TWO NON-SEPARABLE COMPLETE METRIC SPACES DEFINED ON [0, 1]

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Let $\mathfrak{M}$ be the set of all Lebesgue measurable subsets of the closed interval [0, 1], and let $A, B \in \mathfrak{M}$. It is well-known that $\mathfrak{M}$ becomes a pseudo-metric space if distance is defined by

$$d(A, B) = m(A - B) + m(B - A) = m[(A - B) \cup (B - A)] ,$$

m denoting the Lebesgue measure. See [**1**, pp. 31–32]. It is the purpose of this paper to extend $\mathfrak{M}$ to include the non-measurable sets and to examine some of the properties of the resulting space.

If we remove the restriction that A and B be measurable, and let them be any subsets of [0, 1], then if

$$\rho(A, B) = m^*(A - B) + m^*(B - A), \text{ and } \delta(A, B) = m^*[A - B) \cup (B - A)]$$

(where m^* denotes the exterior Lebesgue measure), it is easily seen that pseudo-metric spaces $\mathfrak{S}$ and $\mathfrak{T}$ are obtained, corresponding to ρ and δ respectively. The properties which we discuss of $\mathfrak{S}$ and $\mathfrak{T}$ are the same and are proved analogously, so we shall state and prove our results for the space $\mathfrak{S}$ only, it being understood that similar theorems and proofs hold for $\mathfrak{T}$.

LEMMA 1. *A necessary and sufficient condition that $\rho(A, B) = 0$ is the existence of sets Z_1 and Z_2, both of Lebesgue measure zero, such that $A \cup Z_1 = B \cup Z_2$.*

Necessity. If $\rho(A, B) = 0$, then $m(A - B) = m(B - A) = 0$. Since $A \cup (B - A) = A \cup B = B \cup A = B \cup (A - B)$, Z_1 and Z_2 may be taken as $B - A$ and $A - B$, respectively.

Sufficiency. If $A \cup Z_1 = B \cup Z_2$, then

$$\rho(A, B) \leqq \rho(A, A \cup Z_1) + \rho(A \cup Z_1, B \cup Z_2) + \rho(B \cup Z_2, B) = 0$$

The relation $\rho(A, B) = 0$ is seen to be an equivalence relation defined on the elements of $\mathfrak{S}$; hence, those elements are partitioned into equivalence classes. Let $[A]$ denote the equivalence class which contains A. It is clear that if $C \in [A]$ and $D \in [B]$, then $\rho(A, B) = \rho(C, D)$. If $\mathfrak{S}^*$ is the set of all equivalence classes defined above, and if $\rho([A], [B]) = \rho(A, B)$, then $\mathfrak{S}^*$ becomes a metric space with the metric $\rho([A], [B])$.

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LEMMA 2. *If* $B_n \in [A_n]$ *for* $n = 1, 2, \cdots,$ *then* $[\bigcup_{n=1}^{\infty} A_n] = [\bigcup_{n=1}^{\infty} B_n]$ and $[\bigcap_{n=1}^{\infty} A_n] = \bigcap_{n=1}^{\infty} B_n]$.

LEMMA 3. *If* A *is measurable and* $B \in [A]$, *then* B *is measurable.*

There exist Z_1 and Z_2 such that $A \cup Z_1 = B \cup Z_2$ with $m(Z_1) = m(Z_2) = 0$. Let $\tilde{B}$ denote $[0, 1] - B$. Then $B \cup Z_2$ is measurable and since $B = (B \cup Z_2) - (\tilde{B} \cap Z_2)$, B is measurable.

It follows from Lemma 3 that the sets in each equivalence class are either all measurable or all non-measurable. Thus the space $\mathfrak{S}^* = \mathfrak{M}^* \cup \mathfrak{N}^*$, where $\mathfrak{M}^*$ is the space of all equivalence classes of measurable sets, and $\mathfrak{N}^*$ is the space of all equivalence classes of non-measurable sets. It should be noted that $\mathfrak{M}^*$ is the metric space corresponding to the well-known pseudo-metric space $\mathfrak{M}$ defined at the beginning of the paper.

In the following we will omit the asterisks and square brackets, and will write $\mathfrak{S}$ for $\mathfrak{S}^*$, etc., and $\rho(A, B)$ for $\rho([A], [B])$. When we write $A \in \mathfrak{S}$, A may be considered either as an equivalence class or as a representative element of that class.

THEOREM 1. *The space* $\mathfrak{S}$ *is complete.*

The proof is similar to that given in [1, p. 32].

THEOREM 2. *For every* $A \in \mathfrak{S}$ *and every positive number* $\varepsilon < 1$, *there exists* $B \in \mathfrak{S}$ such that $0 < \rho(A, B) < \varepsilon$.

Proof Case I. $m(A) = 0$.

If $m(A) = 0$, then $A \in [\phi]$, ϕ denoting the empty set. Let $B \in \mathfrak{S}$ be an interval of length $< \varepsilon$. Then $\rho(A, B) = \rho(\phi, B) = m(B) < \varepsilon$.

Case II. $m^*(A) > 0$.

Let $I \in \mathfrak{S}$ be an interval of length $< \varepsilon$, such that $m^*(I \cap A) > 0$. If $B = A - I$, then

$$\rho(A, B) = \rho(A, A - I) = m^*[A - (A - I)] = m^*(I \cap A) \leqq m^*(I) < \varepsilon .$$

COROLLARY 1. *If in Theorem* 2, $A \in \mathfrak{M}$, *then* B (*as constructed*) $\in \mathfrak{M}$.

