y(x)$ is continuous and has piecewise continuous derivatives. For brevity, if $[c, d]$ is a compact subinterval of X the symbol $H_+[c, d]$ will signify the condition that $I[y; c, d]>0$ for arbitrary $y(x)$ differentially admissible on $[c, d]$, and such that $y(x)\not\equiv 0$ on $[c, d]$, $y(c)=0=y(d)$. We shall also denote by H_R the condition that $R(x)>0$ on X; in view of the basic assumption that $R(x)$ is non-singular on X the condition H_R holds whenever there is a single s of X such that $R(s)>0$.

For the subsequent discussion the following known variational results are basic.

THEOREM 4.1. *If $[c, d]$ is a compact subinterval of X then a necessary and sufficient condition for $H_+[c, d]$ is that H_R hold, together with one of the following conditions*:

(i) (2.4) *is non-oscillatory on* $[c, d]$;

(ii) *there exists a matrix* $(U(x); V(x))$ *of conjoined solutions of* (2.4) *with* $U(x)$ *non-singular on* $[c, d]$.

THEOREM 4.2. *If $[c, d]$ is a compact subinterval of X such that $H_+[c, d]$ holds, then for arbitrary vectors y_c, y_d there is a unique solution $(u(x); v(x))$ of* (2.4) *satisfying $u(c)=y_c$, $u(d)=y_d$, and $I[y; c, d]>I[u; c, d]$ for arbitrary differentially admissible $y(x)$ with $y \not\equiv u$ on $[c, d]$, $y(c)=u(c)$, $y(d)=u(d)$.*

THEOREM 4.3. *Suppose that $[c, d]$ is a compact subinterval of X such that $H_+[c, d]$ holds. If $(U_c(x); V_c(x))$, $[(U_d(x); V_d(x))]$, is the solution of* (2.4′) *determined by $U_c(c)=E$, $U_c(d)=0$, $[U_d(d)=E$, $U_d(c)=0]$, and $(U(x); V(x))$ is a solution of* (2.4′) *satisfying $U(c)=E$, $V(c)>V_c(c)$, $[U(d)=E$, $V(d)<V_d(d)]$, then $(U(x); V(x))$ is a matrix of conjoined solutions of* (2.4) *with $U(x)$ non-singular on $[c, d]$.*

For the case in which the coefficient matrices of (2.1) are real-valued the results of Theorems 4.1 and 4.2 are classical results in the calculus of variations, (see, for example, Morse [10; Chapter I], or Bliss [3; Chapter IV]; for the general case of complex coefficients these results are contained in Theorems 2.1 and 2.2 of Reid [13]. In connection with Theorem 4.2 it is to be commented that if

$$I[\eta, u; c, d]=\int_c^d [\eta^{*\prime}(Ru'+Qu)+\eta^*(Q^*u'+Pu)]dx$$

for differentially admissible $\eta(x)$, $u(x)$, then in case $(u(x)\,;\,v(x))$ is a solution of (2.3) on $[c,\,d]$ we have

$$I[\eta,\,u\,;\,c,\,d]=\eta^*(x)v(x)\Big|_c^d\,. \tag{4.1}$$

A ready consequence of (4.1) is that if $(u(x)\,;\,v(x))$ and $y(x)$ satisfy the conditions of Theorem 4.2 then

$$I[y\,;\,c,\,d]=I[u\,;\,c,\,d]+I[y-u\,;\,c,\,d]\,, \tag{4.2}$$

which is the well-known "integral formula of Weierstrass" for the functional (2.2).

Theorem 4.3 is a comparison theorem of Sturmian type that is a special case of results of Morse [9 ; §10, or 10 ; Chapter IV, §8] in case the coefficients of (2.1) are real-valued, and Morse's method may be extended readily to prove the stated result. The method introduced by Hestenes [6], (see also Bliss [3 ; §§86–87]), to establish the corresponding result for variational problems of Bolza type yields the following brief and elegant proof of the statement of the theorem involving $(U_c(x)\,;\,V_c(x))$; the statement involving $(U_d(x)\,;\,V_d(x))$ follows by a similar argument. By Theorem 4.2 the condition $H_+[c,\,d]$ implies the existence of the solution $(U_c(x)\,;\,V_c(x))$ of (2.4′) satisfying $U_c(c)=E$, $U_c(d)=0$; the end condition $U_c(d)=0$ clearly implies that $(U_c(x)\,;\,V_c(x))$ is a matrix of conjoined solutions and consequently $V_c(c)=U_c^*(c)\,V_c(c)$ is hermitian. For $(U(x)\,;\,V(x))$ a solution of (2.4′) satisfying $U(c)=E$, $V(c)>V_c(c)$ the matrix $U(d)$ is non-singular, since if $U(d)\xi=0$ then $u(x)=(U(x)-U_c(x))\xi$, $v(x)=(V(x)-V_c(x))\xi$ is a solution of (2.4) satisfying $u(c)=0=u(d)$ so that $u(x)\equiv0$ by Theorem 4.1, and hence $(V(c)-V_c(c))\xi=0$ and $\xi=0$. Moreover, $U(x)$ is non-singular on $c<x<d$, since if $c<b<d$ and $U(b)\xi=0$ then $y(x)$ defined as $y(x)=(U(x)-U_c(x))\xi$, $c\leqq x\leqq b$, and $y(x)=-U_c(x)\xi$, $b\leqq x\leqq d$, satisfies $y(c)=0=y(d)$ and is differentially admissible on $[c,\,d]$, while in view of the hermitian character of $U_c^*(b)V_c(b)$ we have

$$\begin{aligned}I[y\,;\,c,\,d]&=\xi^*[U^*(b)-U_c^*(b)][V(b)-V_c(b)]\xi-\xi^*V_c^*(b)U_c(b)\xi\\&=-\xi^*U_c^*(b)[V(b)-V_c(b)]\xi+\xi^*V_c^*(b)[U(b)-U_c(b)]\xi\\&=-\xi^*\{U_c,\,U-U_c\}\xi\\&=-\xi^*[V(c)-V_c(c)]\xi\,,\end{aligned}$$

and consequently $I[y\,;\,c,\,d]<0$ unless $\xi=0$, so that $\xi=0$ in view of $H_+[c,\,d]$.

5. Systems (2. 4) that are non-oscillatory for large x.

For a system satisfying H_R and non-oscillatory for large x, the following theorem determines a particular matrix of conjoined solutions which subsequently will be shown to be a principal solution, as defined in Section 3.

