A PRÜFER TRASFORMATION FOR DIFFERENTIAL SYSTEMS

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1. Introduction. For a real self-adjoint matrix differential equation

$$(1.1) \qquad (P(x)Y')' + F(x)Y = 0 , \qquad a \leqq x < \infty,$$

in the $n \times n$ matrix $Y(x)$ it has been established recently by J. H. Barrett [1] that there is a transformation analogous to the well-known Prüfer [8] polar-coordinate transformation for a real self-adjoint linear homogeneous differential equation of the second order. In the form for a solution $Y(x)$ of (1.1) obtained by Barrett the roles of the sine and cosine functions in the Prüfer transformation are assumed by the respective $n \times n$ matrices $S(x)$, $C(x)$ satisfying a matrix differential system

$$(1.2) \qquad S = Q(x)C , \quad C' = -Q(x)S , \quad S(a) = 0 , \quad C(a) = E ,$$

where $Q(x)$ is an associated real symmetric matrix. Barrett uses the method of successive approximations to determine $Q(x)$ as a solution of the functional equation $Q = CP^{-1}C^* + SFS^*$, where S and C are related to Q by (1.2).

The present paper is concerned with the derivation of similar results for a matrix differential system

$$(1.3) \qquad Y' = G(x)Z , \quad Z' = -F(x)Y$$

where $G(x)$, $F(x)$ are continuous $n \times n$ hermitian matrices; in particular, if $G(x)$ is of constant rank and $G(x) \geqq 0$ then (1.3) is equivalent to a differential system with complex coefficients that is of the general form of the canonical accessory differential equations for a variational problem of Bolza type. The method of the present paper for the determination of the associated matrix $Q(x)$ is more direct than that employed by Barrett [1]; in particular, the present method affords a ready determination of the most general form of $Q(x)$. In addition, it is shown that certain criteria of oscillation and non-oscillation obtained by Barrett for an equation (1.1) may be improved and extended.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector (y_α), $(\alpha = 1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{\frac{1}{2}}$. The symbol E is used for

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the $n \times n$ identity matrix, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is designated by M^*. If M is an $n \times n$ matrix the symbol $|M|$ is used for the supremum of $|My|$ on the unit sphere $|y| = 1$. The notations $M \geqq N$, $(M > N)$, are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a non-negative (positive) definite hermitian matrix. Finally, if all elements of a matrix $M(x)$ possess a property of continuity or differentiability, for brevity we shall say that $M(x)$ possesses this property.

2. Related matrix differential systems. Consider a matrix differential system

$$U' = A(x)U + B(x)V, \quad V' = C(x)U - A^*(x)V, \tag{2.1}$$

where $A(x)$, $B(x)$, $C(x)$ are $n \times n$ matrices continuous on $[a, \infty): a \leqq x < \infty$, with $B(x)$, $C(x)$ hermitian on this interval. If $U(x) = \| U_{\alpha j}(x) \|$, $V(x) = \| V_{\alpha j}(x) \|$, $(\alpha = 1, \cdots, n; j = 1, \cdots, r)$, are $n \times r$ matrices, for typographical simplicity the symbol $(U(x); V(x))$ will be used to denote the $2n \times r$ matrix whose jth column has elements $U_{1j}(x), \cdots, U_{nj}(x)$, $V_{1j}(x), \cdots, V_{nj}(x)$. If $U(x)$, $V(x)$ is a solution of (2.1) then $(U(x); V(x))$ is of constant rank on $[a, \infty)$; in the major portion of the following discussion we shall be concerned with matrices $(U(x); V(x))$ which are solutions of systems of the from (2.1) and of rank n.

If $(U_1(x); V_1(x))$ and $(U_2(x); V_2(x))$ are individually solutions of (2.1), then (see, Reid [9; Lemma 2.1]), the matrix $U_1^*(x)V_2(x) - V_1^*(x)U_2(x)$ is constant on $[a, \infty)$. In particular, if $(U(x); V(x))$ is a solution of (2.1) such that $U^*(x)V(x) - V^*(x)U(x) \equiv 0$, then $(U(x); V(x))$ is termed a matrix of "conjoined" solutions of this equation, (see, for example, Reid [9, 10] for comments on this terminology).

