NON-ABELIAN ORDERED GROUPS

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1. Introduction. In this paper we prove some theorems about non-abelian o-groups, and give some methods of constructing such groups. Most of the literature on o-groups is concerned with abelian o-groups, and the examples in print of non-abelian o-groups are few. Iwasawa [8] proves that any free group can be ordered, and he also gives some additional examples of o-groups. Vinogradov [15] shows that the free product of two o-groups A and B can be ordered so as to preserve the given orders. Chehata [1] gives an example of an o-group that is simple. [3] and [11] contain examples of o-groups. Most of the theorems in this paper give methods for constructing o-groups. For example, in § 3 we study the o-automorphisms of an o-group G. For every group A of o-automorphisms of G that can be ordered we can construct a new o-group H that contains A and G. H is the natural splitting extension of G by A. In § 5 the relationship between central extensions and bilinear mappings is exploited. It is shown that any skew-symmetric real matrix can be used to construct o-groups. In § 6 some o-groups of rank 2 are constructed. In § 4 a study is made of the ordered extensions of a subgroup of the reals. One of the main results is a necessary and sufficient condition for such an extension to split. The principal tool used throughout is the extension theory of Schreier [14].

2. Notation and Terminology. The notation of [3] is used throughout. In particular, the notation and results from § 2 [3, pp. 517–518] are used repeatedly. Unless otherwise stated the group operation will always be addition and 0 will denote a group identity. N and N' are o-groups with elements $a, b, c, \cdots$ and $a', b', c', \cdots$ respectively. G is a normal o-extension of N by N'. We identify G with its representation $G' = N' \times N$, where

$$(a', a) + (b', b) = (a' + b', f(a', b') + ar(b') + b)$$

and (a', a) is positive if $a' > 0$ or $a' = 0$ and $a > 0$. See [3] for the properties of the factor mapping f and the representative function r.

θ will always denote a trivial homomorphism of a group onto the identity element of some other group. For an o-group H, let $A(H)$ be the group of all o-automorphisms of H. For an abelian o-group K, let $D(K)$ be the d-closure or completion of K. In particular, $D(K)$ is a vector space over the rationals and there is a natural extension of the order

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of K to an order of $D(K)$. Finally let $\mathbf{R}$ be the additive group of all real numbers, $\mathbf{P}$ be the multiplicative group of all positive real numbers, R be the additive group of all rational numbers, $\mathbf{P}$ be the multiplicative group of all positive rational numbers, and I be the additive group of integers—*all with their natural order.*

3. Order preserving automorphisms of G. If H is an o-group and A is a group of o-automorphisms of H that can be ordered, then the group $H' = A \times H$, where $(\alpha, a) + (\beta, b) = (\alpha\beta, a\beta + b)$ for α, β in A and a, b in H, can be ordered. Simply define (α, a) positive if α is positive in A or α is the identity and a is positive in H. Then clearly H' is a splitting o-extension of H by A. Thus if A contains more than one element, then H' is a non-abelian o-group. If A is the group of all o-automorphisms of H, then H' is called the o-*holomorph* of H. In [5] it has been shown that a certain class of o-groups with well ordered rank have ordered o-holomorphs. In this section we investigate the o-automorphisms of G.

Let π be an o-automorphism of G for which $(0 \times N)\pi = 0 \times N$. and let $\mathscr{A}$ be the group of all these o-automorphisms. If G has well ordered rank or if N' or N has finite rank, then $\mathscr{A} = A(G)$. For (a', a) and (b', b) in G we have

$$(a', a)\pi = [(a', 0) + (0, a)]\pi = (a', 0)\pi + (0, a)\pi$$
$$= (a'\alpha, a'\beta) + (0, a\gamma) = (a'\alpha, a'\beta + a\gamma) ,$$

where

$$0\beta = 0 . \tag{1}$$

$$[(a', a) + (b', b)]\pi = (a' + b', f(a', b') + ar(b') + b)\pi$$
$$= ((a' + b')\alpha, (a' + b')\beta + (f(a', b') + ar(b') + b)\gamma) .$$
$$(a', a)\pi + (b', b)\pi = (a'\alpha, a'\beta + a\gamma) + (b'\alpha, b'\beta + b\gamma)$$
$$= (a'\alpha + b'\alpha, f(a'\alpha, b'\alpha) + (a'\beta + a\gamma)r(b'\alpha) + b'\beta + b\gamma) .$$

Thus $\alpha \in A(N')$ and

$$(a' + b')\beta + (f(a', b') + ar(b') + b)\gamma$$
$$= f(a'\alpha, b'\alpha) + (a'\beta + a\gamma)r(b'\alpha) + b'\beta + b\gamma .$$

When $a' = b' = 0$ this reduces to $(a + b)\gamma = a\gamma + b\gamma$. Thus $\gamma \in A(N)$. The following two equations are the result of letting $a' = b = 0 (a = b = 0)$.

$$b'\beta + ar(b')\gamma = a\gamma r(b'\alpha) + b'\beta \tag{2}$$

$$(a' + b')\beta + f(a', b')\gamma = f(a'\alpha, b'\alpha) + a'\beta r(b'\alpha) + b'\beta . \tag{3}$$

Conversely suppose that $\alpha \in A(N')$, $\gamma \in A(N)$, $\beta : N' \to N$, and (1), (2), (3) are satisfied. For $(a' a)$ in G define $(a', a)\pi = (a'\alpha, a'\beta + a\gamma)$. Then by straightforward computation it follows that $\pi \in \mathscr{A}$.