THEOREM 5.1. *Suppose that* (2.4) *satisfies* H_R *and is non-oscillatory on a subinterval* X_0: $a_0<x<\infty$ *of* X. *If* $s \in X_0$ *and for* $t \in X_0$, $t \neq s$, *the matrix* $(U_{st}(x); V_{st}(x))$ *is the solution of* (2.4′) *determined by* $U_{st}(s) = E$, $U_{st}(t)=0$, *then* $U_{s,\infty}(x)=\lim_{t\to\infty} U_{st}(x)$, $V_{s,\infty}(x)=\lim_{t\to\infty} V_{st}(x)$ *exist and* $(U_{s,\infty}(x); V_{s,\infty}(x))$ *is a matrix of conjoined solutions of* (2.4) *with* $U_{s,\infty}(x)$ *non-singular on* X_0; *moreover*, $U_{r,\infty}(x)=U_{s,\infty}(x)U_{r,\infty}(s)$ *and* $V_{r,\infty}(x) = V_{s,\infty}(x)U_{r,\infty}(s)$ *for* $r, s, x \in X_0$.

As the initial condition $U_{st}(t)=0$ implies $\{U_{st}, U_{st}\}=0$, it follows that if $s, t \in X_0$, $s \neq t$, then $(U_{st}(x); V_{st}(x))$ is a matrix of conjoined solutions, so that the matrix $U^*_{st}(x)V_{st}(x)$ is hermitian for $x \in X$; in particular, $V_{st}(s)$ is hermitian. For a given $s \in X_0$ let r, t be points of X_0 satisfying $r<s<t$, and for an arbitrary non-zero constant vector ξ let $y(x)$ denote the vector function defined on $[r, t]$ as

$$(5.1) \qquad y(x)=U_{sr}(x)\xi \ \text{on} \ [r, s]\,; \ y(x)=U_{st}(x)\xi \ \text{on} \ [s, t]\,.$$

Now this vector function $y(x)$ is differentially admissible and $y(r)=0 =y(t)$, so that under the hypothesis that (2.4) satisfies H_R and is non-oscillatory on X_0 it follows from Theorem 4.1 that

$$0<I[y\,;\, r,\, t]=\xi^*U^*_{sr}(s)V_{sr}(s)\xi-\xi^*U^*_{st}(s)V_{st}(s)\xi=\xi^*[V_{sr}(s)-V_{st}(s)]\xi\,.$$

As this relation holds for arbitrary non-zero vectors ξ we have

$$(5.2) \qquad V_{st}(s)<V_{sr}(s) \ \text{for} \ r,\, s,\, t \in X_0,\ r<s<t\,.$$

For $s<t<d$, and ξ an arbitrary non-zero constant vector, let $u(x) =U_{sd}(x)\xi$, $v(x)=V_{sd}(x)\xi$ and $y(x)=U_{st}(x)\xi$ on $[s, t]$, $y(x)\equiv 0$ on $[t, d]$. Then $(u(x); v(x))$ is a solution of (2.4), while $y(x)$ is differentially admissible and satisfies $y(s)=u(s)$, $y(d)=u(d)$, $y(x)\not\equiv u(x)$ on $[s, d]$, so that

$$(5.3) \qquad -\xi^*V_{sd}(s)\xi=I[u\,;\, s,\, d]<I[y\,;\, s,\, d]=I[y\,;\, s,\, t]=-\xi^*V_{st}(s)\xi$$

in view of Theorem 4.2; that is,

$$(5.4) \qquad V_{st}(s)<V_{sd}(s) \ \text{for} \ s,\, t,\, d \in X_0,\ s<t<d\,.$$

By a similar argument it follows that

$$(5.5) \qquad V_{sc}(s)<V_{sr}(s) \ \text{for} \ c,\, r,\, s \in X_0,\ c<r<s\,.$$

From (5.2), (5.4) it follows that for fixed $s \in X_0$ the one-parameter family of hermitian matrices $V_{st}(s)$, $s<t<\infty$, is monotone increasing and bounded, so that there is an hermitian matrix $V_{s,\infty}$ such that $V_{sd}(s)\to V_{s,\infty}$ as $d\to\infty$. Moreover, in view of (5.2), (5.4), (5.5) it follows that

$$(5.6) \qquad V_{st}(s)<V_{s,\infty}<V_{sr}(s) \ \text{for} \ r,\, s,\, t \in X_0,\ r<s<t\,,$$

If $(U_{s,\infty}(x);\ V_{s,\infty}(x))$ is the solution of (2.4′) determined by the initial values $U_{s,\infty}(s)=E$, $V_{s,\infty}(s)=V_{s,\infty}$ then clearly $(U_{st}(x);\ V_{st}(x))\rightarrow(U_{s,\infty}(x);\ V_{s,\infty}(x))$, while the hermitian character of $V_{s,\infty}=U^*_{s,\infty}(s)V_{s,\infty}(s)$ implies that $\{U_{s,\infty},\ U_{s,\infty}\}=0$, and $(U_{s,\infty}(x);\ V_{s,\infty}(x))$ is a matrix of conjoined solutions. Moreover, in view of Theorem 4.3, inequality (5.6) implies that $U_{s,\infty}(x)$ is non-singular on each subinterval $[r,\ t]$ of X_0 with $r<s<t$, and hence $U_{s,\infty}(x)$ is non-singular on X_0.

The final statement of the theorem is an immediate consequence of the fact that $U_{st}(x)=U_{rt}(x)U_{rt}^{-1}(s)$, $V_{st}(x)=V_{rt}(x)U_{rt}^{-1}(s)$ for $r,\ s,\ t\in X_0$, $r\neq t$, $s\neq t$.

If (2.4) is oscillatory on X then there exists a t such that there are points s of X which precede t and are conjugate to t, and consequently there is a largest such conjugate point $s=c(t)$ preceding t. For a system (2.4) satisfying H_R it follows from Theorem 4.1 that if $c(t)$ exists for a value $t=t_1$ then $c(t)$ exists for $t_1<t<\infty$ and increases with t. In accordance with the terminology introduced by Morse and Leighton [11] for a scalar second order linear differential equation, the first conjugate point $c(\infty)$ of $x=\infty$ on X is defined as the limit of $c(t)$ as $t\rightarrow\infty$. Clearly such a system (2.4) is non-oscillatory for large x if and only if either (2.4) is non-oscillatory on X or $c(\infty)$ exists and is finite. If $c(\infty)$ exists and is finite then (2.4) is non-oscillatory on $(c(\infty),\ \infty)$, so that the interval X_0 of Theorem 5.1 may be chosen as this interval, and consequently for $c(\infty)<s<\infty$ the matrix of conjoined solutions $(U_{s,\infty}(x);\ V_{s,\infty}(x))$ has $U_{s,\infty}(x)$ non-singular on $(c(\infty),\ \infty)$. On the other hand, the definition of $c(\infty)$ implies that (2.4) is oscillatory on an arbitrary subinterval $(a_0,\ \infty)$ of X with $a_0<c(\infty)$, and Theorem 4.1 implies that $U_{s,\infty}(x)$ is singular at some point of such a subinterval $(a_0,\ \infty)$, so that by continuity $U_{s,\infty}(x)$ is singular for $x=c(\infty)$. That is, if H_R holds and (2.4) is non-oscillatory for large x then the matrix of conjoined solutions $(U_{s,\infty}(x);\ V_{s,\infty}(x))$ of Theorem 5.1 is such that $c(\infty)$ exists on X if and only if $U_{s,\infty}(x)$ is singular at some point of X, in which case $c(\infty)$ is the largest value of x for which $U_{s,\infty}(x)$ is singular.