If $H(x)$ is a fundamental matrix for the differential equation

$$H' = A(x)H, \tag{2.2}$$

then under the transformation

$$U(x) = H(x)Y(x), \quad V(x) = H^{*-1}(x)Z(x) \tag{2.3}$$

the system (2.1) reduces to

$$Y' = G(x)Z, \quad Z' = -F(x)Y \tag{2.4}$$

where $G(x)$, $F(x)$ are the hermitian matrices $G(x) = H^{-1}(x)B(x)H^{*-1}(x)$, $F(x) = -H^*(x)C(x)H(x)$. Moreover, if $(U; V)$ and $(Y; Z)$ are solutions of the respective systems (2.1) and (2.4), related by (2.3), then $U^*V - V^*U \equiv Y^*Z - Z^*Y$, and (U, V) is a matrix of conjoined solutions of (2.1) if and only if $(Y; Z)$ is a matrix of conjoined solutions of (2.4).

In view of this equivalence between systems (2.4) and the formally more general system (2.1), in the following discussion specific attention will be limited to systems of the form (2.4).

Corresponding to (1.2) we shall consider the differential system

$$\Phi' = Q(x)\Psi \ , \quad \Psi' = - Q(x)\Phi \ , \tag{2.5}$$

where $Q(x)$ is a continuous hermitian matrix on $[a, \infty)$. Now if $\Phi = \Phi_0(x)$, $\Psi = \Psi_0(x)$ is a solution of (2.5) then $\Phi = \Psi_0(x)$, $\Psi = - \Phi_0(x)$ is also a solution. Consequently, from the above remarks it follows that if $(\Phi(x) \, ; \, \Psi(x))$ is a solution of (2.5) then the matrices $\Phi^*\Psi - \Psi^*\Phi$ and $\Phi^*\Phi + \Psi^*\Psi$ are constant. In particular, if $\Phi^*\Psi - \Psi^*\Phi = 0$, and $\Phi^*\Phi + \Psi^*\Psi = E$ on $[a, \infty)$ then

$$\begin{Vmatrix} \Phi(x) & \Psi(x) \\ \Psi(x) & - \Phi(x) \end{Vmatrix} \tag{2.6}$$

is unitary on this interval, and also $\Phi\Phi^* + \Psi\Psi^* = E, \Phi\Psi^* - \Psi\Phi^* = 0$ on $[a, \infty)$. If $\Phi = S(x) = S(x \, ; \, a, Q), \Psi = C(x) = C(x \, ; \, a, Q)$ is the solution of (2.5) satisfying $S(a) = 0$, $C(a) = E$, then

$$S^*S + C^*C = E, \quad S^*C - C^*S = 0, \quad SS^* + CC^* = E, \quad SC^* - CS^* = 0 \ ,$$

on $[a, \infty)$, corresponding to the result of Theorem 1.1 of Barrett [1] for the case of $Q(x)$ real and symmetric.

3. **The "Prüfer" transformation.** The principal result of the present paper is the following theorem ; if $G(x), F(x)$ are real-valued, $G(x)$ non-singular, and $Y(a)=0$, the result of this theorem is equivalent to Theorem 2.1 of Barrett [1].