For mappings u and v of N' into N and $a' \in N'$ we define $a'(u + v) = a'u + a'v$. Then each $\pi \in \mathscr{A}$ has a matrix representation

$$\begin{bmatrix} \alpha\beta \\ \theta\gamma \end{bmatrix}$$

where θ is the trivial homomorphism of N, into N', and the mapping of π onto its matrix representation is an isomorphism of $\mathscr{A}$ onto

$$\left\{ \begin{bmatrix} \alpha\beta \\ \theta\gamma \end{bmatrix} : \alpha \in A(N), \gamma \in A(N'), \beta : N' \to N, \text{ and (1), (2), (3) are satisfied} \right\}.$$

For, let $\pi = (\alpha, \beta, \gamma)$ and $\bar{\pi} = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$, then

$$\begin{aligned}(a', a)\bar{\pi}\pi &= (a'\bar{\alpha}, a'\bar{\beta} + a\bar{\gamma})\pi = (a'\bar{\alpha}\alpha, a'\bar{\alpha}\beta + (a'\bar{\beta} + a\bar{\gamma})\gamma) \\ &= (a'\bar{\alpha}\alpha, a'(\bar{\alpha}\beta + \bar{\beta}\gamma) + a\bar{\gamma}\gamma)\end{aligned}$$

and

$$(4) \qquad \begin{bmatrix} \bar{\alpha}\bar{\beta} \\ \theta\bar{\gamma} \end{bmatrix}\begin{bmatrix} \alpha\beta \\ \theta\gamma \end{bmatrix} = \begin{bmatrix} \bar{\alpha}\alpha & \bar{\alpha}\beta + \bar{\beta}\gamma \\ \theta & \bar{\gamma}\gamma \end{bmatrix}$$

We shall frequently identity the elements of $\mathscr{A}$ with their matrix representation. Let $\mathscr{B}$ be the set of all $\beta : N' \to N$ that satisfy (1), (2), (3) when α and γ are the identity automorphisms of N' and N respectively.

LEMMA 3.1. *$\mathscr{B}$ is an additive group that can be ordered.*

Proof. From the matrix representation of $\mathscr{A}$ it follows that $\mathscr{B}$ is a group. Well order the elements of N' and define $\beta \in \mathscr{B}$ positive if $\beta \neq \theta$ and $a'\beta > 0$, where a' is the first element in the well ordering for which $a'\beta \neq 0$. It is easy to check that this definition orders $\mathscr{B}$.

COROLLARY I. *The group of all mappings of a set onto an o-group can be ordered.*

COROLLARY II. *The group of all o-automorphisms of G that induce the identity automorphism on $G/(0 \times N)$ and on $0 \times N$ can be ordered.*

Now suppose that $\mathscr{B}$, $A(N')$ and $A(N)$ are o-groups and let

$$\pi = \begin{bmatrix} \alpha\beta \\ \theta\gamma \end{bmatrix} \quad \bar{\pi} = \begin{bmatrix} \bar{\alpha}\bar{\beta} \\ \theta\bar{\gamma} \end{bmatrix}$$

be elements of $\mathscr{A}$. Then

$$(5) \quad \pi^{-1} = \begin{bmatrix} \alpha^{-1} & -\alpha^{-1}\beta\gamma^{-1} \\ \theta & \gamma^{-1} \end{bmatrix} \quad \pi^{-1}\bar{\pi}\pi = \begin{bmatrix} \alpha^{-1}\bar{\alpha}\alpha & \alpha^{-1}(\bar{\alpha}\beta + \bar{\beta}\gamma) - \alpha^{-1}\beta\gamma^{-1}\bar{\gamma}\gamma \\ \theta & \gamma^{-1}\bar{\gamma}\gamma \end{bmatrix}$$

DEFINITION 3.1. π is *positive* if α is positive in $A(N')$ or α is the identity and γ is positive in $A(N)$ or α ond γ are identity automorphisms and β is positive in $\mathscr{B}$.

Let $\mathscr{P}$ be the set of all positive elements in $\mathscr{A}$. It follows from (4) that $\mathscr{P}$ is closed with respect to multiplication. It follows from the first part of (5) that for each $\pi \in \mathscr{A}$, either π is the identity or $\pi \in \mathscr{P}$ or $\pi^{-1} \in \mathscr{P}$. Unfortunately $\mathscr{P}$ is not in general normal. For suppose that $\bar{\pi} \in \mathscr{P}$, then if $\bar{\alpha}$ is positive or $\bar{\gamma}$ is positive, then $\pi^{-1}\bar{\pi}\pi$ is positive. Finally assume that $\bar{\alpha}$ and $\bar{\gamma}$ are identity automorphisms, then

$$\pi^{-1}\bar{\pi}\pi = \begin{bmatrix} \phi' & \alpha^{-1}(\beta + \bar{\beta}\gamma - \beta) \\ \theta & \phi \end{bmatrix},$$

where $\phi'(\phi)$ is the identity of $A(N')(A(N))$. Thus our definition orders $\mathscr{A}$ if and only if $\alpha^{-1}(\beta + \bar{\beta}\gamma - \beta) = \alpha^{-1}\beta + \alpha^{-1}\bar{\beta}\gamma - \alpha^{-1}\beta$ is positive. If we use the ordering of $\mathscr{B}$ defined in the proof of Lemma 3.1, then it suffices to show that $a'\alpha^{-1}\bar{\beta} > 0$, where a' is the first element in the well ordering of N' such that $a'\alpha^{-1}\bar{\beta} \neq 0$.

THEOREM 3.1. *If $A(N)$ can be ordered, then the group of all o-automorphisms π of G such that $(0 \times N)\pi = 0 \times N$ and π induces the identity automorphism on $G/(0 \times N)$ can be ordered.*

We next consider the special cases where G is a central extension of N or where G is a splitting extension of N. First assume that N (actually $0 \times N$) is in the center of G. Then $r = \theta$ and N is abelian. In particular, (1), (2), (3) reduce to

$$(a' + b')\beta + f(a', b')\gamma = f(a'\alpha, b'\alpha) + a'\beta + b'\beta$$

and $0\beta = 0$. Thus $\mathscr{B}$ is the torsion free abelian group $H(N', N)$ of all homomorphisms of N' into N. Let Γ be the set of all ordered pairs of convex subgroups N'^{γ}, N'_{γ} of N' such that N'^{γ} covers N'_{γ}.

