SUBDIRECT SUMS AND INFINITE ABELIAN GROUPS

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1. Definitions. Let G be a group, and suppose G is a subgroup of the direct sum $\sum_{a\in I}\oplus H_a$ of the collection of groups $\{H_a\}_{a\in I}$. If the projection of G into H_a is onto H_a for each $a\in I$, then G is said to be a *subdirect sum* of the groups $\{H_a\}_{a\in I}$. (Only weak direct and subdirect sums are considered here.) If a group G is isomorphic to a subdirect sum of the groups $\{H_a\}_{a\in I}$, then G is said to be *represented* as a subdirect sum of the groups $\{H_a\}_{a\in I}$. A group is called a *rational group* if it is a subgroup of a $Z(p^\infty)$ group or a subgroup of the additive group of rational numbers.

2. THEOREM. *Every Abelian group can be represented as a subdirect sum of rational groups where the subdirect sum intersects each of the rational groups non-trivially.*

Proof. G is isomorphic to a subgroup of some divisible group, and thus can be represented as a subdirect sum G' of rational group $\{H_a\}_{a\in I}$. Let $(h_1, h_2, \cdots, h_a, \cdots)$ be an element of G'. Let $(h_1, h_2, \cdots, h_a, \cdots)\beta_1 = (k_1, h_2, \cdots, h_a, \cdots)$, where $k_1 = h_1$ if $G'\cap H_1 \neq 0$, and $k_1 = 0$ if $G'\cap H_1 = 0$. Assume β_c has been defined for $c < b$. Define

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_b = (k_1, k_2, \cdots, k_b, h_{b+1}, \cdots)$$

where $k_b = h_b$ if $H_b \cap (\bigcup_{c<b} G'\beta_c) \neq 0$, and $k_b = 0$ otherwise. Each β_a preserves addition because each is a projection. Let $(h_1, h_2, \cdots, h_a, \cdots) \neq (0, 0, \cdots, 0, \cdots)$ and let

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_a = (k_1, k_2, \cdots, k_a, h_{a+1}, h_{a+2}, \cdots) .$$

Only a finite number of the coordinates of $(h_1, h_2, \cdots, h_a, \cdots)$ are not 0. Let them be $h_{a_1}, h_{a_2}, \cdots, h_{a_n}$, where $a_1 < a_2 < \cdots < a_n$. If $a < a_n$, then

$$\begin{aligned}&(h_1, h_2, \cdots, h_a, \cdots)\beta_a \\ &\quad = (k_1, k_2, \cdots, k_a, h_{a+1}, \cdots, h_{a_n}, h_{a_n+1}, \cdots) \neq (0, 0, \cdots, 0, \cdots)\end{aligned}$$

since $h_{a_n} \neq 0$. Assume $a \geqq a_n$. If $n=1$ and $a_1=1$, then $(h_1, h_2, \cdots, h_a, \cdots) = (h_{a_1}, 0, 0, \cdots, 0, \cdots) \in G'$ and $G'\cap H_1 \neq 0$ so that $(h_{a_1}, 0, 0, \cdots$ [illegible] $= (h_{a_1}, 0, 0, \cdots, 0, \cdots)$. That is, $k_{a_1} = h_{a_1} \neq 0$, and hence $(h_1, h_2, \cdots$ [illegible] $\neq (0, 0, \cdots, 0, \cdots)$. If $n = 1$ and $a_n \neq 1$, then $(0, 0, \cdots, h_{a_1}, 0, 0, \cdots) \in G'$ and also in $G'\beta_c$ for all $c < a_1$. Thus $H_{a_1} \cap (\bigcup_{c<a_1} G'\beta_c) \neq 0$, and

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$$(h_1, h_2, \cdots, h_a, \cdots)\beta_a = (0, 0, \cdots, 0, h_{a_1}, 0, 0, \cdots)\beta_a$$
$$= (0, 0, \cdots, 0, h_{a_1}, 0, 0, \cdots)\beta_{a_1} = (0, 0, \cdots 0, h_{a_1}, 0, 0, \cdots)$$
$$\neq (0, 0, \cdots, 0, \cdots) .$$

Assume $n > 1$. If $(h_1, h_2, \cdots, h_a, \cdots)\beta_a = (0, 0, \cdots, 0, \cdots)$, then $k_c = 0$ for $c \leqq a_n$, and

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_{a_{n-1}} = (0, 0, \cdots, 0, h_{a_n}, 0, 0, \cdots) .$$

Therefore $H_{a_n} \cap (G'\beta_{a_{n-1}}) \neq 0$, and so $H_{a_n} \cap (\bigcup_{c<a} G'\beta_c) \neq 0$. Hence $k_{a_n} = h_{a_n} \neq 0$, and this contradicts $k_c = 0$ for $c \leqq a_n$. Therefore