THEOREM 3.1. *If $(Y(x) \, ; \, Z(x))$ is a matrix of conjoined solutions of* (2.4) *that is of rank n, then there exist on $[a, \infty)$ a continuously differentiable non-singular matrix $R(x)$ and a continuous hermitian matrix $Q(x)$ for which* (2.5) *has a solution $(\Phi(x) \, ; \, \Psi(x))$ such that on $[a, \infty)$,*

$$\Phi(x)\Phi^*(x) + \Psi(x)\Psi^*(x) = E \ , \tag{3.1}$$

$$Y(x) = \Phi^*(x)R(x) \ , \quad Z(x) = \Psi^*(x)R(x) \ ; \tag{3.2}$$

moreover, if these conclusions hold for $R(x) = R_1(x)$, $Q(x) = Q_1(x)$ then the most general forms of these matrices are $R(x) = \Gamma R_1(x)$, $Q(x) = \Gamma Q_1(x)\Gamma^*$, where Γ is a constant unitary matrix.

In view of the hypotheses of the theorem, $Y(x)$ and $Z(x)$ are $n \times n$ matrices such that $Y^*Z - Z^*Y = 0$, $Y^*Y + Z^*Z > 0$ on $[a, \infty)$, and consequently if $R(x)$, $\Phi(x)$, $\Psi(x)$ are continuously differentiable matrices satisfying (3.1) (3.2) on $[a, \infty)$, then on this interval we must have

$$R^*(x)R(x) = Y^*(x)Y(x) + Z^*(x)Z(x) \,, \tag{3.3}$$

$$\Phi(x)\Psi^*(x) - \Psi(x)\Phi^*(x) = 0 \,. \tag{3.4}$$

Moreover, the substitution (3.2) implies the identities

$$\begin{aligned} Y' - GZ &= (\Phi^{*\prime} - \Psi^* Q)R + (\Psi^* Q - G\Psi^*)R + \Phi^* R' \,, \\ Z' + FY &= (\Psi^{*\prime} + \Phi^* Q)R - (\Phi^* Q - F\Phi^*)R + \Psi^* R' \,, \end{aligned} \tag{3.5}$$

and for $(Y; Z)$, $(\Phi; \Psi)$ solutions of the respective systems (2.4), (2.5) satisfying (3.1), (3.2) it follows directly, with the aid of (3.4), that the equations

$$R^* R' = Y^* GZ - Z^* FY \,, \tag{3.6}$$

$$R^* QR = Z^* GZ + Y^* FY \,, \tag{3.7}$$

hold on $[a, \infty)$. It is to be noted that (3.3), (3.6) may be considered as definitive equations for $R(x)$, and that (3.7) defines $Q(x)$ in terms of $Y(x)$, $Z(x)$ and $R(x)$. Indeed, if $(Y(x); Z(x))$ is a matrix of conjoined solutions of (2.4) that is of rank n, and $R(x)$ is a continuously differentiable matrix satisfying (3.3), (3.6), then $R(x)$ is non-singular and $\Phi(x)$, $\Psi(x)$ are specified uniquely by (3.2). From the relations $R^*R = Y^*Y + Z^*Z = R^*(\Phi\Phi^* + \Psi\Psi^*)R$, $0 = Y^*Z - Z^*Y = R^*(\Phi\Psi^* - \Psi\Phi^*)R$ it then follows that Φ, Ψ satisfy (3.1), (3.4), so that (2.6) is unitary on $[a, \infty)$, and on this interval we have also

$$\Phi^*\Phi + \Psi^*\Psi = E \,, \quad \Psi^*\Phi - \Phi^*\Psi = 0 \,. \tag{3.8}$$

For $Q(x)$ defined by (3.7), or the equivalent relation

$$Q = \Psi G \Psi^* + \Phi F \Phi^* \,, \tag{3.7'}$$

it then follows that

$$\begin{aligned} (\Psi^* Q - G\Psi^*)R + \Phi^* R' &= (\Psi^*\Psi + \Phi^*\Phi - E)G\Psi^* R \\ &\qquad + (\Psi^*\Phi - \Phi^*\Psi)F\Phi^* R = 0 \,, \\ -(\Phi^* Q - F\Phi^*)R + \Psi^* R' &= (\Psi^*\Phi - \Phi^*\Psi)G\Psi^* R \\ &\qquad + (E - \Phi^*\Phi - \Psi^*\Psi)F\Phi^* R = 0 \,, \end{aligned}$$

and hence (3.5), (3.8) imply that $(\Phi(x); \Psi(x))$ is a matrix of conjoined solutions of (2.5) which is of rank n and satisfies (3.1).