6. Principal solutions. From Theorem 5.1 it follows that if (2.4) satisfies H_R and is non-oscillatory on X_0: $a_0<x<\infty$ then there exist matrix solutions $(U(x);\ V(x))$ of (2.4′) with $U(x)$ non-singular on X_0. The basic result on principal solutions for such a system (2.4) is contained in the following theorem.

THEOREM 6.1. *Suppose that the equation* (2.4) *satisfies* H_R *and is non-oscillatory on a subinterval* X_0: $a_0<x<\infty$ *of* X. *If* $(U(x);\ V(x))$ *is a solution of* (2.4′) *with* $U(x)$ *non-singular on an interval* X_U: $a_U<x<\infty$ *then for* s *a point common to* X_0 *and* X_U *the matrix*

$$M(s\,;\,U)=\lim_{t\to\infty} S^{-1}(t,\,s\,;\,U) \tag{6.1}$$

exists and is finite. Moreover, $M(s\,;\,U)=0$ *and* $(U(x)\,;\,V(x))$ *is a principal solution of* (2.4′) *if and only if* $U(x)=U_{r,\infty}(x)C$, $V(x)=V_{r,\infty}(x)C$, *where* r *is any fixed value on* X_0, $(U_{r,\infty}(x)\,;\,V_{r,\infty}(x))$ *is the matrix of conjoined solutions as determined by Theorem* 5.1, *and* C *is a non-singular constant matrix.*

In view of Theorems 3.2 and 5.1 it clearly suffices to establish the result of the above theorem for $s=r$ a point common to X_0 and X_U. For such a value s it follows from Theorem 3.1 that

$$U_{s,\infty}(x)=U(x)T(x,\,s\,;\,U)[U^{-1}(s)+S(x,\,s\,;\,U)\{U,\,U_{s,\infty}\}]\,,$$

$$U_{st}(x)=U(x)T(x,\,s\,;\,U)[E-S(x,\,s\,;\,U)S^{-1}(t,\,s\,;\,U)]U^{-1}(s)\,,$$

and since $U_{st}(x)\to U_{s,\infty}(x)$, $V_{st}(x)\to V_{s,\infty}(x)$ as $x\to\infty$ it follows that $M(s\,;\,U)$ defined by (6.1) exists and has the finite value

$$M(s\,;\,U)=-\{U,\,U_{s,\infty}\}U(s)\,. \tag{6.2}$$

In particular, (6.2) implies that $M(s\,;\,U)=0$ if and only if $\{U,\,U_{s,\infty}\}=0$. As $0=\{U_{s,\infty},\ U_{s,\infty}\}=V_{s,\infty}(s)-V^*_{s,\infty}(s)$ it follows that $0=\{U,\,U_{s,\infty}\}=U^*(s)V_{s,\infty}(s)-V^*(s)U_{s,\infty}(s)=U^*(s)V^*_{s,\infty}(s)-V^*(s)$ if and only if $(U(s)\,;\,V(s))$ satisfies with the non-singular matrix $C=U(s)$ the initial conditions $U(s)=U_{s,\infty}(s)C$, $V(s)=V_{s,\infty}(x)C$, and therefore $U(x)\equiv U_{s,\infty}(x)C$, $V(x)\equiv V_{s,\infty}(x)C$.

In particular, under the hypotheses of Theorem 6.1 it follows that if $(U(x)\,;\,V(x))$ is a principal solution of (2.4′) then $(U(x)\,;\,V(x))$ is a matrix of conjoined solutions of (2.4), and therefore $T(x,\,s\,;\,U)\equiv E$. As the first conclusion of Theorem 3.3 with $U_0(x)=U(x)$ implies that if (2.4) has a solution $(U(x)\,;\,V(x))$ with $U(x)$ non-singular for large x, and $T(x,\,s\,;\,U)\to 0$ as $x\to\infty$, then $(U(x)\,;\,V(x))$ is a principal solution, the following corollary is direct consequence of the results of Theorems 3.3, 6.1, and formula (6.2).

COROLLARY. *In case* (2.4) *satisfies* H_R, *and is non-oscillatory for large* x, *then*:

(i) *if* $(U(x)\,;\,V(x))$ *is a solution of* (2.4′) *with* $U(x)$ *non-singular on* X_0: $a_0<x<\infty$, *and* $s\in X_0$, *then it is not true that* $T(x,\,s\,;\,U)\to 0$ *as* $x\to\infty$;

(ii) *if* $(U(x)\,;\,V(x))$ *is a principal solution of* (2.4′), *then for a solution* $(U_0(x)\,;\,V_0(x))$ *of* (2.4′) *the matrix* $\{U,\,U_0\}$ *is non-singular if and only if* $U_0(x)$ *is non-singular for large* x *and* $U_0^{-1}(x)U(x)\to 0$ *as* $x\to\infty$, *moreover, if* $\{U,\,U_0\}$ *is non-singular then, for* s *sufficiently large,* $\lim_{t\to\infty} S(t,\,s,\,U_0)$ *exists and is non-singular.*

Finally, we shall establish the following result; in particular, conclusion (v) generalizes a result of Hartman [5].

THEOREM 6.2. *Suppose that* (2.4) *satisfies* H_R *and is non-oscillatory on a subinterval* $X_0: a_0 < x < \infty$ *of* X, *while* $(U_{s,\infty}(x); V_{s,\infty}(x))$, $s \in X_0$, *is the matrix of conjoined solutions as determined by Theorem* 5.1. *If* $(U(x); V(x))$ *is a solution of* (2.4′) *with* $U(x)$ *non-singular on* X_0, *and* $S(\infty, r; U) = \lim_{x\to\infty} S(x, r; U)$ *exists and is finite for some* $r \in X_0$, *then for arbitrary* $s \in X_0$:

(i) $S(\infty, s; U)$ *exists, and*

(6.3) $S(\infty, s; U) = T(s, r; U)[S(\infty, r; U) - S(s, r; U)]$ *for* $s, x \in X_0$;

(ii) $\{U, U_{s,\infty}\}$ *is non-singular*;

(iii) $U^{-1}(x)U_{s,\infty}(x) \to 0$ *as* $x \to \infty$;

(iv) $\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(s)$ *is non-singular, and* $T(\infty, s; U) = \lim_{x\to\infty} T(x, s; U)$ *exists and is equal to the non-singular matrix* $\{U_{s,\infty}, U\}^{-1}[\{U_{s,\infty}, U\} - U^{*-1}(s)\{U, U\}]$;

(v) $U_{s,\infty}(x) = -U(x)S(\infty, x; U)\{U, U_{s,\infty}\}$.

