MODULES WHOSE ANNIHILATORS ARE DIRECT SUMMANDS

CHARLES W. CURTIS

Introduction. Let B be a ring with an identity element, and let M be a right B-module. The set of all elements b in B such that $Mb = (0)$ is called the annihilator of M, and will be denoted by $(0 : M)$. It is a natural question to ask under what circumstances the ideal $(0 : M)$ is a direct summand of B. If B is a semi-simple ring with minimum condition, for example, then every ideal is a direct summand, and there is no problem. We shall be concerned with a ring B, not assumed to be semi-simple, which is a crossed product $\Delta(G, H, \rho)$ of a finite group G and a division ring Δ, with factor set ρ. In particular, B may be the group algebra of a finite group with coefficients in a field. The purpose of this note is to obtain necessary and sufficient conditions on the structure of the module M in order that its annihilator $(0 : M)$ be a direct summand of B.

Our interest in the problem stems chiefly from the fact that the the modules whose annihilators are direct summands turn out to be precisely the modules for which the pairing defined in § 2 of [1] is regular in the sense of [1, p. 281]. The main results of [1], given in § 5 and § 6, are based upon the assumption that the pairing is regular, and establish a connection between the structure of the module M relative to the set of B-endomorphisms of M and the structure of a certain ideal in B, called the nucleus of M, which is the uniquely determined complementary ideal to $(0 : M)$ when $(0 : M)$ is a direct summand.

2. Familiarity with crossed products and their connection with projective representations of finite groups is assumed (see [1, § 2]). In this section we recall some of the properties of a crossed product, and introduce, in a more general, and at the same time, much simpler fashion, the pairing defined in a special case by formula (7) of [1]. Let $G = \{1, s, t, \cdots\}$ be a finite group, Δ a division ring and $B = \Delta(G, H, \rho)$ a crossed product of G and Δ with correspondence $s \to \bar{s} = s^H$ from G to the group of automorphisms of Δ, and factor set $\{\rho_{s,t}\}$. There exist elements $\{b_1, b_s, \cdots\}$ in B in one-to-one correspondence with the elements of G, such that every element of B can be expressed uniquely in the form $\sum b_s \xi_s$, with coefficients ξ_s in Δ. The multiplication in B is determined by the equations

$$(1) \qquad b_s b_t = b_{st}\rho_{s,t}\ ;\ \xi b_s = b_s \xi^{\bar{s}}, \qquad \xi \in \Delta .$$

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The fact that B is an associative ring implies that the factor set $\{\rho_{s,t}\}$ satisfies the equations

$$(2) \qquad \rho_{s,tu}\rho_{t,u} = \rho_{st,u}\rho_{s,t}^{\bar{u}} ,$$

for all s, t, u in G. We shall assume that the factor set ρ is normalized so that $\rho_{1,t} = \rho_{t,1} = 1$ for all t in G; then b_1 is the identity element in B.

The additive group of B is a right vector space over Δ which we shall denote by $B^{(r)}$, if we define scalar multiplication by $\xi \in \Delta$ by means of the right multiplication $\xi_r : x \to x\xi$. Similarly the additive group of B can be regarded as a left vector space $B^{(l)}$ over Δ. The elements $b_1, b_s, \cdots$ form bases for both of these spaces. Because both spaces are finite dimensional, B satisfies both chain conditions for left and right ideals.

The mapping $\lambda : \sum b_s \xi_s \to \xi_1$ is a linear function on both vector spaces $B^{(r)}$ and $B^{(l)}$ whose kernel contains no left or right ideal different from zero. Therefore the mapping $\Lambda : \Lambda(a, b) = \lambda(ab)$ is a non-degenerate bilinear form on $B^{(l)} \times B^{(r)} \to \Delta$. Using the bilinear form Λ it is easy to verify (cf. [1, p. 279]) that B is a quasi-Frobenius ring, that is, B satisfies the minimum condition, and every right ideal in B is the right annihilator of its left annihilator, and similarly for left ideals.

