THE FREE LATTICE GENERATED BY A SET OF CHAINS

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1. Introduction. P. M. Whitman [4] defined an ordering of the set of lattice polynomials generated by a set of unrelated elements. R. P. Dilworth [3] generalized this ordering to apply to the case of lattice polynomials generated by an arbitrary partly ordered set P. Dilworth proved that this ordering gives a lattice isomorphic to the free lattice, $\mathrm{FL}(P)$, which is generated by P and which preserves bounds of pairs of elements of P. R. A. Dean [2] considered the ordering of lattice polynomials which preserves order of pairs of elements in P and which leads to the completely free lattice $\mathrm{CF}(P)$. He shows that $\mathrm{CF}(P)$ and $\mathrm{FL}(P)$ are identical in the case in which P is a set of unrelated chains.

This article is a further study of $\mathrm{FL}(P)$ where P is a set of unrelated chains. An arbitrary element of P will be denoted by p or q. The set of chains consisting of

$$a_{11}<a_{12}< \cdots <a_{1n_1}\,;\ a_{21}<a_{22}< \cdots <a_{2n_2}\,;\ \cdots\,;\ a_{m1}<a_{m2}< \cdots <a_{mn_m}\,;$$

where a_{ij} and a_{kl} are unrelated when $i\neq k$, will be denoted by $n_1+n_2+\cdots+n_m$.

DEFINITION 1. *Lattice polynomials* over P are defined inductively as follows.

(1) The elements $p, q, \cdots$, of P are lattice polynomials over P.

(2) If A and B are lattice polynomials over P, then so are $A\cup B$ and $A\cap B$.

DEFINITION 2. The *rank*, $r(A)$, of a lattice polynomial A is defined inductively as follows.

(1) $r(A)=0$ if and only if A is in P.

(2) $r(A\cup B)=r(A\cap B)=r(A)+r(B)+1$.

DEFINITION 3. The dual polynomial, A', of a polynomial A of $\mathrm{FL}(n_1+n_2+\cdots+n_m)$ is defined inductively as follows.

(1) If $A\equiv a_{ij}$, then $A'\equiv a_i(n_i-j+1)$.

(2) If $A\equiv A_1\cup A_2$, then $A'\equiv A_1'\cap A_2'$.

(3) If $A\equiv A_1\cap A_2$, then $A'\equiv A_1'\cup A_2'$.

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From Definition 2, Lemma 1, and Lemma 2 of [2], and the fact that FL(P) and CF(P) are identical in the case under consideration, we have the following theorem.

THEOREM 1. *Let P be a partly ordered set consisting of a set of unrelated chains. In* FL(P), *each relation $A \geqq B$ is one of six types. These types and necessary and sufficient conditions which apply to each case are the following.*

(A) $p \geqq q$ *if and only if* $p \geqq q$ *in* P.

(B) $p \geqq B_1 \cap B_2$ *if and only if* $p \geqq B_1$ *or* $p \geqq B_2$.

(C) $A_1 \cup A_2 \geqq p$ *if and only if* $A_1 \geqq p$ *or* $A_2 \geqq p$.

(D) $A \geqq B_1 \cup B_2$ *if and only if* $A \geqq B_1$ *and* $A \geqq B_2$.

(E) $A_1 \cap A_2 \geqq B$ *if and only if* $A_1 \geqq B$ *and* $A_2 \geqq B$.

(F) $A_1 \cup A_2 \geqq B_1 \cap B_2$ *if and only if* $A_1 \geqq B_1 \cap B_2$ *or* $A_2 \geqq B_1 \cap B_2$ *or* $A_1 \cup A_2 \geqq B_1$ *or* $A_1 \cup A_2 \geqq B_2$.

2. **FL(2+2).** Let $a_1 < a_2$ and $b_1 < b_2$ be the generators of FL(2+2). The notation of the elements of FL(2+2) is defined in the following recursive manner.

$$A_1 = a_2 , \quad B_1 = b_2$$

and, for $n > 1$,

$$A_n = a_2 \cap (a_1 \cup B_{n-1}) , \quad B_n = b_2 \cap (b_1 \cup A_{n-1}) .$$
$$C_n = a_1 \cup B_n .$$
$$D_n = b_1 \cup A_n .$$
$$P_n = A_n \cup B_n .$$
$$Q_n = C_n \cap D_n .$$
$$M_1 = a_1 \cup b_1 .$$
$$M_2 = (a_2 \cap b_2) \cup a_1 \cup b_1 .$$
$$V_1 = b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) .$$
$$V_2 = (a_2 \cap b_2) \cup (b_2 \cap (a_1 \cup b_1)) .$$
$$V_3 = b_2 \cap (a_1 \cup b_1) .$$
$$W_1 = a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) .$$
$$W_2 = (a_2 \cap b_2) \cup (a_2 \cap (a_1 \cup b_1)) .$$
$$W_3 = a_2 \cap (a_1 \cup b_1) .$$