Conclusion (i) is an immediate consequence of relation (3.10). Now, as established in the proof of Theorem 6.1, the matrix $M(s; U) = \lim_{t\to\infty} S^{-1}(t, s; U)$ exists and has the finite value $-\{U, U_{s,\infty}\}\, U(s)$, so if $S(\infty, s; U)$ exists and is finite we have

$$(6.4) \qquad E = -S(\infty, s; U)\{U, U_{s,\infty}\}U(s),$$

and hence $\{U, U_{s,\infty}\}$ is non-singular; in turn it follows from the Corollary to Theorem 6.1 that (ii) implies (iii).

In order to establish conclusion (iv), it is noted that the non-singularity of $U(x)$ on X_0 implies the validity of (3.8) with $U_0 = U_{s,\infty}(x)$, so that

$$(6.5) \qquad \begin{aligned} &\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(x)U_{s,\infty}(x) \\ &\qquad = T^{*-1}(x, s; U)[\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(s)] \end{aligned}$$

for $s, x \in X_0$. From conclusions (ii), (iii) and relation (6.5) it follows that if ξ is a constant vector satisfying $[\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(s)]\xi = 0$ then $\xi = 0$, so that $\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(s)$ is non-singular for $s \in X_0$. This result, together with conclusions (ii), (iii) and relation (6.5), imply that for $s \in X_0$ the matrix $T^{*-1}(x, s; U)$ approaches the non-singular matrix $\{U, U_{s,\infty}\}[\{U, U_{s,\infty}\} - \{U, U\}U^{-1}(s)]^{-1}$, which is equivalent to the final statement of conclusion (iv).

Finally, it is to be noted that (6.4) is equivalent to

$$E = -U(x)S(\infty, x; U)\{U, U_{x,\infty}\}, \text{ for } x \in X_0,$$

and as $U_{s,\infty}(t) = U_{x,\infty}(t)U_{s,\infty}(x)$, $V_{s,\infty}(t) = V_{x,\infty}(t)U_{s,\infty}(x)$ for $s, t, x \in X_0$ it follows that $\{U, U_{x,\infty}\}U_{s,\infty}(x)=\{U, U_{s,\infty}\}$ and $U_{s,\infty}(x)=-U(x)S(\infty, x; U)$ $\{U, U_{s,\infty}\}$ for $x, s\in X_0$, thus establishing conclusion (v).

7. An example. In the notation of the preceding sections, the example of Section 11 of Hartman [5] shows that for an equation (2.4) which satisfies H_R, and is non-oscillatory for large x, there may exist solutions $(U(x); V(x))$ of (2.4′) with $U(x)$ non-singular for large x and such that

$$\left[\int_s^x U^{-1}(t)B(t)U^{*-1}(t)dt\right]^{-1}\to 0 \quad as\ x\to\infty\ , \tag{7.1}$$

while $(U(x); V(x))$ is not a principal solution. As shown by Theorem 6.1, for general solutions $(U(x); V(x))$ of (2.4′) with $U(x)$ non-singular for large x the discriminating property for principal solutions is not (7.1), but rather $S^{-1}(x, s; U)\to 0$ as $x\to\infty$. We shall proceed to illustrate the results of the preceding sections by the example of Hartman.

For typographical simplification a 2×2 matrix $\|M_{\alpha\beta}\|$, $(\alpha, \beta=1, 2)$, will be displayed as $M=(M_{11}; M_{12}; M_{21}; M_{22})$. In this notation the two-dimensional vector equation of Hartman's example is

$$u''+P(x)u=0,\ 0<x<\infty,\ with\ P(x)=(0\ ;\ 0\ ;\ 0\ ;\ (4x^2)^{-1})\ . \tag{7.2}$$

For (7.2) the matrix solutions $(U_{st}(x); V_{st}(x)=U'_{st}(x))$ of Theorem 5.1 have

$$U_{st}(x)=((x-t)/(s-t)\ ;\ 0\ ;\ 0\ ;\ (x/s)^{1/2}\ (\ln t-\ln x)/(\ln t-\ln s))\ .$$

and consequently $(U_{s,\infty}(x); V_{s,\infty}(x))$ has $U_{s,\infty}(x)=(1; 0; 0; (x/s)^{1/2})$. Hartman's example involves the principal solution $(U_{1,\infty}(x); V_{1,\infty}(x))$ for which $U_{1,\infty}(x)=(1; 0; 0; x^{1/2})$, and the matrix solution $(U(x); V(x))$ having $U(x)=(1; x; 0; x^{1/2})$. For these matrix solutions one may compute readily the following quantities ;

$$\begin{aligned}
&S(x, s; U_{s,\infty})=(x-s\ ;\ 0\ ;\ 0\ ;\ s(\ln x-\ln s))\ ,\\
&\{U, U\}=(0\ ;\ 1\ ;\ -1\ ;\ 0),\ \{U_{1,\infty}, U\}=(0\ ;\ 1\ ;\ 0\ ;\ 0)\ ,\\
&T(x, 1; U)=(1-x\ln x\ ;\ 1-x-x\ln x\ ;\ \ln x\ ;\ 1+\ln x)\ ,\\
&S(x, 1; U)=(x-1+x\ln x\ ;\ -\ln x\ ;\ -x\ln x\ ;\ \ln x)\ ,\\
&M(1; U)=(0\ ;\ 0\ ;\ 1\ ;\ 1)\ ;\ U^{-1}(x)U_{1,\infty}(x)=(1\ ;\ -x\ ;\ 0\ ;\ 1)\ .
\end{aligned}$$

It is to be noted that $\{U_{1,\infty}, U\}$ is singular, so that the corollary to Theorem 6.1 implies that the matrix $U^{-1}(x)U_{1,\infty}(x)$ does not tend to 0 as $x\to\infty$, a fact that is obvious from the specific value of this matrix.