Thus the proof of Theorem 3.1 is reduced to the determination of a continuously differentiable matrix $R(x)$ satisfying (3.3), (3.6) on $[a, \infty)$. In turn, this determination is attained readily with the aid of the following auxiliary result, which will be proved in the next section.

LEMMA 3.1. *If $M(x)$ is a continuously differentiable hermitian matrix* *such that $M(x) > 0$ on $[a, \infty)$, and $N(x)$ is the positive definite hermitian square root of $M(x)$, then $N(x)$ is continuously differentiable on $[a, \infty)$.*

Indeed, under the hypotheses of Theorem 3.1 the hermitian matrix $Y^*(x)Y(x) + Z^*(x)Z(x)$ is positive definite and continuously differentiable on $[a, \infty)$. If $R_0(x)$ is the positive definite hermitian square root of this matrix, then by the above lemma $R_0(x)$ is continuously differentiable on $[a, \infty)$. Moreover, the most general solution of (3.3) is clearly of the form $R(x) = W(x)R_0(x)$, where $W(x)$ is a unitary matrix, and the condition that $R(x)$ be continuously differentiable and satisfy (3.6) is equivalent to the condition that $W(x)$ be a unitary matrix that is continuously differentiable and satisfies

$$W' = WK(x) , \tag{3.9}$$

where $K(x) = R_0^{-1}(Y^*GZ - Z^*FY - R_0R_0')R_0^{-1}$. As $Y^*GZ - Z^*FY = Y^*Y' + Z^*Z'$, and $R_0 = R_0^*$, it follows that $R_0(K + K^*)R_0 = (Y^*Y + Z^*Z)' - (R_0R_0)' = 0$. Hence $K(x)$ is skew-hermitian on $[a, \infty)$, and if $W = W_0(x)$ is the solution of (3.9) satisfying $W_0(a) = E$ then $W_0(x)$ is unitary on $[a, \infty)$, and the most general solution $W(x)$ of (3.9) that is unitary on this interval is of the form $\Gamma W_0(x)$, where Γ is a constant unitary matrix. Thus the conclusions of the theorem hold for $R = R_1 = W_0R_0$, $Q = Q_1 = R_1^{*-1}(Z^*GZ + Y^*FY)R_1^{-1}$, and the most general $R(x)$, $Q(z)$ satisfying these conclusions are $R(x) = \Gamma R_1(x)$, $Q(x) = \Gamma Q_1(x)\Gamma^*$, where Γ is a constant unitary matrix.

4. Proof of Lemma 3.1. If M is an $n \times n$ matrix satisfying $M \geqq 0$, then the existence of a unique $N \geqq 0$ satisfying $N^2 = M$ is well-known; indeed, this result is a special case of a theorem on non-negative symmetric transformations in Hilbert space, (see, for example, Riesz-Nagy [**11**, pp. 263-265]). The author is unaware of any previous proof of the differentiability result of Lemma 3.1, however, so a proof will be given here. As a preliminary step in the proof of this lemma, the following result is established.

LEMMA 4.1. *If $U(x)$ is an $n \times n$ matrix which is continuous and $|U(x)| \leqq 1$ on a compact interval $[a, b]$, then*

$$V(x) = E - \sum_{k=1}^{\infty} c_k U^k(x) , \tag{4.1}$$

with $c_1 = 1/2$, $c_k = (1 \cdot 2 \cdots (2k-3))/(k!\, 2^k)$, $(k = 2, 3, \cdots)$, is such that:

(i) *$V(x)$ is continuous on $[a, b]$;*

(ii) *$V^2(x) = E - U(x)$;*

(iii) *if $U(x) \geqq 0$ then $V(x) \geqq 0$;*

(iv) *if $U(x)$ is continuously differentiable and $|U(x)| < 1$ on $[a, b]$ then $V(x)$ is also continuously differentiable on this interval.*