A right B-module[1] M is a fortiori a right vector space over Δ since $\Delta \subset B$. For each s in G, the mapping $T_s : x \to xb_s$ is a semi-linear transformation belonging to the automorphism $\bar{s}$ in this vector space. The correspondence $s \to T_s$ defines a projective representation of G. Each transformation T_s has an inverse T_s^{-1} which is a semilinear transformation with automorphism $\bar{s}^{-1}$. Let M' be any left vector space over Δ which is paired with M to Δ by a non-degenerate bilinear form f. Let us assume also that the semi-linear transformations T_s all possess transposes T_s^* with respect to the form f, such that

$$(3) \qquad f(\psi, xT_s) = f(T_s^* \psi, x)^{\bar{s}} ,$$

for all $x \in M$, $\psi \in M'$. If we define $(\sum b_s \xi_s)\psi = \sum T_s^*(\xi_s \psi)$, then M' becomes a left B-module (see [1, p. 274]). When these conditions are satisfied, we shall call the system (M', M, f) a *pair of dual B-modules*.

LEMMA 1. *Let (M', M, f) be a pair of dual B-modules. Then the function*

$$(4) \qquad \tau_f(\psi, x) = \sum_{s \in G} f(\psi, xT_s) b_s^{-1}$$

is a non-degenerate B-bilinear function on $M' \times M \to B$ (cf. [1, Proposition 1]).

[1] We shall assume that the identity element of B acts as the identity operator on all modules we shall consider.

Proof. For any $u \in G$ we have

$$\begin{aligned} b_u^{-1}\tau_f(T_u^*\psi, x) &= \sum_{s\in G} b_u^{-1} f(T_u^*\psi, xT_s)b_s^{-1} \\ &= \sum_{s\in G} f(\psi, xT_sT_u)b_u^{-1}b_s^{-1} = \tau_f(\psi, x) \end{aligned}$$

by (1) and (3). Similarly, for all u,

$$\tau_f(\psi, xT_u)b_u^{-1} = \tau_f(\psi, x) .$$

Since the function τ_f is obviously bilinear as far as Δ is concerned, these calculations establish that for all $b \in B$,

$$b\tau_f(\psi, x) = \tau_f(b\psi, x) \text{ and } \tau_f(\psi, xb) = \tau_f(\psi, x)b .$$

The non-degeneracy of τ_f follows at once from the non-degeneracy of f.

To each right B-module M corresponds a two-sided ideal B_M in B, defined as follows. Find a left B-module M' which is paired with M to Δ by a non-degenerate bilinear form f such that (M', M, f) is a pair of dual B-modules (for example, the space M' of all linear functions on M can be used). Then by Lemma 1, the set B_M consisting of all finite sums $\sum \tau_f(\psi_i, x_i)$, $\psi_i \in M'$, $x_i \in M$, is a two-sided ideal in B. We shall call B_M the *nucleus* of M. We leave it to to the reader to verify that, as our notation indicates, B_M is independent of the choice of M' and f.

We now define a right B-module M to be a *regular module* if B_M contains an element ε such that $\varepsilon b = b\varepsilon = b$ for all $b \in B_M$. We remark that the statement that M is a regular module is equivalent to the statement, in the terminology of [**1**], that (M', M, τ_f) is a regular pairing (see [**1**, p. 281]).

3. This section contains some lemmas on regular modules. We remark first that if M_1 and M_2 are isomorphic B-modules, then $B_{M_1} = B_{M_2}$, and hence regularity is preserved under isomorphism.

LEMMA 2. *The nucleus B_M and the annihilator $(0 : M)$ of a regular module M are two-sided ideals in B generated by central idempotents, and $B = (0 : M) \oplus B_M$.*[2]

Proof. Let (M', M, f) be a pair of dual B-modules, where M is the given regular module. By Theorem 1, p. 282, of [**1**], we have $B_r = (B_M)_r$, and consequently $B = B_M + (0 : M)$. Let $\varepsilon = \sum \tau_f(\psi_i, x_i)$ be the identity element in B_M. Then $a \in B_M \cap (0 : M)$ implies $a = \varepsilon a = \sum \tau_f(\psi_i\, x_i a) = 0$, and the sum is direct. We have $\varepsilon' = 1 - \varepsilon \in (0 : M)$, and because B_M and $(0 : M)$ are ideals whose intersection is zero, ε and ε' are orthogonal central idempotents which generate B_M and $(0 : M)$

[2] We take this opportunity to correct an error in [**1**]. The assertion made in example (c) in § 11, p. 291, of [**1**] that $0 \neq (0 : M) \subset B_M$ for a certain regular module M is false and the assertion (c) should be deleted from [**1**].

respectively.