These elements and their dual elements are all the elements of FL(2+2). This is shown by considering the $\cap$ and $\cup$ tables of the above elements and their dual elements. Since the generators of FL(2+2) are among these elements and their duals, in order to show that these are all the elements of FL(2+2) it is sufficient to show that this set is closed under $\cup$ and $\cap$. Actually, it is sufficient to show the set consisting of the above elements and their duals is closed under $\cup$ (or $\cap$). This follows from the fact that the intersection of two elements, $A\cap B$, can be expressed as $A\cap B=(A')'\cap(B')'=(A'\cup B')'$, the latter being found from the $\cup$ table. The diagram of FL(2+2) as shown in Figure 1 is obtained from the relations found in the $\cup$ ($\cap$) table. Rather than give the entire $\cup$ table, the diagram of FL(2+2) is given and a typical element, $A_i\cup B_j$, of the $\cup$ table is considered. The other parts of the $\cup$ table are obtained in a similar manner. First, we consider the following theorem.

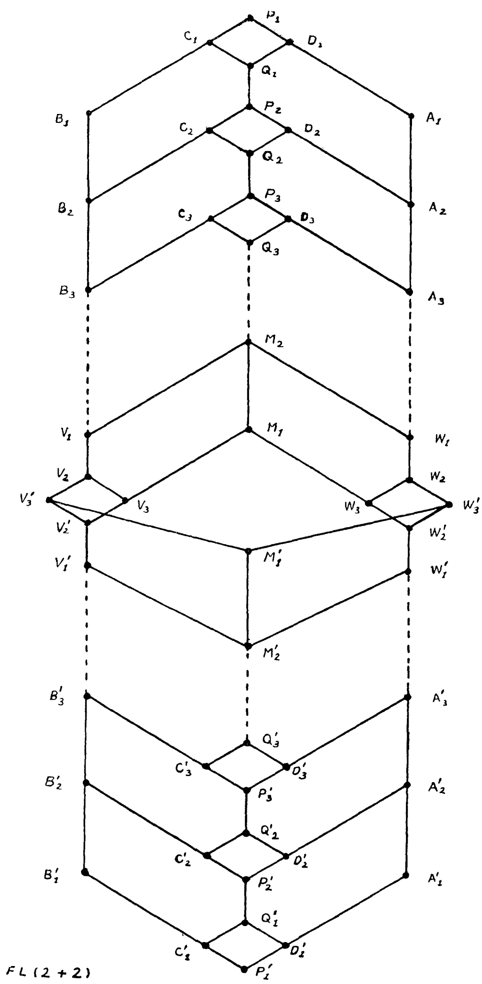

Fig. 1

THEOREM 2. *In* FL(2+2) *we have* $A_1>A_2>\cdots$, $B_1>B_2>\cdots$, $C_1>C_2>\cdots$, *and* $D_1>D_2>\cdots$.

Proof. The proof of this theorem is similar to the proof, in §4 of [2], that FL(2+2) contains four infinite chains. In [2] the symbols, A_n, B_n, C_n, and D_n represent the same elements as A_{2n-1}, D_{2n-1}, B_{2n}, and C_{2n}, respectively, of this paper. Thus we conclude from the results of [2] that $A_1>A_3>A_5>\cdots$, $B_2>B_4>B_6>\cdots$, $C_2>C_4>C_6>\cdots$, and $D_1>D_3>D_5>\cdots$. The conclusion of this theorem follows in a similar manner.

We now show that

$$A_i\cup B_j=\begin{cases}P_i, & i=j\ ,\\ D_i, & i<j\ ,\\ C_j, & j<i\ .\end{cases}$$

$A_i\cup B_i=P_i$ by definition. Since $B_j\geqq b_1$, it follows by (D) of Theorem 1 that $A_i\cup B_j\geqq b_1\cup A_i=D_i$. Now consider the relation

$$b_1 \cup A_i \geqq A_i \cup B_j = (b_2 \cap (b_1 \cup A_{j-1})) \cup A_i\ , \qquad i<j\ .$$

Since $A_n \geqq A_m$, $n \leqq m$ we have $b_1 \cup A_i \geqq b_1 \cup A_{j-1}$, $i<j$. Hence by (F) of Theorem 1, $b_1 \cup A_i \geqq B_j$. Since $b_1 \cup A_i \geqq A_i$, it follows by (D) of Theorem 1 that $b_1 \cup A_i \geqq B_j \cup A$. This completes the proof that $D_i = b_1 \cup A_i = B_j \cup A_i$, $i<j$.