THEOREM 3.2. *Suppose that G is a central extension of N, $A(N)$ can be ordered, Γ is well ordered, and for each pair $\alpha \in A(N')$ and $\gamma \in \Gamma$ there exists a pair of positive integers m and n such that $ng\alpha \equiv mg$ modulo N'_{γ} for all $g \in N'^{\gamma}$. Then $A(N')$ and $\mathscr{A}$ can be ordered.*

Proof. By the theorem in [5], $A(N')$ can be ordered. As in the proof of Theorem 3 [4 p. 388] we well order the elements of N' so that

$$\underbrace{0 \to g_{11} \to g_{12} \to \cdots}_{N'^{1}} \quad \underbrace{g_{21} \to g_{22} \to \cdots}_{N'^{2}\backslash N'_{2}} \quad \cdots \quad \underbrace{g_{\omega 1} \to g_{\omega 2} \to \cdots}_{N'^{\omega}\backslash N'_{\omega}} \quad \cdots$$

For each $\theta \neq \beta \in \mathscr{B}$ there exists a least element $L(\beta)$ in this well ordering such that $L(\beta)\beta \neq 0$. Define β positive if $L(\beta)\beta > 0$. As before this orders $\mathscr{B}$. Thus to complete the proof it suffices to show that if β is positive, then $\alpha\beta$ is positive for all $\alpha \in A(N')$. Let $g \in N'^{\gamma}/N'_{\gamma}$. Then there exist positive integers m and n such that $n(g\alpha) = mg + d$, where $d \in N'_{\gamma}$, hence $d \to g$. If $g \to L(\beta)$, then

$$n(g\alpha\beta) = (mg + d)\beta = m(g\beta) + d\beta = 0.$$

Thus $g\alpha\beta = 0$. If $g = L(\beta)$, then

$$n(L(\beta)\alpha\beta) = (mL(\beta) + d)\beta = m(L(\beta)\beta) + d\beta = m(L(\beta)\beta) > 0 .$$

Thus $L(\beta)\alpha\beta > 0$.

COROLLARY. *If N is in the center of G, $A(N)$ can be ordered and $N' = R$, then $A(G)$ can be ordered.*

One should be careful not to place too many restrictions on G. For $A(G)$ may become trivial (consist of the identity only). de Groot [6] has shown that exist 2^c non-isomorphic archimedean o-groups that admit only the identity automorphism. Suppose that G admits no proper o-automorphism and that N' and N are non-trivial. Then, since an inner automorphism is an o-automorphism, G is abelian. Hence N is in the center of G. Thus in order to construct a non-archimedean o-group that admits only the trivial o-automorphism, it suffices to find non-trivial subgroups N' and N of $\mathbf{R}$ such that neither admit proper o-automorphisms and the only homomorphism of N' into N is θ. Then $G = N' \oplus N$ will do. One such pair is

$$N = I \text{ and } N' = \{m/2^n : m, n \in I\}e + \{p/3^q : p, q \in I\} ,$$

where e is trancendental.

For the remainder of this section assume that G is a splitting extension of N by N' and that $N \subseteq \mathbf{R}$. Without loss of generality $f(a', b') = 0$ for all a', b' in N' and $A(N) \subseteq \mathbf{P}$. Thus $r(b'), \gamma \in \mathbf{P}$, and $ar(b'), a\gamma$ represent ordinary multiplication, where $a \in N, b' \in N'$ and $\gamma \in A(N)$. In particular, (2) and (3) reduce to

(2′) $$r(b') = r(b'\alpha), \text{ and}$$

(3′) $$(a' + b')\beta = a'\beta r(b') + b'\beta .$$

Pick an element $k \in N$ and define $x'\beta = k(r(x') - 1)$ for all $x' \in N'$. $a'\beta r(b') + b'\beta = k(r(a') - 1)r(b') + k(r(b') - 1) = k(r(a')r(b') - 1) = k(r(a' + b') - 1) = (a' + b')\beta$. Thus $\beta \in \mathscr{B}$. Suppose that there exists an element a' in the center of N' such that $r(a') \neq 1$. Let x' be any other element of N', and let $\beta \in \mathscr{B}$. Then $x'\beta r(a') + a'\beta = (x' + a')\beta = (a' + x')\beta = a'\beta r(x') + x'\beta$. Thus $x'\beta(r(a') - 1) = a'\beta(r(x') - 1)$ or

$$x'\beta = \left[\frac{a'\beta}{r(a') - 1}\right][r(x') - 1] . \tag{6}$$

Therefore β is determined by $a'\beta$.

LEMMA 3.2. *If there exists an element a' in the center of N' such that $r(a') \neq 1$, then $\mathscr{B}$ is isomorphic to a subgroup of* **R** *that contains N.*

Proof. For $\beta \in \mathscr{B}$ we define $\beta\sigma = (a'\beta)/(r(a') - 1)$. Then

$$\begin{aligned}(\beta_1 + \beta_2)\sigma = a'(\beta_1 + \beta_2)/(r(a') - 1) = (a'\beta_1)/(r(a') - 1) \\ + (a'\beta_2)/(r(a') - 1) = \beta_1\sigma + \beta_2\sigma .\end{aligned}$$

If $0 = \beta\sigma = (a'\beta)/(r(a') - 1)$, then $a'\beta = 0$. Thus by (6), $\beta = \theta$. Therefore σ is an isomorphism of $\mathscr{B}$ into **R**, and by the preceding discussion $\mathscr{B}\sigma \supseteq N$.

If $r(a') < 1$, then $1 < r(a')^{-1} = r(-a')$. Thus we may assume that $r(a') - 1 > 0$. Define $\beta \in \mathscr{B}$ positive (notation) $\beta > \theta$) if $\beta\sigma > 0$. Then $\mathscr{B}$ is ordered and $A(N) \subseteq \mathbf{P}$ has a natural order. $\beta\sigma = (a'\beta)/(r(a') - 1) > 0$ if and only if $a'\beta > 0$. Thus $\beta > \theta$ if and only if $a'\beta > 0$. Suppose that $A(N')$ is also ordered. Then Definition 3.1 orders $A(G)$ if we can show that $\bar{\beta} > \theta$ implies that $\alpha^{-1}(\beta + \bar{\beta}\gamma - \beta) > \theta$ for all $\bar{\beta} \in \mathscr{B}$, and all $\pi = (\alpha, \beta, \gamma) \in A(G)$. But

$$\begin{aligned}a'\alpha^{-1}(\beta + \bar{\beta}\gamma - \beta) = a'\alpha^{-1}\bar{\beta}\gamma = ((a'\alpha^{-1})\bar{\beta})\gamma \\ = [(a'\bar{\beta})(r(a'\alpha^{-1}) - 1)/(r(a') - 1)]\gamma = a'\bar{\beta}\gamma .\end{aligned}$$

But since $a'\bar{\beta} > 0$ we have $a'\bar{\beta}\gamma > 0$.