To illustrate further the results of the preceding section, consider the solution $(U_1(x); V_1(x))$ of (2.4′) with $U_1(x)=(x; 1; 0; x^{1/2}\ln x)$. For this solution $U_1(x)$ is non-singular for $x>1$, and one has

$$U_1^{-1}(x)U_{s,\infty}(x)=(1/x\,;\ -1/(s^{1/2}\ x\ \ln\ x)\,;\ 0\,;\ 1/(s^{1/2}\ \ln\ x))\ ,$$
$$\{U_1,\ U_1\}=(0\,;\ -1\,;\ 1\,;\ 0),\ \{U_{s,\infty},\ U_1\}=(1\,;\ 0\,;\ 0\,;\ s^{-1/2})\ ,$$
$$x^3(\ln\ x)^2U_1^{-1}(x)B(x)U_1^{*-1}(x)=(1+x(\ln\ x)^2\,;\ -x\,;\ -x\,;\ x^2)\ .$$

Moreover, if $\theta=\theta(x,\ s)=(1/\ln\ x)-(1/\ln\ s)$, it may be verified that

$$T(x,\ s\,;\ U_1)=(1-\theta/x\,;\ (x-s-\theta)/(sx)\,;\ \theta\,;\ 1+\theta/s)\ ,$$
$$S(x,\ s\,;\ U_1)=((x-s-\theta)/(sx)\,;\ \theta/s\,;\ \theta/x\,;\ -\theta)\ ,$$
$$(x-s)S^{-1}(x,\ s\,;\ U_1)=(xs\,;\ x\,;\ s\,;\ 1-(x-s)/\theta)\ ,$$

from which one may verify readily that for $1<s<\infty$,

$$T(\infty,\ s\,;\ U_1)=(1\,;\ 1/s\,;\ -1/\ln\ s\,;\ 1-1/(s\ \ln\ s))\ ,$$
$$S(\infty,\ s\,;\ U_1)=(1/s\,;\ -1/(s\ \ln\ s)\,;\ 0\,;\ 1/\ln\ s)\ ,$$
$$M(s\,;\ U_1)=(s\,;\ 1\,;\ 0\,;\ \ln\ s)\ .$$

8. **Further properties of principal solutions.** Suppose that (2.4) satisfies H_R, and is non-oscillatory on a subinterval $X_0:\ a_0<x<\infty$ of X; for $s,\ t\in X_0,\ s<t$, let $Y_{st}(x)=U_{st}(x)$ on $x\leqq t$, and $Y_{st}(x)\equiv 0$ on $x\geqq t$, where, as in Theorem 5.1, $(U_{st}(x)\,;\ V_{st}(x))$ is the solution of (2.4′) satisfying $U_{st}(s)=E,\ U_{st}(t)=0$.

For brevity, if $y(x)$, $u(x)$ are differentially admissible vector functions on $[s.\ \infty)$ such that

$$\lim_{t\to\infty}I[y,\ u\,;\ s,\ t] \tag{8.1}$$

exists and is finite, the value of (8.1) will be denoted by $I[y,\ u\,;\ s]$, moreover, for brevity we shall write $I[y\,;\ s]$ in place of $I[y,\ y\,;\ s]$. In particular, for arbitrary constant vectors ξ we have $I[Y_{st}\xi\,;\ s]=I[U_{st}\xi\,;\ s,\ t]$. Now from relations (5.3) and (4.2) it follows that

$$0<\xi^*[V_{sd}(s)-V_{st}(s)]\xi=I[Y_{st}\xi\,;\ s]-I[Y_{sd}\xi\,;\ s]=I[Y_{st}\xi-Y_{sd}\xi\,;\ s]$$

for $s<t<d,\ s\in X_0$, and since $V_{st}(s)\to V_{s,\infty}$ as $t\to\infty$ it follows that for $s\in X_0$, and ξ an arbitrary constant vector,

$$I[Y_{st}\xi-Y_{sd}\xi\,;\ s]\to 0\ \textit{as}\ t,\ d\to\infty. \tag{8.2}$$

It is to be emphasized that in general it is not true that

$$-\xi^*V_{s,\infty}(s)\xi=I[U_{s,\infty}\xi\,;\ s],\ \textit{for}\ s\in X_0\ , \tag{8.3}$$

although $-\xi^*V_{st}(s)\xi=I[Y_{st}\xi\,;\ s]$ for $t>s$, and $Y_{st}(x)\xi\to U_{s,\infty}(x)\xi$ as $t\to\infty$; moreover, in general it is not true that the vector function $U_{s,\infty}(x)\xi$ is bounded on $[s,\ \infty)$, although $Y_{st}(x)\xi\equiv 0$ for $x\geqq t$. The statements are illustrated by the well-known scalar second order equation $u''+u/(4x^2)=0$, which is non-oscillatory on $(0,\ \infty)$; for this equation $u_{1,\infty}(x)=x^{1/2}$

and $v_{1,\infty}(1)=1/2$, while $\omega(x, u_{1,\infty}, u'_{1,\infty})\equiv 0$. However, much more can be said about the principal solutions $(U_{s,\infty}(x)\,;\ V_{s,\infty}(x))$ in case the hermitian integrand function ω is such that

(8.4) $\omega(x, y, \pi)\geqq 0$ *for arbitrary* x, y, π *with* $x \in X_0$.

In view of the continued understanding that $R(x)$ is non-singular on X, it is clear that (8.4) implies H_R, as well as the result that $H_+[s, t]$ holds for arbitrary compact subintervals $[s, t]$ of X_0, so that (2.4) is non-oscillatory on X_0.

THEOREM 8.1. *If condition* (8.4) *holds on a subinterval* $X_0 : a_0<x<\infty$ *of* X *then* (8.3) *is valid; moreover,* $U^*_{s,\infty}(x)V_{s,\infty}(x)\leqq 0$ *on* $s\leqq x<\infty$ *and* $U^*_{s,\infty}V_{s,\infty}\to 0$ as $x\to\infty$.

Since $V_{st}(s)\to V_{s,\infty}(s)$, and the vector function $Y_{st}(x)\xi$ tends to $U_{s,\infty}(x)\xi$ uniformly on each compact subinterval of $[s, \infty)$ as $t\to\infty$, whenever condition (8.4) holds on X_0 it follows readily from the relation $-\xi^* V_{st}(s)\xi=I[Y_{st}\xi\,;\ s]$ that $I[U_{s,\infty}\xi\,;\ s]$ exists and

$$-\xi^* V_{s,\infty}(s)\xi\geqq I[U_{s,\infty}\xi\,;\ s]\ .$$

Now $V_{s,\infty}(s)$ is hermitian and by (4.1) we have

$$-\xi^* V_{s,\infty}(s)\xi=I[Y_{st}\xi,\ U_{s,\infty}\xi\,;\ s,\ t]=I[Y_{st}\xi,\ U_{s,\infty}\xi\,;\ s]\ .$$