It follows in a similar manner that $A_i \cup B_j = C_j$, $j<i$.

3. Order-convergence.

DEFINITION 4. In a lattice, $\{b_i\}$ is said to *order-converge* to b if sequences $\{u_i\}$ and $\{v_i\}$ exist such that

$$v_i \geqq v_{i+1} \geqq b_{i+1} \geqq u_{i+1} \geqq u_i$$

for all i, and lub $\{u_i\}$ = glb $\{v_i\} = b$.

As seen from Figure 1, or as can be shown directly using Theorem 1, it is clear that $A_n \geqq W_1$ for each n, $B_n \geqq V_1$ for each n, $C_n \geqq M_2$ and $D_n \geqq M_2$ for each n. Thus we conclude that W_1 is a lower bound to the set $\{A_n\}$, V_1 is a lower bound to the set $\{B_n\}$, and M_2 is a lower bound to each of the sets $\{C_n\}$ and $\{D_n\}$.

THEOREM 3. *In* FL(2+2) W_1 *is* glb $\{A_n\}$, V_1 *is* glb $\{B_n\}$, *and* M_2 *is* glb $\{C_n\}$ *and* glb $\{D_n\}$.

Proof. Since each of W_1, V_1, and M_2 is a lower bound to the indicated sets, in order to prove the theorem it is sufficient to prove the following four statements.

(1) If $A_n \geqq K$ for each n, then $W_1 \geqq K$.

(2) If $B_n \geqq K$ for each n, then $V_1 \geqq K$.

(3) If $C_n \geqq K$ for each n, then $M_2 \geqq K$.

(4) If $D_n \geqq K$ for each n, then $M_2 \geqq K$.

The proof is as follows. Let $r(K)=0$. If $A_n \geqq K$ for each n, then $K \equiv a_1$. In this case $W_1 \geqq K$. Similarly if $B_n \geqq K$ for each n, then $K \equiv b_1$ and hence $V_1 \geqq K$. If $C_n \geqq K$ for each n, then $K \equiv a_1$ or $K \equiv b_1$. In either case $M_2 \geqq K$. Similarly if $D_n \geqq K$ for each n, then $M_2 \geqq K$.

Proceeding by induction, we assume, when $r(K)<k$, that the four conditions (1), (2), (3), and (4) each hold. We now consider the cases when $r(K)=k$ and $K \equiv K_1 \cap K_2$ or $K \equiv K_1 \cup K_2$. First, let $K \equiv K_1 \cap K$. If $A_n \geqq K$ for each n, then $a_2 \geqq K$ and $a_1 \cup B_{n-1} \geqq K$ for each $n>1$. The latter is true if and only if one of the following holds.

(a) $a_1 \geqq K_1 \cap K_2$,

(b) $B_{n-1} \geqq K_1 \cap K_2$,

(c) $C_{n-1} \geqq K_1$,

(d) $C_{n-1} \geqq K_2$.

If (a) holds, then $W_1 \geqq K_1 \cap K_2$. We now show that if (a) does not hold, then one of (b), (c), or (d) must hold for each $n > 1$. Since $B_m > B_{m+1}$, if $B_m \not\geqq K_1 \cap K_2$, then $B_n \not\geqq K_1 \cap K_2$, $n > m$. Otherwise $B_m > B_n \geqq K_1 \cap K_2$ implies $B_m > K_1 \cap K_2$. Similarly $C_m \not\geqq K$ implies $C_n \not\geqq K$ when $n > m$. Thus if (b), (c), and (d) fail to hold for some $n = i, j$, or k, respectively, then (b), (c), and (d) fail to hold for $n = \max(i, j, k)$. This result with the assumption that (a) is false contradicts the fact that $a_1 \cup B_{n-1} \geqq K_1 \cap K_2$ for each $n > 1$. Thus one of (b), (c), or (d) holds for each $n > 1$ if (a) fails to hold. If (*b*) is true, then $b_2 \geqq K_1 \cap K_2$. This, with $a_2 \geqq K_1 \cap K_2$, implies $W_1 \geqq K_1 \cap K_2$. By the induction hypothesis, (c) or (d) implies $M_2 \geqq K_1$ or K_2, thus $M_2 \geqq K_1 \cap K_2$. This, with $a_2 \geqq K_1 \cap K_2$, implies $W_1 \geqq K_1 \cap K_2$.