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_a \neq (0, 0, \cdots, 0, \cdots) ,$$

and the kernel of β_a is 0. Hence each β_a is an isomorphism. Now let $(h_1, h_2, \cdots, h_a, \cdots)\beta = (k_1, k_2, \cdots, k_a, \cdots)$. Clearly β is a homomorphism of G' into $\sum_{a \in I} \oplus H_a$. But the kernel of β is 0 because every element in G' has only a finite number of non-zero coordinates. Let I' be the set of indices such that $a \notin I'$ implies that the image of the projection of $G'\beta$ into H_a is 0. $G'\beta$ is isomorphic to a subdirect sum of the groups $\{H_a\}_{a \in I'}$. If $G'\beta \cap H_1 = 0$, then for $(h_1, h_2, \cdots, h_a, \cdots) \in G'$ we have $(h_1, h_2, \cdots, h_a, \cdots)\beta_1 = (0, h_2, \cdots, h_a, \cdots)$, so that

$$(h_1, h_2, \cdots, h_a, \cdots)\beta = (0, k_2, k_3, \cdots, k_a, \cdots) .$$

Hence the image of the projection of $G'\beta$ into H_1 is 0. Therefore $1 \notin I'$. Let $a > 1$. Suppose $G'\beta \cap H_a = 0$ and $H_a \cap (\bigcup_{c<a} G'\beta_c) \neq 0$. Then there exists $b<a$ such that $H_a \cap G'\beta_b \neq 0$. Let $(0, 0, \cdots, 0, k_a, 0, 0, \cdots) \in H_a \cap G'\beta_b$, where $k_a \neq 0$. Let $(h_1, h_2, \cdots, h_a, \cdots)\beta_b = (0, 0, \cdots, 0, k_a, 0, 0, \cdots)$. Then $(h_1, h_2, \cdots, h_a, \cdots)\beta = (0, 0, \cdots, 0, k_a, 0, 0, \cdots)$, and so $G'\beta \cap H_a \neq 0$. Therefore if $G'\beta \cap H_a = 0$, then $H_a \cap (\bigcup_{c<a} G'\beta_c) = 0$. This implies for every $(h_1, h_2, \cdots h_a, \cdots) \in G'$ that

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_a = (k_1, k_2, \cdots, k_a, h_{a+1}, h_{a+2}, \cdots) ,$$

where $k_a = 0$, and hence that

$$(h_1, h_2, \cdots, h_a, \cdots)\beta = (k_1, k_2, \cdots, 0, k_{a+1}, k_{a+2}, \cdots) .$$

Thus the image of the projection of $G'\beta$ into H_a is 0 so that $a \notin I'$. Hence for $a \in I'$, $G'\beta \cap H_a \neq 0$. Since G is isomorphic to $G'\beta$, the theorem follows.

3. REMARKS. Theorem 9 in [1] is an immediate corollary of the preceding theorem, as are some other known theorems in Abelian group theory. In [2], Scott proves that every uncountable Abelian group G has, for every possible infinite index α, $2^{o(G)}$ subgroups of order equal to $o(G)$ and of index α, and that for each given infinite index, their intersection is 0. The following theorem shows that if G is torsion free, one can say more.

4. THEOREM. *Every torsion free Abelian group G of infinite rank has, for every possible infinite index α, $2^{o(G)}$ pure subgroups of order equal to $o(G)$ and of index α. Furthermore, the intersection of these pure subgroups of index α is* 0.

Proof. Represent G as a subdirect sum G' of rational groups $\{H_a\}_{a \in I}$ such that for each $a \in I$, $G' \cap H_a \neq 0$. Let α be an infinite cardinal such that $\alpha \leqq o(G)$. $o(I) = o(G)$ since G has infinite rank. Let $I = S_1 \cup S_2$ where $o(S_1) = \alpha$, $o(S_2) = o(G)$, and $S_1 \cap S_2 = \phi$. Let T be a subset of S_2 such that $o(S_2 - T) = o(G)$. There are $2^{o(G)}$ such T's. Let $(h_1, h_2, \cdots, h_a, \cdots)$ be in G', and let

$$(h_1, h_2, \cdots, h_a, \cdots)t = \left(\sum_{j \in T} h_j, k_1, k_2, \cdots, k_a, \cdots\right),$$

where $k_i = h_i$ if $i \in S_1$ and $k_i = 0$ otherwise. The mapping t is a homomorphism and the order of its image is equal to $o(S_1)$. That is, the index of the kernel of t is α. The order of the kernel of t is equal to $o(G)$ since $o(S_2 - T) = o(G)$, and $G' \cap H_a \neq 0$ for all $a \in I$. Let $T, T' \supseteqq S_2$, $T \neq T'$. Then there is a $j \in T$ such that $j \notin T'$, say. Let $h_j \in G'$, $h_j \neq 0$. Then

$$(0, 0, \cdots, h_j, 0, 0, \cdots)t = (h_j, 0, \cdots).$$

However, $(0, 0, \cdots, h_j, 0, 0, \cdots)t' = (0, 0, 0, \cdots)$. Hence the kernel of t is not the same as the kernel of t'. These kernels are pure in G' since the quotient groups are torsion free. Thus G has $2^{o(G)}$ pure subgroups of index α, and of order equal to $o(G)$. Suppose $(h_1, h_2, \cdots, h_a, \cdots)$ is in the intersection of all these pure subgroups of index α. Then if $b \in S_1$, $h_b = 0$. Hence if $h_c \neq 0$, letting $T = \{c\}$, we have

$$(h_1, h_2, \cdots, h_c, \cdots, h_a, \cdots)t = (h_c, 0, 0, \cdots) \neq 0,$$

which is impossible. Therefore for each $a \in I$, $h_a = 0$, and this shows that the intersection of these subgroups is 0.