Moreover, whenever (8.4) holds we have the Schwarz inequality

$$|I[Y_{st}\xi,\ U_{s,\infty}\xi\,;\ r]|^2\leqq I[Y_{st}\xi\,;\ r]I[U_{s,\infty}\xi\,;\ r]\ \textit{for}\ s<r<\infty,$$

and as $I[Y_{st}\xi\,;\ r]\leqq I[Y_{st}\xi\,;\ s]\leqq I[Y_{sp}\xi\,;\ s]$ for $t\geqq p>s$ it follows that for given $p>s$, $\varepsilon>0$ there exists a value $r=r_\varepsilon>s$ such that

$$-\xi^* V_{s,\infty}(s)\xi\leqq \Re(I[Y_{st}\xi,\ U_{s,\infty}\xi\,;\ s,\ r])+\varepsilon\ \textit{for}\ t\geqq p\ .$$

As $\Re(I[Y_{st}\xi,\ U_{s,\infty}\xi\,;\ s,\ r])\to I[U_{s,\infty}\xi\,;\ s,\ r]$ as $t\to\infty$, and $I[U_{s,\infty}\xi\,;\ s,\ r]\leqq I[U_{s,\infty}\xi\,;\ s]$ by (8.4), it follows that $-\xi^* V_{s,\infty}(s)\xi\leqq I[U_{s,\infty}\xi\,;\ s]$, thus completing the proof of (8.3). Finally, condition (8.4) implies that for ξ a non-zero constant vector the integral $I[U_{s,\infty}\xi\,;\ s,\ r]=\xi^*[U^*_{s,\infty}(r)V_{s,\infty}(r)-V_{s,\infty}(s)]\xi$ is a monotone increasing function of r on $s<r<\infty$ which tends to $I[U_{s,\infty}\xi\,;\ s]=-\xi^* V_{s,\infty}(s)\xi$ as $r\to\infty$, and consequently $U^*_{s,\infty}(r)V_{s,\infty}(r)\leqq 0$ on (s, ∞) and $U^*_{s,\infty}(r)V_{s,\infty}(r)\to 0$ as $r\to\infty$.

In particular, if $R(x)\equiv E$, $Q(x)\equiv 0$ and $P(x)\geqq 0$ on X, then the above theorem implies that $(|U_{s,\infty}(x)\xi|^2)'=2\xi^* U^*_{s,\infty}(x)V_{s,\infty}(x)\xi\leqq 0$, so that for such an equation (2.4) the norm of the vector function $U_{s,\infty}(x)\xi$ tends to a limit as $x\to\infty$. This particular result has been established by Wintner [16].

It is to be emphasized that condition (8.4) does not imply that

$U_{s,\infty}\to 0$ as $x\to\infty$. For example, (8.4) holds for the scalar equation

$$(u'/(e^x+2))'-2u/(e^x+2)^2=0$$

with general solution $u=c_1(1+e^{-x})+c_2e^x$, and principal solution $u_{0,\infty}(x)=(1+e^{-x})/2$.

THEOREM 8.2. *If H_R holds and (2.4) is non-oscillatory on a subinterval $X_0: a_0<x<\infty$ of X then $U_{s,\infty}(x)\to 0$ as $x\to\infty$ if there exists a constant $k>0$ and a continuous positive function $h(x)$ such that if s, $d\in X_0$, $s<d$, then*

$$I[y\,;\ s,\ d]\geqq k\int_s^d [h(x)|y'|^2+|y|^2/h(x)]dx \tag{8.5}$$

for arbitrary $y(x)$ which are differentially admissible on $[s, d]$ and satisfy $y(s)=0=y(d)$.

If the vector function $y(x)$ is differentially admissible on $[s, d]$, and $y(s)=0=y(d)$, then

$$\begin{aligned}2|y(x)|^2&=\int_s^x(y^*y'+y^{*\prime}y)dx-\int_x^d(y^*y'+y^{*\prime}y)dx\,,\\&\leqq 2\int_s^d|y|\ |y'|dx\leqq\int_s^d[h(x)|y'|^2+|y|^2/h(x)]dx\,,\end{aligned}$$

the last inequality holding for arbitrary continuous positive functions $h(x)$. Consequently the hypothesis of Theorem 8.2 implies that there is a positive constant k such that

$$2k|y(x)|^2\leqq I[y\,;\ s,\ d]\ \textit{for}\ s\leqq x\leqq d \tag{8.6}$$

holds if s, $d\in X_0$, $s<d$, and $y(x)$ is a differentially admissible vector function on $[s, d]$ with $y(s)=0=y(d)$. In particular, if $s<t<d$ and ξ is a constant vector, then $y(x)=Y_{st}(x)\xi-Y_{sd}(x)\xi$ is such a vector function with $y(x)\equiv 0$ for $x\geqq d$ and $I[y\,;\ s,\ d]=I[y\,;\ s]$, so that

$$2k|Y_{st}(x)\xi-Y_{sd}(x)\xi|2\leqq I[Y_{st}\xi-Y_{sd}\xi\,;\ s],\quad s\leqq x<\infty\,. \tag{8.7}$$

Inequalities (8.2), (8.7) then imply that as $t\to\infty$ the convergence of $Y_s(x)\xi$ to $U_{s,\infty}(x)\xi$ is uniform on $s\leqq x<\infty$. As $Y_{st}(x)\xi\equiv 0$ for $x\geqq t$ it then follows that $U_{s,\infty}(x)\xi\to 0$ as $x\to\infty$ for arbitrary constant vectors ξ, so that $U_{s,\infty}(x)\to 0$ as $x\to\infty$.

THEOREM 8.3. *If on a subinterval $X_0: a_0<x<\infty$ of X we have $Q(x)\equiv 0$, $R(x)$ of class C' with $R(x)>0$, $R'(x)\leqq 0$, and there is a non-negative continuous function $k(x)$ such that $\int^\infty k(x)dx$ is divergent and $y^*P(x)y\geqq k(x)y^*R(x)y$ for arbitrary vectors y, then $U^*_{s,\infty}(x)R(x)U_{s,\infty}(x)\to 0$ as $x\to\infty$.*

The hypotheses of the theorem clearly imply condition (8.4) on X_0. Now if $Q(x)\equiv 0$ and $R(x)$ is of class C' we have $V_{s,\infty}(x)=R(x)U'_{s,\infty}(x)$, and as $\{U_{s,\infty},\ U_{s,\infty}\}=0$ it follows that $(U^*_{s,\infty}RU_{s,\infty})'=2U^*_{s,\infty}V_{s,\infty}+U^*_{s,\infty}R'U_{s,\infty}$, so that in view of the condition $R'(x)\leqq 0$ and the last conclusion of Theorem 8.1 we have $(U^*_{s,\infty}RU_{s,\infty})'\leqq 0$ on X_0. Consequently, for an arbitrary constant vector ξ the non-negative function $\xi^*U^*_{s,\infty}(x)R(x)U_{s,\infty}(x)\xi$ is non-increasing on X_0, and thus tends to a non-negative limit as $x\to\infty$. Moreover, by Theorem 8.1 the integral $I[U_{s,\infty}\xi\,;\,s]$ exists and is finite, so that in view of the relation

$$I[U_{s,\infty}\xi\,;\,s]\geqq\int_s^\infty \xi^*U^*_{s,\infty}PU_{s,\infty}\xi\ dx\geqq\int_s^\infty k(x)[\xi^*U^*_{s,\infty}RU_{s,\infty}\xi]dx\ ,$$

and the divergent character of $\int^\infty k(x)dx$, it follows that $\xi^*U^*_{s,\infty}(x)R(x)U_{s,\infty}(x)\xi\to 0$ as $x\to\infty$, for ξ an arbitrary constant vector.