Thus we conclude that $A_n \geqq K_1 \cap K_2$ for each n and $r(K_1 \cap K_2) = k$ imply $W_1 \geqq K_1 \cap K_2$. Similarly $K \equiv K_1 \cup K_2$ and $A_n \geqq K_1 \cup K_2$ for each n imply $W_1 \geqq K_1 \cup K_2$. It is shown in a similar manner that $D_n \geqq K$ for each n implies $M_2 \geqq K$; $B_n \geqq K$ for each n implies $V_1 \geqq K$; and $C_n \geqq K$ for each n implies $M_2 \geqq K$ where $r(K) = k$ in each case. Thus, by induction, the proof of the theorem is complete.

COROLLARY. *In* FL(2+2) *the sequence* $\{A_n\}$ *order-converges to* W_1, $\{B_n\}$ *order-converges to* V_1, $\{C_n\}$ *and* $\{D_n\}$ *each order-converge to* M_2.

Proof. In the case of $\{A_n\}$ we let $u_n = W_1$ and $v_n = A_n$. Then each of the conditions of Definition 4 is satisified where lub $\{u_n\} =$ glb $\{v_n\} = W_1$. Thus $\{A_n\}$ order-converges to W_1. The other conclusions of the corollary follow in like manner.

We may generalize these results in the following manner. Let $n_1 + n_2 + \cdots + n_m$ be a set of chains in which two chains each have two or more elements. From each of these two chains take the least elements, a_{i1}, a_{i2} and a_{j1}, a_{j2}. If we replace a_r with a_{ir} and b_r with a_{jr}, $r = 1, 2$, in A_n, B_n, C_n, D_n, W_1, V_1, and M_2, the resulting elements will be mutually related in the same manner as A_n, B_n, C_n, D_n, W_1, V_1, and M_2 since the set a_{i1}, a_{i2}, a_{j1}, a_{j2} is isomorphic to $2+2$.

If we substitute a_{ir} and a_{jr} in A_n, B_n, etc. as indicated above, and if we designate the resulting elements by the same symbols as the symbols from which they are obtained, we obtain the following theorem.

THEOREM 4. *In* $\mathrm{FL}(n_1 + n_2 + \cdots + n_m)$, *where* $n_i \geqq 2$ *and* $n_j \geqq 2$ *for some unequal* i, j, *the set* $\{A_n\}$ *order-converges to* W_1 ; $\{B_n\}$ *order-converges to* V_1 ; $\{C_n\}$ *and* $\{D_n\}$ *each order-converge to* M_2.

4. FL(4+1). The notation for the elements of FL(4+1) is defined recursively in the following manner.

$$A_1 = a_3\ , \quad B_1 = a_1 \cup b\ , \quad A_2 = a_3 \cap (a_2 \cup B_1)\ , \quad B_2 = (a_1 \cup b) \cap a_4\ ,$$

and for $n > 2$,

$$A_n = a_3 \cap (a_2 \cup B_{n-1})\ , \quad B_n = B_1 \cap ((a_4 \cap b) \cup A_{n-2})\ .$$

$$C_n = (a_4 \cap b) \cup A_n\ .$$

$$D_n = a_2 \cup B_n\ .$$

$$F_n = A_n \cup B_n\ .$$

$$G_n = A_n \cup B_{n+1}\ .$$

$$H_1 = a_4 \cap D_1 \text{ and, for } n > 1,\ H_n = C_{n-1} \cap D_n\ .$$

$$E_1 = a_4 \text{ and, for } n > 1,\ E_n = C_{n-1} \cap D_{n-1}\ .$$

$$S_n = A_n \cup H_n\ .$$

$$T_n = D_n \cap G_n\ .$$

$$P_1 = a_4 \cap F_1 \text{ and, for } n > 1,\ P_n = C_{n-1} \cap F_n\ .$$

$$Q_n = B_n \cup E_n\ .$$

$$V_1 = a_3 \cap ((a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_4 \cap b))\ .$$