Conclusion (i) is an immediate consequence of the fact that $c_k > 0$ and $\sum_{k=1}^{\infty} c_k$ is convergent, so that the matrix series of (4.1) converges uniformly on $[a, b]$; indeed, the convergence of this series is also uniform on the class of matrices $U(x)$ satisfying $|U(x)| \leqq 1$ on $[a, b]$. Conclusion (ii) follows from the fact that if $g(z) = \sum_{k=1}^{\infty} c_k z^k$ then $1 - g(z)$ is the Maclaurin expansion for the branch of $(1 - z)^{\frac{1}{2}}$ that is equal to 1 at $z=0$. In fact, if $W_j(x) = \sum_{k=1}^{j} c_k U^k(x)$, then $c_k > 0$, $(k = 1, 2, \cdots)$, and the identity $g(z) = (z + g^2(z))/2$ imply that $(U(x) + W_j^2(x))/2 = \sum_{k=1}^{\infty} d_{kj} U^k(x)$ with $d_{kj} = c_k$, $(k = 1, \cdots, j + 1)$, $0 < d_{kj} < c_k$, $(j + 1 < k \leqq 2j)$, and $d_{kj} = 0$ for $k > 2j$, so that

$$\tfrac{1}{2}\,|(E - W_j(x))^2 - (E - U(x))| = |\tfrac{1}{2}(U(x) + W_j^2(x)) - W_j(x)| \leqq \sum_{k=j+1}^{\infty} c_k ,$$

for arbitrary $U(x)$ satisfying $|U(x)| \leqq 1$. If $U(x) \geqq 0$, then $W_j(x) \geqq 0$ $(j=1, 2, \cdots)$, and as $\sum_{k=1}^{\infty} c_k = g(1) = 1$ we have $0 \leqq W_j(x) \leqq I$ and $V(x) \geqq 0$, thus establishing conclusion (iii).

If $U(x)$ is continuously differentiable on $[a, b]$ then $U^k(x)$ is also continuously differentiable on this interval, and $[U^k(x)]'$ is the sum of k terms $U^{\alpha}(x)U'(x)U^{\beta}(x)$ with $\beta = k - 1 - \alpha$, $\alpha = 0, 1, \cdots, k - 1$, so that $|[U^k(x)]'| \leqq k\,|U(x)|^{k-1}\,|U'(x)|$, $(k = 1, 2, \cdots)$. Consequently, if $|U(x)| \leqq r < 1$ the uniform convergence of $\sum_{i=1}^{\infty} kc_k z^{k-1}$ on $0 \leqq z \leqq r$ implies that $\sum_{k=1}^{\infty} c_k[U^k(x)]'$ converges uniformly on $[a, b]$, so that $V(x)$ is continuously differentiable on $[a, b]$ and $V'(x) = -\sum_{k=1}^{\infty} c_k[U^k(x)]'$. It is to be remarked that if $U(x)$ is merely absolutely continuous on $[a, b]$, and $|U(x)| \leqq r < 1$ on this interval, then the above argument shows that $|W_j'(x)| \leqq |U'(x)|(\sum_{k=1}^{\infty} kc_k r^{k-1})$, $(j = 1, 2, \cdots)$, almost everywhere on $[a, b]$, and it follows readily that $V(x)$ is absolutely continuous on this interval.