THEOREM 3.3. *If G splits over N, $N \subseteq$* **R**, *$A(N')$ can be ordered and there exists an element a' in the center of N' such that $r(a') \neq 1$, then $A(G)$ can be ordered.*

COROLLARY. *If H is a non-abelian splitting o-extension of a subgroup of* **R** *by a subgroup of* **R**, *then $A(H)$ can be ordered.*

This is an immediate consequence of the theorem. If $N' = \mathbf{R}$, then (2′) is equivalent to $1 = r(b'(\alpha - 1))$. Hence either $r = \theta$ or $\alpha = 1$. Thus if $N' = \mathbf{R}$, then this corollary is an immediate consequence of Theorem 3.1.

4. Ordered extension of subgroups of R. Throughout this section assume that N is a subgroup of $\mathbf{R}$ and that N' is abelian. In particular, r is a homomorphism of N' into the group $A(N)$, and without loss of generality $A(N) \subseteq \mathbf{P}$ and $ar(b')$ is ordinary multiplication, where $a \in N$ and $b' \in N$.

$$(a', a) + (0, b) = (a', a + b) \text{ and } (0, b) + (a', a) = (a', br(a') + a) .$$

These are equal if and only if $br(a') = b$. Thus G is a central extension of N by N' if and only if $r = \theta$.

LEMMA 4.1. *Suppose that N' is d-closed. Then there exists a non-central o-extension of N by N' if and only if there exists $1 \neq p \in \mathbf{P}$ such that $p^s N = N$ for all $s \in R$.*

Proof. First suppose that G is a non-central o-extension of N by N'. Then $r \neq \theta$. Pick $a' \in N'$ so that $1 \neq r(a') = p \in \mathbf{P}$. For each positive integer n there exists $b' \in N'$ such that $nb' = a'$. Hence $p = r(a') = r(nb') = r(b')^n$. Thus $r(b') = p^{1/n}$. For $m \in I$, we have $r(mb') = r(b')^m = p^{m/n}$. Thus $p^{m/n}N = N$ for all rational numbers m/n.

Conversely suppose that there exists $1 \neq p \in \mathbf{P}$ such that $p^s N = N$ for all $s \in R$. Pick $0 \neq b' \in N'$. Then $N' = Rb' \oplus D$, where Ra' is the one dimensional subspace of N' that contains a' and D is a subspace of N'. Each $a' \in N'$ has a unique representation $a' = sb' + d$, where $s \in R$ and $d \in D$. Define $q(a') = p^s$. Then $H = N' \times N$, where $(a', a) + (b', b) = (a' + b', aq(b') + b)$ is a splitting extension of N by N' that is not a central extension.

COROLLARY. *If N' is d-closed and $N \subseteq R$, then G is a central extension of N by N'.*

THEOREM 4.1. *Suppose that $r \neq \theta$. Then G splits over N if and only if there exist $a' \in N'$ and $a \in N$ such that*

(a) $r(a') \neq 1$

(b) $[1/(r(a') - 1)][a(r(b') - 1) + f(a', b') - f(b', a')] \in N$ for all $b' \in N'$.

Proof. First suppose that G splits. Choose a group H of representatives of G/N, and pick one element (a', a) of H such that $r(a') \neq 1$. Let (b', b) be any other element of H. Then since H is abelian,

$$\begin{aligned}(b' + a', f(b', a') + br(a') + a) &= (b', b) + (a', a) = (a', a) + (b', b) \\ &= (a' + b', f(a', b') + ar(b') + b) .\end{aligned}$$

Thus

$$b(r(a') - 1) = a(r(b') - 1) + f(a', b') - f(b', a') .$$

(b) is satisfied because

$$[1/(r(a') - 1)][a(r(b') - 1) + f(a', b') - f(b', a')] = b .$$

Note that

$$H = \{(b', [1/(r(a') - 1)][a(r(b') - 1) + f(a', b') - f(b', a')]) : b' \in N'\} .$$

Thus H is uniquely determined by (a', a).

Conversely suppose that $a' \in N'$ and $a \in N$ satisfy (a) and (b). Let

$$S = \{(b', b) \in G : (b', b) + (a', a) = (a', a) + (b', b)\} .$$

Clearly S is a group. By the above computation it follows that $(b', b) \in S$ if and only if

$$b = [1/(r(a') - 1)][a(r(b') - 1) + f(a', b') - f(b', a')] .$$

Thus for each $b' \in N'$ there is one and only one (b', b) in S. Therefore S is a group of representatives for G/N.

The factor mapping f is *symmetric* (*skew-symmetric*) if $f(a', b') = f(b', a')$ $(f(a', b') = -f(b', a'))$ for all a', b' in N'.

COROLLARY I. *If $r \neq \theta$ and f is symmetric, then G splits. Moreover $f(a', b') = 0$ for all a', b' in N'.*

Proof. Pick $a' \in N'$ such that $r(a') \neq 1$ and let $a = 0$. Then (a) and (b) are satisfied, hence G splits. Also by the proof of the converse of the theorem, $S = \{(b', 0) : b' \in N'\}$ is a group of representatives. Thus $(a', 0) + (b', 0) = (a' + b', f(a', b')) \in S$. Therefore $f(a', b') = 0$

Let $f(N', N')$ denote the range of f.

COROLLARY II. *If there exists an $a' \in N'$ such that $r(a') \neq 1$ and $[1/(r(a') - 1)]f(N', N') \subseteq N$, then G splits.*

Proof. Let $a = 0$. Then (a) and (b) are satisfied. Moreover, $\{(b', [1/(r(a') - 1)][f(a', b') - f(b', a')])\}$ is a group of representatives.