$$V_2 = (a_3 \cap (a_1 \cup b)) \cup (a_3 \cap (a_2 \cup (a_4 \cap b))\ .$$

$$V_3 = a_3 \cap (a_2 \cup (a_4 \cap b))\ .$$

$$W_1 = (a_1 \cup b) \cap ((a_3 \cap (a_1 \cup b) \cup a_2 \cup (a_4 \cap b))\ .$$

$$W_2 = (a_3 \cap (a_1 \cup b)) \cup ((a_1 \cup b) \cap (a_2 \cup (a_4 \cap b)))\ .$$

$$W_3 = (a_1 \cup b) \cap (a_2 \cup (a_4 \cap b)).$$

$$M_1 = a_2 \cup (a_4 \cap b)\ .$$

$$M_2 = (a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_4 \cap b)\ .$$

$$b\ .$$

As in the case of FL(2+2), that these elements and their dual elements are all the elements of FL(4+1) follows from the fact that they include the generators of FL(4+1) and are closed under $\cup$ and $\cap$. The relations between the elements of FL(4+1) as shown by the diagram in Figure 2 are proved similar to the way the relations of the elements of FL(2+2) are proved. The following results are stated without proof since the proofs are similar to the proofs of the corresponding statements regarding FL(2+2).

THEOREM 5. *FL(4+1) contains the infinite chains* $A_1 > A_2 > \cdots$, $B_1 > B_2 > \cdots$, $C_1 > C_2 > \cdots$, *and* $D_1 > D_2 > \cdots$.

THEOREM 6. *In FL(4+1),* $\{A_n\}$ *order-converges to* V_1, $\{B_n\}$ *order-converges to* W_1, $\{C_n\}$ *and* $\{D_n\}$ *each order-converge to* M_2.

Theorem 6 can be generalized in the same manner as was the corollary to Theorem 3. Let $n_1+n_2+\cdots+n_m$ be a set of two or more chains in which one chain contains four or more elements. From the chain containing four elements, take the four least elements a_{i1}, a_{i2}, a_{i3}, and a_{i4}. From another chain, take the least element a_{j1}. If we substitute a_{ir} for a_r, $r=1, 2, 3, 4$, and a_{j1} for b in A_n, B_n, C_n, D_n, M_2, W_1, and V_1, and if we designate the resulting elements by the same symbols as the symbols from which they are obtained then we get the following corollary in the same way as Theorem 4 was obtained.

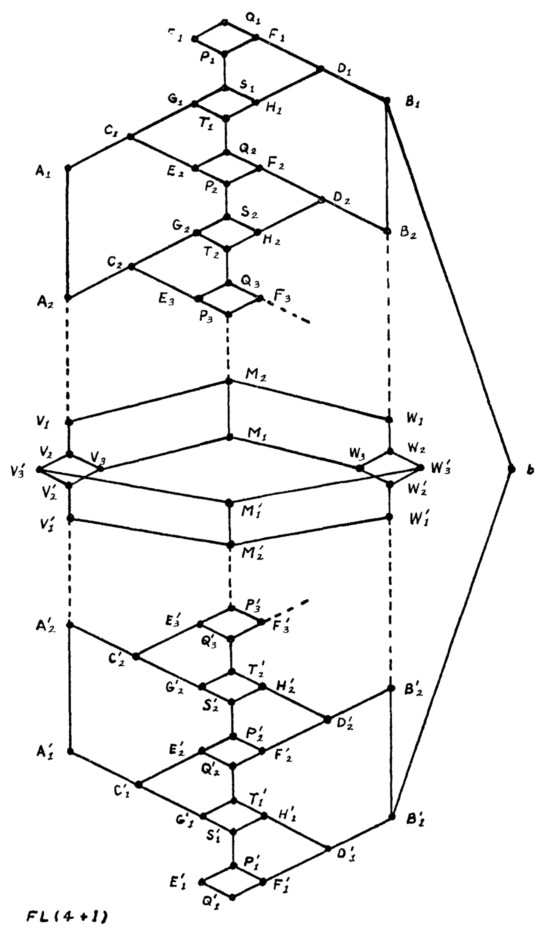

Fig. 2

COROLLARY. *In* $\mathrm{FL}(n_1+n_2+\cdots+n_m)$, *where* $n_i \geqq 4$ *for some* i *and* $m \geqq 2$, $\{A_n\}$ *order-converges to* V_1, $\{B_n\}$ *order-converges to* W_1, $\{C_n\}$ *and* $\{D_n\}$ *each order-converge to* M_2.