LEMMA 3. *Let M be a right B-module such that $M = M_1 \oplus M_2$, where M_1 and M_2 are submodules. Let M' be the space of all linear functions on M, paired with M to Δ by the function f defined by $f(\psi, x) = \psi(x)$, $\psi \in M'$, $x \in M$. Then (M', M, f) is a pair of dual B-modules, Let $M_1^{\perp}$ and $M_2^{\perp}$ be the subspaces of M' which annihilate M_1 and M_2 respectively. Then $M' = M_1^{\perp} \oplus M_2^{\perp}$; the restrictions f_1 and f_2 of f to $M_2^{\perp} \times M_1$ and $M_1^{\perp} \times M_2$, respectively, are non-degenerate; and $(M_2^{\perp}, M_1, f_1)$, and $(M_1^{\perp}, M_2, f_2)$ are pairs of dual B-modules.*

Proof. The semi-linear transformations T_s all possess transposes T_s^* relative to the form f, such that formula (3) holds, and consequently (M', M, f) is a pair of dual B-modules. The sets $M_1^{\perp}$ and $M_2^{\perp}$ are subspaces of M' such that $M_1^{\perp} \cap M_2^{\perp} = (0)$. If $\psi \in M'$, then $\psi | M_1 = \psi_1$ is a linear function on M_1, which can be extended to a linear function ψ_1 in M' by setting $\psi_1 | M_2 = 0$. Similarly we define ψ_2. Then $\psi = \psi_1 + \psi_2$, and we have proved that $M' = M_2^{\perp} \oplus M_1^{\perp}$. The restrictions f_1 and f_2 defined in the statement of the lemma are clearly non-degenerate. Finally, since M_1 and M_2 are B-submodules, it follows from (3) that $T_s^*(M_i^{\perp}) \subseteqq M_i^{\perp}$, $i = 1, 2$, and hence $T_s | M_i$ has the transpose $T_s^* | M_{2-i}^{\perp}$, $i = 1, 2$, and the proof is complete.

LEMMA 4. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are B-submodules of M. Then $B_M = B_{M_1} + B_{M_2}$.*

Proof. Let M' be the space of all linear functions on M, and define f, f_1, f_2 as in Lemma 3. Let $\tau_f, \tau_{f_1}, \tau_{f_2}$ be the corresponding functions defined by (4). For $x \in M_1$, $\psi \in M_2^{\perp}$, we have $\tau_{f_1}(\psi, x) = \tau_f(\psi, x)$ and $B_{M_1} \subseteqq B_M$. Similarly $B_{M_2} \subseteqq B_M$. Now let $x \in M$, and write $x = x_1 + x_2$, $x_i \in M_i$; and let $\psi \in M'$, $\psi = \psi_1 + \psi_2$, $\psi_1 \in M_2^{\perp}$, $\psi_2 \in M_1^{\perp}$. Then since M_1 and M_2 are submodules we have

$$\begin{aligned} \tau_f(\psi, x) &= \sum f(\psi_1 + \psi_2, (x_1 + x_2)b_s)b_s^{-1} \\ &= \sum f_1(\psi_1, x_1 b_s)b_s^{-1} + \sum f_2(\psi_2, x_2 b_s)b_s^{-1} \\ &= \tau_{f_1}(\psi_1, x_1) + \tau_{f_2}(\psi_2, x_2) , \end{aligned}$$

and the lemma is proved.

LEMMA 5. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 are regular B-modules. Then M is a regular B-module.*

Proof. By Lemma 4, $B_M = B_{M_1} + B_{M_2}$. By Lemma 2, we have $B_{Mi} = \varepsilon_i B$ where ε_i is a central idempotent, $i = 1, 2$. Then $\varepsilon = \varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2 \in B_M$, and $b\varepsilon = \varepsilon b = b$ for all $b \in B_M$, proving our assertion.