Now if $M(x)$ satisfies the hypotheses of Lemma 3.1 then for an arbitrary compact subinterval $[a, b]$ of $[a, \infty)$ there exists a constant $k > 0$ such that $0 < k^2M(x) < E$ on $[a, b]$. The hermitian matrix $U(x) = E - k^2M(x)$ satisfies $0 < U(x) < E$ on $[a, b]$, and if $V(x)$ is defined by (4.1) then $N(x) = (1/k)V(x)$ is a positive definite hermitian square root of $M(x)$ which is continuously differentiable on $[a, b]$. As indicated at the beginning of this section, it is well known that for a given $M \geqq 0$ there is a unique $N \geqq 0$ satisfying $N^2 = M$. A pertinent step in the proof of uniqueness is the determination of one $N \geqq 0$ satisfying $N^2 = M$, and which is such that N permutes with all matrices which permute with M; the N determined by the above method clearly possesses this property since it is the limit of polynomials in M. Now if N_1 is any square root of M then $N_1M = N_1^3 = MN_1$, and hence $NN_1 = N_1N$. Consequently, $(N_1 + N)(N_1 - N) = N_1^2 - N^2 = 0$, and if also $N_1 \geqq 0$ then individually $N_1(N_1 - N) = 0$ and $N(N_1 - N) = 0$, so that $(N_1 - N)^2 = 0$ and $N_1 = N$. As we have established above that the positive definite square root of

$M(x)$ is continuously differentiable on an arbitrary compact subinterval $[a, b]$ of $[a, \infty)$, we have that this square root is continuously differentiable on $[a, \infty)$.

Finally, it is to be remarked that with proper attention to detail the above method of proof for Lemma 4.1 may be used to establish results corresponding to conclusions (i), (ii), (iv) when $U(x)$ is an operator function on the compact interval $[a, b]$ to the set of endomorphisms of a Banach space, with continuity interpreted as continuity in either the strong or uniform operator topology, and differentiability is correspondingly strong or uniform differentiability for the involved operator functions; for the sense in which these terms are employed, the reader is referred, for example, to Hille-Phillips [5, p. 59].

5. Criteria of oscillation and non-oscillation. In the present section, *specific attention will be limited to a system* (2.4) *with* $G(x)>0$ *on* $[a, \infty)$, so that the considered system is entirely equivalent to a linear second order matrix differential equation; in view of the comments at the beginning of § 2, the derived criteria may be translated immediately into criteria for a system (2.1) with $B(x) > 0$. It is to be commented, moreover, that certain corresponding results hold for such systems satisfying merely $G(x) \geqq 0$ or $B(x) \geqq 0$, although we shall not treat here this more general case.

Two points s, t, of $[a, \infty)$ are said to be (mutually) *conjugate*, with respect to (2.4), if there exists a vector solution $(y(x); z(x))$ of this system with $y(s) = 0 = y(t)$ and $y(x) \not\equiv 0$ on $[s, t]$. A system (2.4) is termed *non-oscillatory on a given interval* provided no two distinct points of this interval are conjugate; moreover, the system will be called *non-oscillatory for large* x if there exists a subinterval $[s, \infty)$ on which this system is non-oscillatory. For brevity, a vector function $y(x)$ will be termed *differentially admissible on a subinterval of* $[a, \infty)$ if on this subinterval $y(x)$ is continuous and has piecewise continuous derivatives.

For the case in which the coefficient matrices of (2.4) are real-valued the statements of the following theorem are classical results in the calculus of variations, (see, for example, Morse [7; Chapter I], or Bliss [3; Chapter IV]); for the general case of complex coefficients the results are contained in Theorem 2.1 of Reid [9].

LEMMA 5.1. *If* $G(x) > 0$ *on* $[a, \infty)$, *then for each compact subinterval* $[c, d]$ *the following conditions are equivalent*:

(i) (2.4) *is non-oscillatory on* $[c, d]$;

(ii) *there exists a matrix* $(Y(x); Z(x))$ *of conjoined solutions of* (2.4) *with* $Y(x)$ *non-singular on* $[c, d]$;

(iii) $I[y; c, d] \equiv \int_c^d [y^{*\prime} G^{-1}(x) y' - y^* F(x) y]\, dx > 0$ *for arbitrary* $y(x)$

differentially admissible on $[c, d]$, *with* $y(c)=0=y(d)$ *and* $y\not\equiv 0$ *on this subinterval.*

THEOREM 5.1. *If* $G(x) > 0$ *on* $[a, \infty)$, *and on this interval there exists a continuous real-valued scalar function* $r(x)$ *such that* $F(x) \leqq r(x)E$, $G(x) \leqq r(x)E$, *and* $\int_a^\infty r(x)dx < \pi$, *then* (2.4) *is non-oscillatory on* $[a, \infty)$.