5. REMARKS. Every torsion free divisible group D of rank α is a direct sum of α copies of the additive group of rational numbers, and D contains an isomorphic copy of every torsion free Abelian group of rank α. The following theorem says that if α is infinite, every torsion free Abelian group of rank α is represented in a special way in D.

6. THEOREM. *Every torsion free Abelian group G of infinite rank can be represented as a subdirect sum G' of copies of the additive group of rational numbers, and in such a way that G' intersects each subdirect summand non-trivially.*

Proof. Represent G as a subdirect sum G' of the rational groups

$\{H_a\}_{a\in I}$ such that for each $a \in I$, $G' \cap H_a \neq 0$. Suppose first that G has countably infinite rank. That is, suppose I is the set of positive integers. Each H_a is a subgroup of the additive group of rational numbers, since G is torsion free. Let $k_1, k_2, k_3, \cdots$ be a sequence of non-zero rational numbers such that $k_i \in G' \cap H_i$. Let $r_1, r_2, r_3, \cdots$ be the non-zero rational numbers arranged in a sequence. Let $s_i = r_i/k_i$. Let $(h_1, h_2, \cdots, h_n, \cdots)$ be an element of G'. Let

$$(h_1, h_2, \cdots, h_n, \cdots)\beta = \left(\sum_{i=1}^{\infty} s_i h_i, \sum_{i=2}^{\infty} s_i h_i, \cdots, \sum_{i=n}^{\infty} s_i h_i, \cdots\right).$$

Since only a finite number of the h_i's are non-zero, for each k, $\sum_{i=k}^{\infty} s_i h_i$ is a rational number, and for only a finite number of k's is $\sum_{i=k}^{\infty} s_i h_i$ non-zero.

$$\begin{aligned}
&((h_1, h_2, \cdots, h_n, \cdots) + (g_1, g_2, \cdots, g_n, \cdots))\beta \\
&\quad = (h_1 + g_1, h_2 + g_2, \cdots, h_n + g_n, \cdots)\beta \\
&\quad = \left(\sum_{i=1}^{\infty} s_i(h_i + g_i), \cdots, \sum_{i=n}^{\infty} s_i(h_i + g_i), \cdots\right) \\
&\quad = \left(\sum_{i=1}^{\infty} s_i h_i + \sum_{i=1}^{\infty} s_i g_i, \cdots, \sum_{i=n}^{\infty} s_i h_i + \sum_{i=n}^{\infty} s_i g_i, \cdots\right) \\
&\quad = (h_1, h_2, \cdots, h_n, \cdots)\beta + (g_1, g_2, \cdots, g_n, \cdots)\beta .
\end{aligned}$$

Hence β is a homomorphism of G' into a direct sum of copies of the additive group R of rationals. Let R_n be the set of nth coordinates of elements of $G'\beta$. R_n is a subgroup of R since it is the image of the projection of $G'\beta$ onto its nth coordinates. Let $m \geqq n$.

$$(0, 0, \cdots, 0, k_m, 0, 0, \cdots) \in G'$$

and

$$(0, 0, \cdots, 0, k_m, 0, 0, \cdots)\beta = (r_m, r_m, \cdots, r_m, 0, 0, \cdots),$$

so that $r_m \in R_n$. Thus R_n contains all but at most a finite number of elements of R, and being a subgroup of R, must then be R. Therefore $G'\beta$ is a subdirect sum of copies of R. Let $x \in G'$, $x \neq 0$, and let h_r be the last non-zero coordinate of x. Then the rth coordinate of $x\beta$ is $s_r h_r \neq 0$. Hence the kernel of β is 0 and β is an isomorphism of G onto a subdirect sum of copies of R. Now consider the case where I is not countable. Let I be the union of the set of mutually disjoint countably infinite sets $\{I_j\}_{j\in J}$. Denote by S_j the image of the projection of G' into $\sum_{a\in I_j} \oplus H_a$. Then G' is a subdirect sum of the set of groups $\{S_j\}_{j\in J}$, and each S_j is of countably infinite rank. Hence each S_j may be represented as a subdirect sum of copies of the additive group of rational numbers, and it follows that G may be so represented. In light of the proof of 2, this representation may be assumed to intersect each subdirect summand non-trivially.

REFERENCES

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