5. FL(1+1+1) as a sublattice of FL(n_1+n_2), $n_1 \geqq 3$ and $n_2 \geqq 2$, or $n_1 \geqq 5$ and $n_2 \geqq 1$. From Theorem 4 and Theorem 6 of [2] we have the following theorem.

THEOREM 7. *Let* U *be a subset of* $\mathrm{FL}(n_1'+n_2'+\cdots+n_m')$ *and let* $U=\{u_{ij}\}$ *be isomorphic to* $n_1+n_2+\cdots n_m$. *Let* $u_{ij} \geqq u_{pq}$ *if and only if* $i=p$ *and* $j \geqq q$. FL(U) *is isomorphic to* $\mathrm{FL}(n_1+n_2+\cdots+n_m)$ *if and only if* $\cup\, u_{ij} \geqq u_{ab}$ *implies* $i=a$ *and* $j \geqq b$ *for some* i, j, *and dually.*

THEOREM 8. $\mathrm{FL}(n_1+n_2)$, $n_1 \geqq 3$ *and* $n_2 \geqq 2$, *contains a sublattice isomorphic to* FL(1+1+1).

Proof. In $\mathrm{FL}(n_1+n_2)$ let $u_{11}=a_2$, $u_{21}=a_3 \cap (a_1 \cup b_1)$, and $u_{31}=a_1 \cup (a_3 \cap b_2)$. In order to show that the sublattice generated by u_{11}, u_{21}, and u_{31} is isomorphic to FL(1+1+1) it is sufficient to show, by Theorem 7, that the u_{i1} are unrelated and that $u_{i1} \cup u_{j1} \ngeqq u_{k1}$ and $u_{i1} \ngeqq u_{j1} \cap u_{k1}$ when i, j, and k are all different.

A direct application of Theorem 1 shows $u_{i1} \not\geqq u_{j1}$ and $u_{j1} \not\geqq u_{i1}$ when $i \neq j$, thus the u_{i1} form an unrelated set. A straightforward application of Theorem 1 also shows that $u_{i1} \cup u_{j1} \not\geqq u_{k1}$ and $u_{i1} \not\geqq u_{j1} \cap u_{k1}$ when i, j, and k are all different. Hence FL(u_{11}, u_{21}, u_{31}) is isomorphic to FL(1+1+1).

THEOREM 9. FL(n_1+n_2), $n_1 \geqq 5$ *and* $n_2 \geqq 1$, *contains a sublattice isomorphic to* FL(1+1+1).

Proof. A proof similar to the proof of Theorem 8 shows that the sublattice of FL(n_1+n_2) generated by $u_{11}=a_3$, $u_{21}=a_4 \cap (a_2 \cup (a_5 \cap b))$, and $u_{31}=a_2 \cup (a_4 \cap (a_1 \cup b))$ is isomorphic to FL(1+1+1).

6. FL($n_1+n_2+ \cdots +n_m$) as a sublattice of FL(1+1+1). In FL(1+1+1), with generators x_1, x_2, x_3, define $u_0=x_1$, and for $n \geqq 1$,

$$u_n = x_1 \cup (x_3 \cap (x_2 \cup (x_1 \cap (x_3 \cup (x_2 \cap u_{n-1}))))) .$$

Whitman has shown [5, p. 112] that $u_0 < u_1 < u_2 < \cdots$ (In his notation $u_i \equiv t_{6i+1}$).

THEOREM 10. *The free lattice generated by* $3m$ *unrelated elements,* FL($1+1+ \cdots +1$), *contains a sublattice isomorphic to* FL($n_1+n_2+ \cdots +n_m$).

Proof. Denote the generators of FL($1+1+ \cdots +1$) by $x_1, x_2, \cdots, x_{3m}$ and choose m sets of elements of FL($n_1+n_2+ \cdots +n_m$) in the following manner. For each i, $i=1, 2, \cdots, m$, let $u_{i0}=x_{3i-2}$, and for $j \geqq 1$,

$$u_{ij} = x_{3i-2} \cup (x_{3i} \cap (x_{3i-1} \cup (x_{3i-2} \cap (x_{3i} \cup (x_{3i-1} \cap u_{i\,j-1}))))) .$$

We note that the polynomials of each set u_{ij}, i fixed and $j=0, 1, 2, \cdots, n_i$, are the same, except for the subscripts of the x's, as the polynomials u_j defined immediately before this theorem. Since the x's are unrelated, the reasoning that led to the conclusion $u_0 < u_1 < u_2 < \cdots$ applies to the u_{ij}. We then conclude that $u_{i0} < u_{i1} < u_{i2} < \cdots$, $i=1, 2, \cdots, m$.