As the condition $0 < G(x) \leqq r(x)E$ implies that $r(x) > 0$ and $G^{-1}(x) \geqq (1/r(x))E$, if $y(x)$ is differentially admissible on a compact subinterval $[c, d]$ then

$$I[y\,;\, c,\, d] \geqq \int_c^d [(1/r(x))\,|\, y'\,|^2 - r(x)\,|\, y\,|^2]dx\,. \tag{5.1}$$

Now under the hypothesis of the theorem the scalar differential equation $(u'/r(x))' + r(x)u = 0$ admits the solution $u(x) = \sin(-\int_x^\infty r(t)dt)$ which is non-zero on $[a, \infty)$, so that in view of criterion (ii) of Lemma 5.1 we have $I[y\,;\, c,\, d] > 0$ for arbitrary $y(x)$ differentially admissible on $[c, d]$, with $y(c) = 0 = y(d)$ and $y \not\equiv 0$ on $[c, d]$, and thus by criterion (iii) of this lemma the system (2.4) is non-oscillatory on $[a, \infty)$.

It is to be remarked that Theorem 5.1 provides an estimate for the non-existence of conjugate points that is an improvement over that given in Theorem 3.2 and Corollary 3.2.1 of Barrett [1]. In a similar fashion one may prove that if the hypotheses of the theorem hold for an $r(x)$ with $\int_a^\infty r(x)dx < \pi/2$, and $(Y(x)\,;\, Z(x))$ is a matrix of conjoined solutions of (2.4) such that for some value c on $[a, \infty)$ we have $Z(c) = 0$ and $Y(c)$ non-singular, then $Y(x)$ is non-singular throughout $[a, \infty)$; that is, for the differential system (2.4) there is no point on $[a, \infty)$ that is a focal point of $x = c$.

THEOREM 5.2. *If* $G(x) > 0$ *and* (2.4) *is non-oscillatory for large* x, *then there exists a matrix* $(Y(x)\,;\, Z(x))$ *of conjoined solutions of this system such that* $Y(x)$ *is non-singular for large* x *and*

$$\int^\infty Y^{-1}(x)G(x)Y^{*-1}(x)dx < \infty\,. \tag{5.2}$$

Moreover, for any such matrix of conjoined solutions,

$$\int^\infty (|\, G(x)|/|\, Y(x)|^2)dx < \infty\,; \tag{5.3}$$

in particular, if (2.4) *is non-oscillatory for large* x *and all solutions of this system remain bounded as* $x\to\infty$, *then*

$$\int^\infty |\, G(x)|\, dx < \infty\,. \tag{5.4}$$

For systems (2.4) with $G(x) > 0$, and non-oscillatory for large x, the existence of a matrix $(Y(x)\,;\, Z(x))$ of conjoined solutions with $Y(x)$ non-singular for large x and satisfying (5.2) has been established by various authors, (see Hartman [4], Barrett [1], and Reid [10]). In view of the definiteness of the integrand matrix of (5.2), the relation (5.2) is equivalent to convergence at ∞ for the integral of $|\,Y^{-1}(x)G(x)Y^{*-1}(x)|$, and relation (5.3) follows immediately from the inequality $|\,G\,| \leqq |\,Y\,|\,|\,Y^{-1}GY^{*-1}\,|\,|\,Y^{*}\,| = |\,Y\,|^{2}\,|\,Y^{-1}GY^{*-1}\,|$. In case $|\,Y(x)|$ is bounded as $x \to \infty$, relation (5.4) is a direct consequence of (5.3). Since for an $n \times n$ non-negative definite hermitian matrix G the trace of G satisfies the inequality $(1/n)$ tr $G \leqq |\,G\,| \leqq$ tr G, inequalities (5.3), (5.4) may be stated equally well in terms of tr G, as in Corollaries 3.1.1 and 3.1.2 of Barrett [1].