LEMMA 6. *Let e be an idempotent in B. Then (Be, eB, Δ) is a pair of dual B-modules.*

Proof. We recall from § 2 that Δ is a non-degenerate bilinear form on $B^{(l)} \times B^{(r)} \to \Delta$. The restriction of Λ to $Be \times eB$ is also non-degenerate (see [1], p. 279). It remains to verify that for all c, d in B,

$$\Lambda(c, db_s) = \Lambda(b_s c, d)^{\bar{s}} . \tag{5}$$

For this it is sufficient to prove that if $a = \sum \xi_u b_u = \sum b_u \xi_u^{\bar{u}}$, then $\lambda(ab_s) = \lambda(b_s a)^{\bar{s}}$ for all $s \in G$. We have $\lambda(ab_s) = \xi_{s^{-1}} \rho_{s^{-1},s}$, while

$$\lambda(b_s a)^{\bar{s}} = \rho_{s,s^{-1}}^{\bar{s}} \xi_{s^{-1}}^{\overline{s^{-1}}\bar{s}} = \rho_{s,s^{-1}}^{\bar{s}} \rho_{s^{-1},s}^{-1} \xi_{s^{-1}} \rho_{s^{-1},s}$$

by formula (2) of [1], and by (2) above we have

$$\rho_{1,s} \rho_{s,s^{-1}}^{\bar{s}} = \rho_{s,1} \rho_{s^{-1},s} ,$$

and the formula (5) is proved.

4. Now we shall formulate and prove our main result. Because B satisfies the minimum condition, $B = B_1 \oplus \cdots \oplus B_r$, where the B_i are uniquely determined indecomposable two-sided ideals, called the block ideals[3] of B. If we write $1 = \varepsilon_1 + \cdots + \varepsilon_r$, $\varepsilon_i \in B_i$, then the ε_i are mutually orthogonal idempotents belonging to the center of B, and ε_i is the identity element in the block ideal to which it belongs. For any right B-module M, $M\varepsilon_i$ is a submodule of M, and M is the direct sum of the modules $M\varepsilon_i$. These submodules are called the *block components* of M; the block component $M\varepsilon_i$ can also be described as the set of elements of M which are left fixed by ε_i. The block components of $(B, +)$, where $(B, +)$ is viewed as a right B-module in the obvious way, are the block ideals $B\varepsilon_i$. Each block component $B\varepsilon_i$ of B can be expressed as a direct sum of the indecomposable right ideals $e_k B$, $e_k^2 = e_k$, which belong to the block. It is known that two indecomposable right ideals eB and $e'B$ belonging to distinct blocks have no isomorphic composition factors. The direct sum of a full set of non-isomorphic indecomposable right ideals $e_k B$ belonging to the ith block component $B\varepsilon_i$ of B, or any right B-module isomorphic to this module, is called a *reduced block component* of B.

Our theorem is stated as follows.

THEOREM. *Let M be a right B-module with annihilator* $(0 : M)$. *The following statements are equivalent.*

(A) $(0 : M)$ *is a direct summand of* B;

(B) *every non-zero block component $M\varepsilon_i$ of M contains the* ith *reduced block component of B as a direct summand*;

(C) *M is a regular module.*

Proof. The implication (C) → (A) is the content of Lemma 2. We prove next that (A) → (B). Let B' be a two sided ideal in B such that

[3] For the concepts of block ideals and block components see [3], and the references given there.