As a particular instance of the above theorem we have the following result.

COROLLARY. *If on a subinterval X_0: $a_0<x<\infty$ of X we have $Q(x)\equiv 0$, $R(x)$ a constant matrix $R>0$, and there is a non-negative continuous function $k_1(x)$ such that $\int^\infty k_1(x)dx$ is divergent and $y^*P(x)y\geqq k_1(x)|y|^2$ for arbitrary vectors y, then for $s\in X_0$ we have $U_{s,\infty}(x)\to 0$ as $x\to\infty$.*

For the case of a scalar equation the result of the above corollary in essence dates from Kneser [7], as has been pointed out by Wintner [15].

Added November 20, 1957. P. Hartman has pointed out to the author that the following argument establishes the conclusion of Theorem 8.3 with the hypothesis that $\int^\infty k(x)dx$ is divergent replaced by the weaker condition that $\int^\infty xk(x)dx$ is divergent. Since Theorem 8.1 implies that $U^*_{s,\infty}V_{s,\infty}\leqq 0$, from the condition $U^*_{s,\infty}R'U_{s,\infty}\leqq 0$ and the expression given for $(U^*_{s,\infty}RU_{s,\infty})'$ in the proof of Theorem 8.3 it follows that the integral $\int^\infty U^*_{s,\infty}V_{s,\infty}\,dx$ exists. From Theorem 8.1 it follows that $U^*_{s,\infty}V_{s,\infty}\to 0$ and

$$-\xi^*U^*_{s,\infty}(u)V_{s,\infty}(u)\xi=I[U_{s,\infty}\xi\,;\,u]\geqq\int_u^\infty \xi^*U^*_{s,\infty}PU_{s,\infty}\xi dx$$

for $a_0<u<\infty$ and arbitrary constant vectors ξ, and as $U^*_{s,\infty}PU_{s,\infty}\geqq 0$ the integrals $\int_u^\infty U^*_{s,\infty}PU_{s,\infty}dx$ and $\int_u^\infty\left[\int_x^\infty U^*_{s,\infty}PU_{s,\infty}dt\right]dx$ exist for $a_0<u<\infty$; an integration by parts then yields the existence of the integral $\int_u^\infty xU^*_{s,\infty}(x)P(x)U_{s,\infty}(x)dx$. Consequently the condition that $y^*P(x)y\geqq k(x)y^*R(x)y$ for arbitrary vectors y implies that the integral

$\int_u^\infty xk(x)U^*_{s,\infty}(x)R(x)U_{s,\infty}(x)dx$ exists, and in view of the relations $U^*_{s,\infty}RU_{s,\infty} \geqq 0$, $(U^*_{s,\infty}RU_{s,\infty})' \leqq 0$ it follows that $U^*_{s,\infty}RU_{s,\infty} \to 0$ whenever $\int^\infty xk(x)dx$ is divergent.

9. **A more general differential system.** In this section we shall consider a differential system with complex coefficients that is of the general form of the accessory differential equations for a variational problem of Bolza type, (see, for example, Bliss [3 ; § 81] and Reid [12]). As in § 2, $\omega(x, y, \pi)$ will denote an hermitian form (2.1) with $R(x)$, $Q(x)$, $P(x)$ $n \times n$ matrices having complex-valued continuous elements on X: $a < x < \infty$, and $R(x)$, $P(x)$ hermitian on this interval. In addition, consider a vector linear form

$$\Phi(x, y, \pi) \equiv \varphi(x)\pi + \theta(x)y \, , \tag{9.1}$$

where $\varphi(x)$ and $\theta(x)$ are $m \times n$, $(m < n)$, matrices with complex-valued continuous elements on X. Instead of the hypothesis of Section 2 that $R(x)$ is non-singular, it is now assumed that the $(n+m) \times (n+m)$ hermitian matrix

$$\left\| \begin{matrix} R(x) & \varphi^*(x) \\ \varphi(x) & 0 \end{matrix} \right\| \tag{9.2}$$

is non-singular on X; in particular, the non-singularity of (9.2) on X implies that $\varphi(x)$ is of rank m on this interval.

For the variational problem involving the functional (2.2) subject to the auxiliary m-dimensional vector differential equation

$$\Phi(x, y, y') = 0 \tag{9.3}$$

the Euler-Lagrange differential equations are in vector form

$$\begin{gathered} (R(x)u' + Q(x)u + \varphi^*(x)\mu)' - (Q^*(x)u' + P(x)u + \theta^*(x)\mu) = 0 \, , \\ \Phi(x, u, u') = 0 \, , \end{gathered} \tag{9.4}$$

where $u(x)$ is an n-dimensional vector function and $\mu(x)$ is an m-dimensional "multiplier" vector function.

The inverse of the non-singular matrix (9.2) is of the form

$$\left\| \begin{matrix} T(x) & \tau^*(x) \\ \tau(x) & t(x) \end{matrix} \right\| ,$$

where $T(x)$ and $t(x)$ are hermitian matrices of orders n and m, respectively, and $\tau(x)$ is an $m \times n$ matrix. In terms of the canonical variables

$$u(x),\ v(x)=R(x)u'(x)+Q(x)u(x)+\varphi^*(x)\mu(x)$$

the Euler-Lagrange equations (9.4) become a vector differential system (2.4), with now

$$(9.5)\qquad A=-(TQ+\tau^*\theta),\ B=T,\ C=P-Q^*TQ-Q^*\tau^*\theta-\theta^*\tau Q-\theta^*t\theta\ ;$$

the matrices B and C of (9.5) are hermitian on X, while B is a non-negative definite matrix of rank $n-m$ with $B\varphi^*=0$ throughout this interval. Throughout this section we shall continue to refer to the vector equation (2.4) and the corresponding matrix equation (2.4′), *with the understanding that the coefficient matrices are given by* (9.5).