Now for a system (2.5) with $Q(x)$ hermitian all solutions are bounded, in view of the unitary nature of the matrix (2.6). Therefore, as a direct consequence of Theorems 5.1 and 5.2, we have the following result.

THEOREM 5.3. *If* $Q(x) > 0$ *on* $[a, \infty)$ *a system of the form* (2.5) *is non-oscillatory for large* x *if and only if*

$$\int^{\infty} |\,Q(x)|\,dx < \infty \,. \tag{5.5}$$

For a given matrix $(Y(x)\,;\, Z(x))$ of conjoined solutions of (2.4) that is of rank n the relations (3.2) hold for a particular solution $(\Phi(x)\,;\, \Psi(x))$ of the associated system (2.5); moreover, the matrix $Q(x)$ clearly depends upon the particular solution $(Y(x)\,;\, Z(x))$ under consideration. However, for a system (2.4) with $G(x) > 0$ the condition of being non-oscillatory for large x may be phrased in terms of any matrix $(Y(x)\,;\, Z(x))$ of conjoined solutions that is of rank n. This result is a direct consequence of a separation theorem of Sturmian type, to the effect that if $G(x) > 0$ and (2.4) is non-oscillatory on a given interval I then an arbitrary matrix $(Y(x)\,;\, Z(x))$ of conjoined solutions of rank n has $Y(x)$ singular for at most n distinct values on I; in particular, such a system (2.4) is non-oscillatory for large x if and only if an arbitrary matrix of conjoined solutions of rank n has $Y(x)$ non-singular on some interval $[a, \infty)$. For systems (2.4) with real coefficients, and $G(x) > 0$, this result is a special case of a more general separation theorem of Sturmian type due to Morse [6 ; Section 10], (see also Morse [7 ; Chapter IV, Section 8], and Birkhoff and Hestenes [2 ; Section 14]). For systems (2.4) with complex coefficients the result may be established by similar methods of proof, so no details will be presented here.

Now for a matrix $(Y(x)\,;\, Z(x))$ of conjoined solutions of (2.4) that is of rank n the corresponding $Q(x)$ is given by (3.7) with $R(x) = W(x)R_0(x)$, where $W(x)$ is unitary and $R_0(x)$ is the positive definite square root of

$Y^*(x)Y(x) + Z^*(x)Z(x)$. Therefore, for each value of x we have $|Q| = |W^*QW| = |R_0^{-1}(Z^*GZ + Y^*FY)R_0^{-1}| = \sup|\xi^*(Z^*GZ + Y^*FY)\xi|$ on the set of vectors ξ satisfying $\xi^*(Y^*Y + Z^*Z)\xi = 1$, and in view of Theorem 5.3 and the above remarks we have the following result for the system (2.4).

THEOREM 5.4. *If* $(Y(x)\,;\, Z(x))$ *is a matrix of conjoined solutions of* (2.4) *that is of rank* n, *and* $Z^*GZ + Y^*FY > 0$ *for large* x, *then* (2.4) *is non-oscillatory for large* x *if and only if*

$$\int^{\infty} |R_0^{-1}(Z^*GZ + Y^*FY)R_0^{-1}|\, dx < \infty\,, \tag{5.6}$$

where $R_0(x)$ *is the positive definite square root of* $Y^*(x)Y(x) + Z^*(x)Z(x)$.

In particular, if $G(x) > 0$ and $F(x) > 0$ for large x then whenever (2.4) is non-oscillatory for large x the relation (5.6) holds for all matrices $(Y(x)\,;\, Z(x))$ of conjoined solutions of rank n, whereas for each such $(Y(x)\,;\, Z(x))$ the integral of (5.6) diverges in case (2.4) is oscillatory for large x.

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