THEOREM 3. *If* $A \in \mathfrak{M}$ *and* $\varepsilon > 0$, *then there exists* $C \in \mathfrak{N}$ *such that* $0 < \rho(A, C) < \varepsilon$.

Proof Case I. $m(A) = 0$.

Let M be a set of real numbers such that for every measurable set E, $m^*(M \cap E) = m(E)$ and $m_*(M \cap E) = 0$, m_* denoting the interior Lebesgue measure. (See [2], Theorem E, p. 70.) In Case I of Theorem 2, let $C = B \cap M$. Then

$$\rho(A, C) = \rho(\phi, C) = m^*(C) = m(B) < \varepsilon \text{ and } m_*(C) = 0 .$$

Case II. $m(A) > 0$.

In Case II of Theorem 2, let $C = A - (I \cap M)$, M described above. Then $\rho(A, C) = m^*(A - C) = m^*(A \cap I \cap M) \leqq m(I) < \varepsilon$, and $m^*(A \cap I \cap M) = m(A \cap I) > 0$, $m_*(A \cap I \cap M) = 0$. Since $(A \cap I \cap M) \in \mathfrak{N}$, $C \in \mathfrak{N}$.

THEOREM 4. *$\mathfrak{N}$ is open in $\mathfrak{S}$.*

Proof. Assume Theorem 4 is false. Then there exists $N \in \mathfrak{N}$ and sets $M_m \in \mathfrak{M}$, $m = 1, 2, \cdots$, such that $\lim_{m\to\infty} \rho(N, M_m) = 0$. The sequence M_m, $m = 1, 2, \cdots$, is, therefore, a Cauchy sequence in $\mathfrak{S}$ and so by Theorem 1 has a subsequence M_{m_n}, $n = 1, 2, \cdots$, such that $\lim_{m\to\infty} \rho(\limsup_n M_{m_n}, M_m) = 0$. Since $\limsup_n M_{m_n}$ is measurable, this means that N is measurable by Lemma 3, a contradiction.

The last few results can be summarized as follows.

THEOREM 5. *$\mathfrak{M}$ is perfect and nowhere dense in $\mathfrak{S}$; $\mathfrak{N}$ is open and dense in $\mathfrak{S}$.*

The remainder of the work is valid for both spaces, as only the equivalence classes are dealt with (these being the same for $\mathfrak{S}$ and $\mathfrak{T}$).

After having proven completeness for $\mathfrak{S}$ in Theorem 1, a natural question to ask is "Is the space separable ?". The theorem proved here which demonstrates the existence of $2^{\mathfrak{c}}(= \mathfrak{f})$, where $2^{\aleph_0} = \mathfrak{c}$,equivalence classes in $\mathfrak{S}$ answers this question (and a similar one about a countable basis) in the negative. It is also interesting to note that the space $\mathfrak{M}$ has exactly $\mathfrak{c}$ equivalence classes. (In the following work Ω is the first ordinal belonging to $\mathfrak{c}$.)

THEOREM 6. *There exist $\mathfrak{f}$ equivalence classes in the space $\mathfrak{S}$.*

Proof. It will be sufficient to construct a well-ordered family $\{A_\alpha \mid 0 \leqq \alpha < \Omega\}$ of mutually disjoint subsets of [0, 1], each of which has $m^*(A_\alpha) = 1$.

Consider $\{B_\beta \mid 0 \leqq \beta < \Omega\}$ as a well-ordering of all closed subsets B_β of [0, 1] which have a positive Lebesgue measure. For each β, $0 \leqq \beta < \Omega$, let $\{x_\alpha^\beta \mid 0 \leqq \alpha \leqq \beta\}$ be a well-ordered subset of B_β such that $x_\alpha^\beta \neq x_{\alpha'}^{\beta'}$, if $\beta \neq \beta'$ or $\alpha \neq \alpha'$. This selection is possible since, for each β, the set of all $x_{\alpha'}^{\beta'}$ with $0 \leqq \alpha' \leqq \beta' < \beta$ has a cardinal number $< \mathfrak{c}$. Set $A_\alpha = \{x_\alpha^\beta \mid \alpha \leqq \beta < \Omega\}$, for each α, $0 \leqq \alpha < \Omega$. By a simple argument $A_\alpha \cap A_{\alpha'} = \phi$, for $\alpha \neq \alpha'$. Now consider any A_α; if $m^*(A_\alpha) \neq 1$, then A_α is contained in some open set Y such that $m(Y) < 1$. The complement of Y is closed and has $m([0, 1] - Y) > 0$. But $m\{[0, x] \cap ([0, 1] - Y)\}$

is a continuous function of x for $0 \leqq x \leqq 1$; therefore, this function takes on all values between 0 and $m([0, 1] - Y)$, inclusive. This means that there are non-denumerably many closed sets whose measures are greater than 0 and which do not intersect A_α. This is, of course, impossible by the construction of A_α. Therefore, $m^*(A_\alpha) = 1$.

Form the set of all subsets of the set of A_x's, and take the sum of each element of this power set. Any two such sums belong to two different equivalence classes since they disagree in a set of exterior measure 1. This set of sums has cardinal $\mathfrak{f}$. There are, therefore, at least $\mathfrak{f}$ equivalence classes, at most $\mathfrak{f}$ such classes; hence, exactly $\mathfrak{f}$.

References

1. H. Hahn and A. Rosenthal, *Set functions*, University of New Mexico Press, 1948.
2. P. R. Halmos, *Measure theory*, van Nostrand, 1950.

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