As in Section 2, if $(U_1(x)\,;\ V_1(x))$ and $(U_2(x)\,;\ V_2(x))$ are solutions of (2.4′) then the matrix $U_1^*(x)V_2(x)-V_1^*(x)U_2(x)$ is a constant; to denote this matrix by $\{U_1,\ U_2\}$ now in general involves an ambiguity, however, since if $(U(x)\,;\ V(x))$ is a solution of (2.4′) there may exist other matrices $V_0(x)\neq V(x)$ such that $(U(x)\,;\ V_0(x))$ is also a solution of (2.4′). This ambiguity does not exist, however, if (2.4) is such that whenever $u(x)\equiv 0$, $v(x)$ is a solution of this equation on a non-degenerate subinterval of X then $v(x)\equiv 0$ on this subinterval; if this property holds the equation (2.4) is said to be *identically normal*, or to be *normal on every subinterval*, on X. It is to be commented that this condition of normality was used in Section 3 to show that if (2.4) is non-oscillatory on X_0, and $(U(x)\,;\ V(x))$ is a solution of (2.4′) with $U(x)$ non-singular on this interval, then $S(t,\ s\,;\ U)$ is non-singular for $s,\ t\in X_0$, $s\neq t$.

For the equation (2.4) now under consideration one may define the concepts of conjugate point, non-oscillation on a subinterval, and non-oscillation for large x, in precisely the language of Section 2. For the problem involving the functional (2.2) subject to the differential equation (9.3) an n-dimensional vector function $y(x)$ will now be said to be differentially admissible on a subinterval of X if on this subinterval $y(x)$ is continuous, has piecewise continuous derivatives, and satisfies (9.3); for a compact subinterval $[c,\ d]$ of X the symbol $H_+[c,\ d]$ will again denote the condition that $I[y\,;\ c,\ d]>0$ for arbitrary differentially admissible $y(x)$ which are not identically zero on $[c,\ d]$ and satisfy $y(c)=0=y(d)$. For the problem now considered the symbol H_R signifies the condition that for all $x\in X$ we have $\pi^*R(x)\pi>0$ for arbitrary non-zero vectors π satisfying the restraint $\varphi(x)\pi=0$; in view of the basic assumption that (9.2) is non-singular throughout X it follows that H_R holds whenever there is a single $s\in X$ such that $\pi^*R(s)\pi>0$ for arbitrary non-zero vectors π satisfying $\varphi(s)\pi=0$.

With the above definitions, the result of Theorem 4.1 is valid for the equation (2.4) now under consideration. In this connection, it is to be commented that if we write $y=(y_\alpha^1+iy_\alpha^2)$, $(\alpha=1, \cdots, n)$, and denote by z the real $2n$-dimensional vector function with components $(y_1^1, \cdots, y_n^1, y_1^2, \cdots, y_n^2)$, then $\omega(x, y, y')$ is a quadratic form $\omega_0(x, z, z')$ in (z, z') with real coefficients, and (9.3) is equivalent to a real $2m$-dimensional vector differential equation $\Phi_0(x, z, z')=0$. Moreover, $H_+[c, d]$ and H_R are individually equivalent to the corresponding conditions $H_+^0[c, d]$ and H_R^0 for the associated real problem in z, and for this latter problem the conclusion that $H_+^0[c, d]$ implies H_R^0 is a well-known result of the calculus of variations, (see, for example, Bliss [3; Theorem 78.2 and Lemma 81.2]). For a problem of the sort formulated above which satisfies H_R, the method of proof of Lemma 89.1 of Bliss [3] yields the result that $H_+[c, d]$ holds if and only if there is a matrix $(U(x); V(x))$ of conjoined solutions of (2.4) with $U(x)$ non-singular on $[c, d]$, and the method of proof of Lemma 89.2 of Bliss [3] establishes that $H_+[c, d]$ holds if and only if (2.4) is non-oscillatory on $[c, d]$.

For a differential system (2.4) of the type now under consideration, the result of Theorem 4.2 is valid only if this system is normal on the interval $[c, d]$, since if $y(x)$ is differentially admissible then $y(c)$, $y(d)$ must satisfy $v^*(d)y(d)-v^*(c)y(c)=0$ with all vector functions $v(x)$ belonging to abnormal solutions $u\equiv 0$, $v(x)$ of (2.4) on $[c, d]$. On the other hand, if (2.4) is normal on every subinterval of X then Theorems 4.2 and 4.3 hold, as well as relations (4.1) and (4.2) for vector functions that are differentially admissible for the problem of this section.

From the above remarks it follows that for systems (2.4) with coefficient matrices given by (9.5), and *which are normal on every subinterval of* X, the various theorems of Sections 3–6 remain valid, with no changes in proofs required. An important illustration of this class of systems (2.4) is afforded by certain systems (2.4) that are equivalent to self-adjoint scalar differential equations of even order. Indeed, suppose that $p_j(x)$, $(j=0, 1, \cdots, 2n)$, are real-valued functions with $p_{2n}(x)\neq 0$ on X and $p_j(x)$ of class $C^{(j/2)}$ or $C^{((j+1)/2)}$ according as j is even or odd, and let $R(x)$, $Q(x)$, $P(x)$ be diagonal matrices with $P_{\alpha\alpha}(x)=(-1)^{\alpha-1}p_{2\alpha-2}(x)$, $Q_{\alpha\alpha}(x)=i(-1)^\alpha p_{2\alpha-1}(x)$, $(\alpha=1, \cdots, n)$, $R_{\alpha\alpha}(x)\equiv 0$ for $\alpha<n$ and $R_{nn}(x)=(-1)^n p_{2n}(x)$, while $\Phi(x, y, \pi)=(\pi_\beta-y_{\beta+1})$, $(\beta=1, \cdots, n-1)$. The corresponding vector differential system (2.4) is readily seen to be normal on every subinterval, and $(u(x); v(x))$ is a solution of this system if and only if $u_\alpha(x)=y^{(\alpha-1)}(x)$, $(\alpha=1, \cdots, n)$, where $y(x)$ is a solution of the self-adjoint differential equation

$$\sum_{\alpha=0}^{n}[p_{2\alpha}(x)y^{(\alpha)}]^{(\alpha)}+i\sum_{\alpha=1}^{n}([p_{2\alpha-1}(x)y^{(\alpha-1)}]^{(\alpha)}+[p_{2\alpha-1}(x)y^{(\alpha)}]^{(\alpha-1)})=0 .$$

It is to be noted also that for a system (2.4) normal on every subinterval the results of Theorems 8.1 and 8.2 are valid, with (8.4) replaced by the condition that $\omega(x, y, \pi) \geqq 0$ for arbitrary $(x, y; \pi)$ with $x \in X_0$, and satisfying $\Phi(x, y, \pi)=0$.

Finally, it is to be remarked that for an equation (2.4) with coefficients given by (9.5), and which is not normal on every subinterval of X, there do exist suitable modifications of Theorems 4.2 and 4.3 which with an altered definition of principal solution enable one to establish certain results corresponding to those of Sections 5,6 ; however, the details of these results will not be presented here.

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NORTHWESTERN UNIVERSITY.