COROLLARY III. *If N is a field and $r \neq \theta$, then G splits.*

Proof. Pick $a' \in N'$ such that $r(a') \neq 1$. Since $1 \in N$ and $r(a')N = N$, $r(a') \in N$. Thus $1/(r(a') - 1) \in N$ and

$$[1/(r(a') - 1)]f(N', N') \subseteq [1/(r(a') - 1)]N = N .$$

REMARK. Rich [13] proved that if $N \subseteq \mathbf{R}$, $N' = \mathbf{R}$ and $r \neq \theta$, then

G splits. This is a special case of Corollary III. Corollary III can be stated independently of the representation of G as follows: If H is an o-group, C is a convex subgroup of H that is o-isomorphic to the additive group of a subfield of $\mathbf{R}$, and H/C is abelian, then either H is a splitting extension of C or H is a central extension of C.

COROLLARY IV. *If there exists an* $a' \in N'$ *such that* $r(a') = (n+1)/n$ *for some positive integer* n, *then* G *splits.*

Proof. $1/(r(a')-1)=n$. Thus $[1/(r(a')-1)]f(N', N')=nf(N', N')\subseteq N$.

COROLLARY V. *If* N *is d-closed and there exists an* $a' \in N'$ *such that* $1 \neq r(a')$ *is rational, then* G *splits.*

Proof. $1/(r(a')-1)$ is rational, hence $[1/(r(a')-1)]N \subseteq N$.

By Theorem 3.3 [3, p. 522] there exists an a-extension H of G such that the convex subgroup K of H that covers 0 is o-isomorphic to $\mathbf{R}$ and H/K is o-isomorphic to N'. Thus by Theorem 4.1 either H is a splitting extension of K or H is a central extension of K.

REMARK. If H is a splitting o-extension of K, then without loss of generality $H = N' \times \mathbf{R}$, where $(a', a) + (b', b) = (a' + b', \quad as(b') + b)$. s is a homomorphism of N' into $\mathbf{P}$. For each x in $D(N)$ there exists a positive integer n such that $nx \in N'$. Define $t(x) = [s(nx)]^{1/n}$. Then t is the unique extension of s to a homomorphism of $D(N')$ into $\mathbf{P}$. $D(N')$, $\mathbf{R}$ and t determine a splitting o-extension M of $\mathbf{R}$ by $D(N)$. M is an a-extension of H and M is d-closed. Thus by Theorem 3.2 [3 p. 519] there exists an a-closed a-extension Q of M with each component o-isomorphic to $\mathbf{R}$. Q is an a-extension of G.

A mapping g of $N' \times N'$ into N is called *bilinear* if for all x, y, z in N'

$$g(x + y, z) = g(x, z) + g(y, z) ,$$

and

$$g(x, y + z) = g(x, y) + g(x, z) .$$

Yamabe [16] and the Neumanns [12] have shown that if $N = I$, and the cardinality of N' is at most $\aleph_1$, and g is bilinear and satisfies $g(x, x) = 0$ only if $x = 0$, then N' is a free abelian group. Hughes [7] has classified the groups of class 2 in terms of some special bilinear mappings. Iwasawa gives an example ([8] Example 2, p. 7) of an o-group that is determined by a bilinear mapping. For let $N'=I\times I$ and $N=I$. Define $g((a,b), (x,y))= ay$. Then $G=I\times I\times I$, where $(a,b,c)+(x,y,z)=(a+x, b+y, ay+c+z)$, and (a,b,c) is positive if $a>0$ or $a=0$ and $b>0$ or $a=b=0$ and $c>0$, is an o-group of rank 3 that is isomorphic with Iwasawa's example. In fact, G is generated by $a=(0,0,1)$, $b=(0,1,0)$ and $c=(1,0,0)$ and has generating relations $a+b=b+a$, $a+c=c+a$ and $c+b-c=a+b$.

The last example can be generalized because the bilinear form is a product of homomorphisms. For example, let N be the additive group of an ordered ring, and let σ and τ be homomorphisms of N' into N. For a', b' in N' define $g(a',b')=\sigma(a')\tau(b')$. Then $H=N'\times N$, where $(a',a)+(b',b)=(a'+b', g(a',b')+a+b)$ is a central extension of N by N'.

LEMMA 4.2. *If f is bilinear, then G is a splitting extension of N or G is a central extension of N.*

Proof. For x, y, z in N' we have

$$\begin{aligned} f(x,y)+f(x,z)+f(y,z) &= f(x,y+z)+f(y,z)=f(x+y,z)+f(x,y)r(z) \\ &= f(x,z)+f(y,z)+f(x,y)r(z) \,. \end{aligned}$$

Therefore $f(x,y)\equiv f(x,y)r(z)$. Thus either $r(z)\equiv 1$ or $f(x,y)\equiv 0$.

COROLLARY. *If N is abelian* (not necessarily a subgroup of $\mathbf{R}$), *f is bilinear and $f(N',N')$ generates N, then G is a central extension of N.*

5. Central extensions and bilinear mappings. Throughout this section assume that N is in the center of G. Thus G is determined by the o-group N', the abelian o-group N, and the factor mapping $f\colon N'\times N'\to N$ that satisfies

$$f(0,b')=f(a',0)=0, \text{ and} \tag{1}$$

$$f(a'+b',c')+f(a',b')=f(a',b'+c')+f(b',c') \,. \tag{2}$$

In particular, any central extension of N by N' can be ordered. A central extension H of N by N' with factor mapping h is *equivalent* to G (notation $H\sim G$) if there exists an isomorphism α of H onto G such that $(0,a)\alpha=(0,a)$ and $(a',a)\alpha\equiv(a',a)$ modulo $0\times N$ for all a in N and all a' in N'. If H is ordered in the usual way, then α is an o-isomorphism. It is well known that $H\sim G$ if and only if there exists $t\colon N'\to N$ such that $t(0)=0$ and

$$f(a',b')=h(a',b')-t(a'+b')+t(a')+t(b')$$

for all a', b' in N'. In particular, $G\sim N'\oplus N$ if and only if there exists $t\colon N'\to N$ such that $t(0)=0$ and $f(a',b')=-t(a'+b')+t(a')+t(b')$ for all a', b' in N'.