Since $x_{3i-2} \not\geqq x_{3p-2}$ and $x_{3i} \not\geqq x_{3p-2}$ when $i \neq p$, $u_{ij} \not\geqq u_{pq}$. Similarly $u_{pq} \not\geqq u_{ij}$ when $i \neq p$. Thus u_{ij} is unrelated to u_{pq} when $i \neq p$. Letting U denote the set of polynomials u_{ij}, $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n_i$ for each i, we see that U is isomorphic to $n_1+n_2+ \cdots +n_m$. By means of Theorem 7, we shall show that the sublattice generated by U is isomorphic to FL($n_1+n_2+ \cdots +n_m$).

If $u_{ab} \geqq \cap u_{ij}$, then it is necessary that one of the following holds.

(1) $x_{3a-2} \geqq \cap u_{ij}$,

(2) $x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))) \geqq \cap u_{ij}$,

(3) $u_{ab} \geqq u_{ij}$ for some i, j.

Condition (1) is true if and only if $x_{3a-2} \geqq$ some u_{ij}. Since $j \neq 0$, this is false; hence (1) cannot hold. Similarly (2) is false since $x_{3a} \ngeqq$ some u_{ij}. Hence (3) must hold, but this is true if and only if $a=i$ and $b \geqq j$. Thus $u_{ab} \geqq \cap u_{ij}$ implies $a=i$ and $b \geqq j$.

If $\cup u_{ij} \geqq u_{ab}$, then it is necessary that

$$\cup u_{ij} \geqq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))) .$$

This is true if and only if one of the following holds.

(a) $\cup u_{ij} \geqq x_{3a}$,

(b) $\cup u_{ij} \geqq x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))$,

(c) some $u_{ij} \geqq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$.

Conditions (a) and (b) are false, respectively, since neither $u_{ij} \geqq x_{3a}$ nor $u_{ij} \geqq x_{3a-1}$ is ever true. Thus (c) must hold. If $i=a$ and $j<b$, since $u_{aj} \geqq x_{3a-2}$, it follows that (c) must be false, otherwise it implies $u_{aj} \geqq u_{ab}$ when $j<b$ contrary to the known relationship $u_{ab} > u_{aj}$, $j<b$. If $a \neq i$, (c) implies at least one of the following.

(1) $x_{3i-2} \geqq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$,

(2) $x_{3i} \geqq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$,

(3) $u_{ij} \geqq x_{3a}$,

(4) $u_{ij} \geqq x_{3a-1}$.

Since $i \neq a$, each of these four conditions is false. Thus $i \neq a$ contradicts (c). We then conclude that if (c) is true, $i=a$ and $j \geqq b$. Furthermore, we conclude that $\cap u_{ij} \geqq u_{ab}$ implies that $i=a$ and $j \geqq b$ for some i, j, and dually. By Theorem 7, it follows that the sublattice generated by U is isomorphic to $\mathrm{FL}(n_1+n_2+ \cdots +n_m)$.

COROLLARY 1. $\mathrm{FL}(1+1+1)$ *contains a sublattice isomorphic to* $\mathrm{FL}(n_1+n_2+ \cdots +n_m)$.

Proof. $\mathrm{FL}(1+1+1)$ contains a sublattice isomorphic to $\mathrm{FL}(M)$, where M is a set of $3m$ unrelated elements, [5, Theorem 6], and $\mathrm{FL}(M)$ in turn contains a sublattice isomorphic to $\mathrm{FL}(n_1+n_2+ \cdots +n_m)$.

COROLLARY 2. $\mathrm{FL}(m_1+m_2)$, $m_1 \geqq 3$ *and* $m_2 \geqq 2$, *or* $m_1 \geqq 5$ *and* $m_2 \geqq 1$, *contains a sublattice isomorphic to* $\mathrm{FL}(n_1+n_2+ \cdots +n_m)$.

Proof. By Theorems 8 and 9, $\mathrm{FL}(m_1+m_2)$ contains a sublattice isomorphic to $\mathrm{FL}(1+1+1)$. In turn, Corollary 1 implies that $\mathrm{FL}(1+1+1)$ contains a sublattice isomorphic to $\mathrm{FL}(n_1+n_2+ \cdots +n_m)$.