$B = B' \oplus (0 : M)$. By the uniqueness of the decomposition of B into block ideals, B' is a direct sum of certain of the block ideals $B\varepsilon_i$. Let $M\varepsilon_i$ be a non-zero block component of M; then $B\varepsilon_i \subseteqq B'$, and $M\varepsilon_i$ is a faithful $B\varepsilon_i$ module. Let eB be an indecomposable right ideal belonging to the ith block. By Proposition 4 of [1], eB contains a unique minimal right ideal $N \neq (0)$. There exists an element $x \in M$ such that $xN \neq (0)$. It follows that $u \rightarrow xu$ is a B-isomorphism of eB onto the submodule $P = xeB$ of $M\varepsilon_i$. We shall prove that there exists a submodule Q of $M\varepsilon_i$ such that $M\varepsilon_i = Q \oplus P$. Let M' be the set of all linear functions on $M\varepsilon_i$, paired with $M\varepsilon_i$ to Δ by the non-degenerate bilinear form f, so that $(M', M\varepsilon_i, f)$ is a pair of dual B-modules. Let $P^\perp$ be the submodule of M' consisting of all elements $\psi \in M'$ such that $f(\psi, P) = (0)$. Then $(M'/P^\perp, P, \bar{f})$ is a pair of dual B-modules, where $\bar{f}$ is the induced mapping on $M'/P^\perp \times P$. On the other hand, by Lemma 6, (Be, eB, Δ) is a pair of dual B-modules. Using the fact that eB is a finite dimensional space, it is easily verified that Be and $M'/P^\perp$ are isomorphic left B-modules. By Theorem 1 of [2], Be is an (M_0)-module, and consequently there exists a B-submodule Q' of M' such that $M' = P^\perp \oplus Q'$. Let $Q = \{x | x \in M\varepsilon_i,\ f(Q', x) = (0)\}$. Then Q is a submodule such that $P \cap Q = (0)$. Moreover

$$M = (P^\perp \cap Q')^\perp = P^\perp + (Q')^\perp = P + Q ,$$

since P is finite dimensional and $Q = (Q')^\perp$.

The proof that $M\varepsilon_i$ contains the reduced block component of $B\varepsilon_i$ as a direct summand is now proved by induction. Let $M\varepsilon_i = R \oplus S$, where R is isomorphic to a direct sum of a finite number of non-isomorphic indecomposable right ideals belonging to the ith block, and let eB be an indecomposable right ideal in $B\varepsilon_i$ not isomorphic to any of the direct summands of R. Let N be the unique minimal subideal of eB. If $RN \neq (0)$, then by the previous argument R contains a direct summand isomorphic to eB, which contradicts the Krull-Schmidt theorem. Thus $RN = (0)$, and $SN \neq (0)$, so that S contains a direct summand isomorphic to eB. This completes the proof of the induction step, and the implication (A) $\rightarrow$ (B) is established.

Finally we prove that (B) $\rightarrow$ (C). By Lemma 5, it is sufficient to prove that each block component $M\varepsilon_i$ of M is a regular module, and for this it is sufficient to show that $\varepsilon_i \in B_{Mi}$ whenever $M\varepsilon_i \neq (0)$. Let us consider a non-zero component $M\varepsilon_i$. Let $e_1B, \cdots, e_sB$ be a full set of non-isomorphic indecomposable right ideals belonging to the i^{th} block. For each j, $1 \leqq j \leqq s$, there exists a B-direct summand P_j of $M\varepsilon_i$ such that $P_j \cong e_jB$. By Lemma 4, $B_{P_j} = B_{e_jB} \subseteqq B_{M\varepsilon_i}$. We prove that $e_j \in B_{e_jB}$. By Lemma 6, (Be_j, e_jB, Δ) is a pair of dual B-modules. We assert that

$$e_j = \tau_\Lambda(e_j, e_j) . \tag{6}$$

In fact, $\tau_\Lambda(e_j, e_j) = \sum \Lambda(e_j, e_jb_s)b_s^{-1}$, and if $e_j = \sum \xi_u b_u$, then

$$\Lambda(e_j, e_jb_s) = \lambda(e_jb_s) = \xi_{s^{-1}}\rho_{s^{-1},s}$$

while from $b_{s^{-1}}b_s = b_1\rho_{s^{-1},s}$ we have $b_s^{-1} = \rho_{s^{-1},s}^{-1}b_{s^{-1}}$. From these remarks (6) follows.

We have shown that $e_j \in B_{M\varepsilon_i}$. Since ε_i is a sum of idempotents e such that eB is isomorphic to one of the ideals e_jB, $1 \leqq j \leqq s$, we have $\varepsilon_i \in B_{M\varepsilon_i}$, and $M\varepsilon_i$ is a regular module. This completes the proof of the theorem.

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UNIVERSITY OF WISCONSIN