It is easy to verify that if g is a bilinear mapping of $N' \times N'$ onto N, then g satisfies (1) and (2). Moreover, such a g exists if and only if we can choose a representative function $r: N' \to G$ such that

$$r(a' + b' + c') = r(a' + b') + r(a' + c') + r(b' + c') - r(a') - r(b') - r(c')$$

for all a', b', c' in N'. From (2) we have

$$f(a' + b', c') - f(a', c') - f(b', c') = f(a', b' + c') - f(a', b') - f(a', c') .$$

Thus f is bilinear if it is linear in one variable.

LEMMA 5.1. *Suppose that f is bilinear, then for a, b in N and a', b', c' in N' we have:*

(i) $-f(a', b') = f(-a', b') = f(a', -b')$.

(ii) $f(a', b') = f(-a', -b')$.

(iii) $(a', a) + (b', b) - (a', a) - (b', b) = (a' + b' - a' - b', f(a', b') - f(b', a'))$.

For $0 = f(a' - a', b') = f(a', b') + f(-a', b')$. Thus $-f(a', b') = f(-a', b')$ and similarly $-f(a', b') = f(a', -b')$. (ii) is an immediate consequence of (i), and (iii) follows by computing the left hand side.

Let $D(N)$ be the d-closure of N, and let $H = N' \times D(N)$. For (a', a) and (b', b) in H define $(a', a) + (b', b) = (a' + b', f(a', b') + a + b)$. Then H is a central extension of $D(N)$ by N', and G is a subgroup of H. There is a natural extension of the ordering of G to an ordering of H. If $G \sim N' \oplus N$, then $H \sim N' \oplus D(N)$, but the converse is false. For in [2] there is an example where $N' = D(N) = R$, $H \sim N' \oplus N$ and $G \not\sim N' \oplus N$ [2, p. 862].

THEOREM 5.1. *Suppose that N' is abelian and let $H = D(N') \times D(N)$. Also suppose that for all a', b' in N' and for all positive integers n, f satisfies*

$$(3) \qquad nf(a', b') = f(na', b') = f(a', nb') .$$

Then there exists a unique $g: D(N') \times D(N') \to D(N)$ that satisfies (3) and such that $g(a', b') = f(a', b')$ for all a', b' in N'. For (x, y) and (u, v) in H define $(x, y) + (u, v) = (x + u, g(x, u) + y + v)$.

(*a*) *H is a central extension of $D(N)$ by $D(N')$, and G is a subgroup of H.*

(*b*) *H is d-closed.*

(*c*) *For each h in H there exists a positive integer $n = n(h)$ such that $nh \in G$.*

(*d*) *There exists a unique extension of the ordering of G to an ordering of H. H will be called the d-closure of G.*

Proof. For each pair x, y in $D(N')$ there exists a positive integer

$n = n_{x,y}$ such that $nx, ny \in N'$, define $g(x, y) = (1/n^2)f(nx, ny)$. This definition is independent of the particular choice of n. For if $mx, my \in N'$, then $m^2 f(nx, ny) = f(mnx, mny) = n^2 f(mx, my)$. Thus $(1/n^2)f(nx, ny) = (1/m^2)f(mx, my)$. Let $x, y, z \in D(N')$ and choose a positive integer n such that $nx, ny, nz, n(x + y)$, and $n(y + z)$ belong to N'. Then

$$\begin{aligned} g(x + y, z) + g(x, y) &= (1/n^2)[f(nx + ny, nz) + f(nx, nz)] \\ &= (1/n^2)[f(nx, ny + nz) + f(ny, nz)] = g(x, y + z) + g(y, z) . \end{aligned}$$

By a similar argument g satisfies (1) and (3). Also if g' is any other extension of f to $D(N') \times D(N')$ that satisfies (3), then $n^2 g'(x, y) = g'(nx, ny) = f(nx, ny)$. Therefore $g'(x, y) = (1/n^2)f(nx, ny) = g(x, y)$ for all x, y in $D(N')$.

Clearly (a) is satisfied. To prove (b) it suffices to show that $n(x, y) = (a, b)$ has a solution in H, where n is a positive integer and $(a, b) \in H$. By induction

$$n(x, y) = (nx, [(n - 1)n/2]g(x, x) + ny) .$$

Thus $x = (1/n)a$ and

$$y = (1/n)(b - [(n - 1)n/2]g((1/n)a, (1/n)a))$$

is a solution. Consider (x, y) in H, and let m be a positive integer such that $mx \in N'$ and $my \in N$. Then

$$\begin{aligned} 2m^2(x, y) &= (2m(mx), (2m^2 - 1)m^2 g(x, x) + 2m(my)) \\ &= (2m(mx), (2m^2 - 1)f(mx, my) + 2m(my)) \in G . \end{aligned}$$

Thus (c) is satisfied. The orderings of N and N' can be uniquely extended to orderings of $D(N)$ and $D(N')$. Define $(x, y) \in H$ positive if $x > 0$ or $x = 0$ and $y > 0$. This extends the ordering of G to an ordering of H. But for any extension of the order of G, $h \in H$ is positive if and only if nh is positive in G, where n is a positive integer such that $nh \in G$. Thus this extension is unique.

REMARK. If f is bilinear or symmetric or skew-symmetric, then so is g. By Theorem 3.2 [3, p. 519] there exists an a-closed a-extension of H with each component o-isomorphic to $\mathbf{R}$.

Suppose that f is bilinear. Let $x, y, z \in N'$ and let $w = x + y - x - y$. Then

$$f(w, z) + f(y, z) + f(x, z) = f(w + y + x, z) = f(x + y, z) = f(x, z) + f(y, z) .$$

Thus $f(w, z) = 0$. Similarly $f(z, w) = 0$. Therefore $f(c, z) = f(z, c) = 0$ for all z in N' and all c in the commutator subgroup of N'.

LEMMA 5.2. *If f is bilinear and N' coincides with its commutator* *group, then* $G = N' \oplus N$.

Newmann [11] exhibits an o-group that coincides with its commutator group.