We note that the reasoning in the proof of Theorem 10 is valid if m is any cardinal number and each chain contains a finite or countable number of elements. In the corollaries to Theorem 10 m must be countable since FL(1+1+1) contains only a countable number of elements.

7. Order-convergence in $\mathbf{FL}(n_1+n_2+\cdots+n_m)$. By Theorem 4 and the corollary to Theorem 6, we see that $\mathrm{FL}(n_1+n_2+\cdots+n_m)$, where $n_i \geqq 2$ and $n_j \geqq 2$ for some distinct i, j, or some $n_i \geqq 4$ and $m \geqq 2$, contains an infinite subset that order-converges. We now show that in case $m \geqq 3$ there exists an infinite subset that order-converges. We summarize this in the following theorem and prove the case $m \geqq 3$ immediately following.

THEOREM 11. *$\mathrm{FL}(n_1+n_2+\cdots+n_m)$, where $n_i \geqq 2$ and $n_j \geqq 2$ for some distinct i and j; or some $n_i \geqq 4$ and $m \geqq 2$; or $m \geqq 3$, contains an infinite subset that order-converges.*

Proof. Denote the least elements of three different chains of $\mathrm{FL}(n_1+n_1+\cdots+n_m)$ by x_1, x_2, x_3 and define

$$u_0 = x_1 ,$$

$$u_n = x_1 \cup (x_3 \cap (x_2 \cup (x_1 \cap (x_3 \cup (x_2 \cap u_{n-1}))))) , \qquad n \geqq 1 ,$$

$$v_0 = x_1 ,$$

$$v_n = x_1 \cup (x_2 \cap (x_3 \cup (x_1 \cap (x_2 \cup (x_3 \cap v_{n-1}))))) , \qquad n \geqq 1 .$$

As mentioned previously, Whitman has shown $u_0 < u_1 < u_2 < \cdots$ [5]. Similarly $v_0 < v_1 < v_2 < \cdots$.

We now define the following elements in $\mathrm{FL}(n_1+n_2+\cdots n_m)$.

$$a_n = (x_2 \cup (x_1 \cap x_3)) \cap u_n , \qquad n = 1, 2 .$$

$$b_n = (x_3 \cup (x_1 \cap x_2)) \cap v_n , \qquad n = 1, 2 .$$

$A_1 = a_2$, $B_1 = b_2$, and, for $n > 1$,

$$A_n = a_2 \cap (a_1 \cup B_{n-1}) , \quad B_n = b_2 \cap (b_1 \cup A_{n-1}) .$$

$$C_n = a_1 \cup B_n .$$

$$D_n = b_1 \cup A_n .$$

$$W_1 = a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) .$$

$$V_1 = b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) .$$

$$M_2 = (a_2 \cap b_2) \cup a_1 \cup b_1 .$$

These elements correspond to the elements of FL(2+2) designated by the same symbols. By means of Theorem 1, in a rather tedious but straightforward manner, it is shown that the above elements are related in the same manner as their corresponding elements in FL(2+2). Thus

$$A_1 > A_2 > \cdots > W_1\,,$$
$$B_1 > B_2 > \cdots > V_1\,,$$
$$C_1 > C_2 > \cdots > M_2\,,$$
$$D_1 > D_2 > \cdots > M_2\,.$$

A proof similar to the proof of Theorem 3, although more tedious, shows that $W_1 = \text{glb}\,\{A_n\}$, $V_1 = \text{glb}\,\{B_n\}$, and $M_2 = \text{glb}\,\{C_n\}$ and $\{D_n\}$. The first step of the induction is vacuously true. If $A_n \geqq K$ for each n where $r(K)=0$, then it is necessary that $a_2 \geqq K$. In turn this implies $x_2 \cup (x_1 \cap x_3) \geqq K$ and $u_2 \geqq K$. Since K is an element of $n_1 + n_2 + \cdots + n_m$, it follows from these two relations and Theorem 1, that at least two of x_1, x_2, x_3 must be $\geqq K$. Since each x_i is from a different unrelated chain, this is false. Hence $A_n \geqq K$, for each n and $r(K)=0$, vacuously implies $W_1 \geqq K$. Similarly, statements (2), (3), and (4) at the beginning of the proof of Theorem 3 are vacuously true when $r(K)=0$. The remainder of the proof is similar to the proof of Theorem 3.

We have answered, in the affirmative, the question posed by Whitman [5], "Does some infinite set in FL(1+1+ $\cdots$ +1) order-converge?" Theorem 11 states that each infinite free lattice generated by a set of chains contains an infinite subset that order-converges.

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