Suppose that $2N = N$ and f is bilinear. Let $p(x, y) = (1/2)[f(x, y) + f(y, x)]$ and let $q(x, y) = (1/2)[f(x, y) - f(y, x)]$ for all x, y in N'. Then $p(q)$ is a symmetric (skew-symmetric) bilinear mapping of $N' \times N'$ into N, and $f(x, y) = p(x, y) + q(x, y)$. Moreover, as in matrix theory, this representation is unique.

THEOREM 5.2. *If* $2N = N$ *and* f *is bilinear, then* $G \sim H$, *where* H *is the central extension of* N *by* N' *that is determined by the skew-symmetric part* q *of. If* f *is symmetric, then* $G \sim N' \oplus N$. *Thus if* G *is abelian, then* $G \sim N' \oplus N$.

Proof. For each x in N' define $t(x) = (-1/2)f(x, x)$. Then

$$\begin{aligned} &-t(x + y) + t(x) + t(y) + q(x, y) \\ &\quad = (1/2)[f(x + y, x + y) - f(x, x) - f(y, y) + f(x, y) - f(y, x)] = f(x, y) . \end{aligned}$$

Thus $G \sim H$. If f is symmetric, then $H = N' \oplus N$, and if G is abelian, then f is symmetric.

Suppose that N and N' are abelian and that f is bilinear. Then by Theorem 5.1, we can embed G into its d-closure $H = D(N') \times D(N)$. The factor mapping g associated with H is bilinear, and by Theorem 5.2 we may choose g so that it is skew-symmetric and bilinear. Moreover, $sg(x, y) = g(sx, y) = g(x, sy)$ for all $s \in R$ and for all x, y in $D(N)$. For

$$ng((m/n)x, y) = g(n(m/n)x, y) = g(mx, y) = mg(x, y) .$$

Thus $(m/n)g(x, y) = g((m/n)x, y)$. Let $\alpha_1, \alpha_2, \cdots$ be a basis for the rational vector space $D(N')$ and consider $X = x_1\alpha_{s_1} + \cdots + x_m\alpha_{s_m}$ and $Y = y_1\alpha_{t_1} + \cdots + y_n\alpha_{t_n}$ in $D(N')$. Then

$$g(X, Y) = \sum x_i g(\alpha_{s_i}, \alpha_{t_j}) y_j$$

Thus g is determined by the skew symmetric matric $A = [g(\alpha_i, \alpha_j)]$ with components in $D(N)$. Conversely any such matric determines a bilinear skew-symmetric factor mapping of $D(N') \times D(N')$ into $D(N)$.

THEOREM 5.3. *If* N' *is abelian and* f *is bilinear, then* G *is a subgroup of its d-closure* H *and* H *is completely determined by* N, N' *and a skew symmetric matrix with entries from* $D(N)$. *The dimension of this matrix is equal to the rank of the vector space* $D(N')$.

If the rank of $D(N')$ is finite, say n, and $D(N) = R$, then by a suitable choice of coordinates for $D(N')$ we can get a canonical form for A.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ -1 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Thus H is determined by n and the rank of A. For example if $N' = R \times R \times R$ and $N = R$, then we have two non-trivial choices for f. One of which is

$$f((x_1, x_2, x_3), (y_1, y_2, y_3))$$
$$= [x_1 x_2 x_3] \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -x_2 y_1 + (x_1 - x_3) y_2 + x_2 y_3 ,$$

and the other is obtained by using the cannonical matrix of rank 2. Thus for any ordering of N' we have at least two non-trivial central o-extensions of N by N'.

LEMMA 5.3. *If A and B are elements of an ordered semigroup S and $A + B < B + A$, then $nA + nB < n(A + B) < n(B + A) < nB + nA$ for all integers n greater than* 2.

Proof. If

$$A + (n-1)A + (n-1)B + B = nA + nB \geq n(A + B)$$
$$= A + (n-1)(B + A) + B ,$$

then $(n-1)A + (n-1)B \geq (n-1)(B + A)$. If

$$B + (n-1)(A + B) + A = n(B + A) \geq nB + nA$$
$$= B + (n-1)B + (n-1)A + A ,$$

then $(n-1)(A + B) \geq (n-1)B + (n-1)A$. Thus the lemma follows immediately by induction on n.

THEOREM 5.4. *If $1 \in N' \subseteq R$, then G is abelian.*

Proof. By a simple induction argument (see [9] p. 265), $f(x, y) = f(y, x)$ for all integers x and y. Let $A = (a', a)$ and $B = (b', b)$ be elements of G. Then since a' and b' are rational numbers, there exists a positive integer n such that $nA = (x', x)$ and $nB = (y', y)$, where x' and y' are integers.

$$nA + nB = (x' + y', f(x', y') + x + y)$$
$$= (y' + x', f(y', x') + y + x) = nB + nA .$$

Thus by Lemma 5.3, we have $A + B = B + A$.

6. o-groups of rank 2. Throughout this section we assume that N and N' are subgroups of $\mathbf{R}$. By Theorem 3.5 [3 p. 523] there exists an a-closed a-extension H of G such that both components are o-isomorphic to $\mathbf{R}$. By Theorem 4.1, either H is a central extension of $\mathbf{R}$ or H is a splitting extension of $\mathbf{R}$. A splitting o-extension of $\mathbf{R}$ by $\mathbf{R}$ is determined by a homomorphism of $\mathbf{R}$ into $\mathbf{P}$. If H is a central extension of $\mathbf{R}$ by $\mathbf{R}$ with a bilinear factor mapping, then H is determined by a skew-symmetric real matrix.

If N' is cyclic, then G is a splitting extension of N. Thus if N' is cyclic and N admits no proper o-automorphisms, then $G = N' \oplus N$. In particular, if $N' = N = I$, then $G = N' \oplus N$. In fact, as Loonstra [9] shows, there are only two normal extensions of I by I (not necessarily ordered) For if H is a normal extension of I by I, then H splits over I. Thus $H = I \times I$ and $(a', a) + (b', b) = (a' + b', as(b') + b)$, where s is a homomorphism of I into the multiplicative group $\{1, -1\}$. Either $s(1) = 1$ or $s(1) = -1$. If $s(1) = 1$, then $s = \theta$ and $H = I \oplus I$. If $s(1) = -1$, then $s(2n) = 1$ and $s(2n+1) = -1$ for all $n \in I$. Thus the addition rule for H is

$$(x, y) + (2m, n) = (x + 2m, y + n)$$
$$(x, y) + (2m + 1, n) = (x + 2m + 1, n - y) .$$

In this case H can't be ordered because $-(1,0) + (0,1) + (1,0) = -(0,1)$. Thus $(0, 1)$ can't be positive or negative.

If $N = N' = R$, then G is o-isomorphic to $R \oplus R$. For by Lemma 4.1, *G is a central extension of N and by Theorem 5.4, G is abelian.* Thus G is an abelian o-group of rank 2 with both components o-isomorphic to R. By Hahn's embedding theorem (see [2]) G is o-isomorphic to $R \oplus R$.

Example of a non-abelian o-group of rank 2 that is isomorphic to its group of o-automorphims. Let $N = N' = \mathbf{R}$. For $a', b' \in N'$ define $f(a', b') = 0$ and $r(a') = e^{a'}$, where e is transcendental. Then $(a', a) + (b', b) = (a' + b', ae^{b'} + b)$. By the remark at the end of § 3, an o-automorphism π of G has a representation $\pi = \begin{bmatrix} 1 & \beta \\ 0 & C \end{bmatrix}$, where $C \in \mathbf{P}$ and $x'\beta = 1\beta(e^{x'} - 1)/(e-1) = \beta\sigma(e^{x'} - 1)$ for all $x' \in N'$. The mapping of π onto $\begin{bmatrix} 1 & \beta\sigma \\ 0 & C \end{bmatrix}$ is an isomorphism of $A(G)$ onto the multiplicative group $A = \left\{\begin{bmatrix} 1 & B \\ 0 & C \end{bmatrix} : B \in \mathbf{R} \text{ and } C \in \mathbf{P}\right\}$. The mapping of $(a', a) \in G$ onto $\begin{bmatrix} e^{a'} & 0 \\ a & 1 \end{bmatrix}$ is an isomorphism of G onto the multiplicative group $B = \left\{\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} : x \in \mathbf{P} \text{ and } y \in \mathbf{R}\right\}$. The mapping of $\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}$ onto $\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}^{-1}$ is an isomorphism of A onto B. Therefore G is isomorphic to $A(G)$. In particular, there exists a non-trivial splitting o-extension of G by G.

We conclude by giving an *example of an o-group of rank 2 that is not a central extension nor a splitting extension of its convex subgroup.* Let G be the o-group of the last example, and let H be the subgroup of G that is generated by $\{(a, a): a \in R\}$. We have $(-1, -1) + (1, 1) = (0, 1 - e)$. Thus H has rank 2.

$$(1, 1) + (0, 1 - e) = (1, 2 - e) \neq (1, e - e^2 + 1) = (0, 1 - e) + (1, 1) .$$

Thus H is not a central extension.

LEMMA. *If $(b', b) \in H$, then $b = \sum_1^m b_i e^{c_i}$, where $b_i, c_i \in R$ and $\sum_1^m b_i = b'$. For $(b', b) = P_1 + P_2 + \cdots + P_n$, where P_i or $-P_i$ is a generator. A simple induction on n proves the lemma. In particular, $(b', 0) \in H$ only if $b' = 0$. It can be shown that $H = \{(a, \sum a_i e^{b_i}): a, a_i, b_i \in R$ and $\sum a_i = a\}$, but we will not need this.*

Now suppose (by way of contradiction) that H is a splitting extension of its convex subgroup C. Pick a group K of representatives of H/C, and let $(1, a)$ be the element in K with first component 1. $a = \sum_1^j a_i e^{b_i}$, where $\sum_1^j a_i = 1$. In particular, $a \neq 0$. By the proof of Theorem 4.1

$$K = \{(b', a(e^{b'} - 1)/(e - 1)): b' \in R\} .$$

Let d be the least common multiple of the denominators of the a_i and let $b' = 1/p$, where p is a prime and $p > d$. Then $d(\sum a_i e^{b_i}) = \sum c_i e^{b_i}$ has integral coefficients. By the above lemma

$$\frac{\left(\sum_1^j c_i e^{b_i}\right)(e^{b'} - 1)}{e - 1} = \sum_1^k e_i e^{d_i} \tag{1}$$

where $e_i, d_i \in R$. Let q be a positive common multiple of p and the denominators of the b_i and the d_i. Then

$$\frac{\left[\sum_1^j c_i (e^{1/q})^{u_i}\right][(e^{1/q})^v - 1]}{(e^{1/q})^q - 1} = \sum_1^k e_i (e^{1/q})^{wq_i} \tag{2}$$

where $u_i, w_i, v \in I$. Without loss of generality we may assume that the u_i and the w_i are positive integers (multiply both sides of (2) by a suitable power of $e^{1/q}$). $e^{1/q}$ is trancendental. Thus (2) is essentially an equality of elements in the simple transcendental field extension $R(X)$ of R.

$$\frac{\left[\sum_1^j c_i X^{u_i}\right][X^v - 1]}{X^q - 1} = \sum_1^k e_i X^{w_i} \tag{3}$$

$b' = 1/p = v/q = v/pv$. Thus there exists a positive integer n such that p^n divides q, but p^n does not divide v. The cyclotomic polynomial

$$f(X) = 1 + X^{p^{n-1}} + X^{2p^{n-1}} + \cdots + X^{(p-1)p^{n-1}}$$

is an irreducible factor of $X^q - 1$, but it does not divide $X^v - 1$. Therefore $f(X)$ divides $\sum c_i X^{u_i}$. Thus $\sum c_i X^{u_i} = f(X)g(X)$, where $g(X)$ is a polynomial with integral coefficients. Now let $X = 1$. Then $d = \sum_1^j c_i = f(1)g(1) = pg(1)$. Thus since p and d are positive and $g(1)$ is an integer, $d \geq p$. But this contradicts our choice of p.

Note that the example on page 526 of [3] is a splitting extension of N by N'; and that $\{(a', -1): 0 \neq a' \in N'\} \cup \{0, 0)\}$ is a group of representatives.

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