SOME CONNECTIONS BETWEEN CONTINUED FRACTIONS AND CONVEX SETS

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The purpose of this paper is to develop certain connections between the continued fraction solutions and the convex set solutions to some of the moment problems. In particular, we shall develop some relations between the work of Wall [3], [4] on continued fractions and the work of Karlin and Shapley [1] on convex sets. The paper is divided into two parts:

I. Stieltjes-type continued fractions and convex sets.

II. Jacobi-type continued fractions and convex sets.

Two characterizations of the moment problem for the interval (0, 1), one by Riesz [2] in terms of convex closures and one in term of Hankel forms, are well known. The work of Karlin and Shapley [1] shows the equivalence of these two characterizations. A third characterization in terms of a Stieltjes-type continued fraction has been given by Wall [3], [4]. In part I we give an interpretation of the parameters in this continued fraction in terms of "distances" in certain convex bodies. This interpretation, through the work of Karlin and Shapley, immediately shows the equivalence of all three characterizations.

Solutions of the moment problem for the interval $(-1, 1)$, in terms of the Riesz condition and Hankel forms, are also well known. In part II we give a third solution in terms of a Jacobi-type continued fraction. Again, through an interpretation of the parameters in this continued fraction in terms of "distances" in certain convex bodies and an extension of the work of Karlin and Shapley, the equivalence of the three characterizations is immediate.

I. STIELTJES-TYPE CONTINUED FRACTIONS AND CONVEX SETS

1. The monotone Hausdorff moment problem. A sequence of real numbers $\{c_n\}(n = 0, 1, 2, \cdots)$ is called a monotone Hausdorff moment sequence if there exists a monotone nondecreasing real function $\phi(u)$, $0 \leqq u \leqq 1$, such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots .$$

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The problem of determining such a function $\phi(u)$ is known as the monotone Hausdorff moment problem. We shall assume throughout part I unless otherwise designated that $c_0 = 1$.

Wall [3], [4] has shown that a sequence $\{c_n\}$ is a monotone Hausdorff moment sequence if and only if the power series

$$P(z) = \sum_{n=0}^{\infty} c_n z^n$$

has a continued fraction expansion of the form

$$\frac{1}{1} - \frac{(1-g_0)g_1 z}{1} - \frac{(1-g_1)g_2 z}{1} - \cdots, \tag{1.1}$$

where $0 \leqq g_p \leqq 1$, $p = 0, 1, 2, \cdots$. We shall agree that the continued fraction terminates with the first identically vanishing partial quotient. The sequence $\{(1 - g_{p-1})g_p\}$ $(p = 1, 2, 3, \cdots)$ is called a *chain sequence* and the numbers g_p are called the *parameters* of the chain sequence. In general the parameters are not uniquely determined and we designate the minimal set of parameters by m_p. In this case $m_0 = 0$ and (1.1) takes the form

$$\frac{1}{1} - \frac{m_1 z}{1} - \frac{(1-m_1)m_2 z}{1} - \frac{(1-m_2)m_3 z}{1} - \cdots. \tag{1.2}$$

Riesz [2], [1], [3] proved that a sequence $\{c_n\}$ is a monotone Hausdorff moment sequence if and only if the point $(c_1, c_2, \cdots, c_n)$, $n = 1, 2, 3, \cdots$, is in the convex closure of the arc whose parametric equations are

$$\begin{aligned} x_1 &= t, \\ x_2 &= t^2, \\ &\cdots \\ x_n &= t^n,\ 0 \leqq t \leqq 1 . \end{aligned} \tag{1.3}$$

The geometry of these convex bodies is developed rather fully in the work of Karlin and Shapley [1].

2. The connecting theorem. Before stating the theorem which connects continued fractions with convex bodies it is necessary to indicate some special notations for the Hankel determinants. We set

$$\Delta_{2n} = \begin{vmatrix} 1\ c_1 & \cdots & c_n \\ c_1\ c_2 & \cdots & c_{n+1} \\ & \cdots & \\ c_n\ c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \cdots, \tag{2.1}$$

$$
(2.2) \qquad \underline{\Delta}_{2n+1} = \begin{vmatrix} c_1 & c_2 & \cdots c_{n+1} \\ c_2 & c_3 & \cdots c_{n+2} \\ & \cdots & \\ c_{n+1} & c_{n+2} & \cdots c_{2n+1} \end{vmatrix}, \qquad \begin{array}{l} n = 0, 1, 2, \cdots, \\ (\underline{\Delta}_{-1} = 1) , \end{array}
$$

$$
(2.3) \qquad \bar{\Delta}_{2n} = \begin{vmatrix} c_1 - c_2 & c_2 - c_3 & \cdots c_n - c_{n+1} \\ c_2 - c_3 & c_3 - c_4 & \cdots c_{n+1} - c_{n+2} \\ & \cdots & \\ c_n - c_{n+1} & c_{n+1} - c_{n+2} & \cdots c_{2n-1} - c_{2n} \end{vmatrix}, \qquad \begin{array}{l} n = 1, 2, 3, \cdots , \\ (\bar{\Delta}_0 = 1), \text{ and} \end{array}
$$

$$
(2.4) \qquad \bar{\Delta}_{2n+1} = \begin{vmatrix} 1 - c_1 & c_1 - c_2 & \cdots c_n - c_{n+1} \\ c_1 - c_2 & c_2 - c_3 & \cdots c_{n+1} - c_{n+2} \\ & \cdots & \\ c_n - c_{n+1} & c_{n+1} - c_{n+2} & \cdots c_{2n} - c_{2n+1} \end{vmatrix}, \qquad \begin{array}{l} n = 0, 1, 2, \cdots , \\ (\bar{\Delta}_{-1} = 1) . \end{array}
$$

It is well known that a sequence $\{c_n\}$ is a monotone Hausdorff moment sequence if and only if the Hankel forms

$$
\sum_{i,j=0}^{n} c_{i+j} x_i x_j, \qquad \sum_{i,j=0}^{n} c_{i+j+1} x_i x_j ,
$$

$$
\sum_{i,j=0}^{n-1} (c_{i+j+1} - c_{i+j+2}) x_i x_j, \qquad \sum_{i,j=0}^{n} (c_{i+j} - c_{i+j+1}) x_i x_j
$$

are all positive semidefinite. In (2.1) replace c_{2n} by $\underline{c}_{2n}$, and in (2.2) replace c_{2n+1} by $\underline{c}_{2n+1}$. Setting $\underline{\Delta}_{2n}$ and $\underline{\Delta}_{2n+1}$ equal to zero, we have the single relation

$$
(2.5) \qquad \underline{c}_n = c_n - \frac{\underline{\Delta}_n}{\underline{\Delta}_{n-2}}, \qquad n = 1, 2, 3, \cdots ,
$$

provided $\underline{\Delta}_{n-2} \neq 0$. Similarly, (2.3) and (2.4) yield

$$
(2.6) \qquad \bar{c}_n = c_n + \frac{\bar{\Delta}_n}{\bar{\Delta}_{n-2}}, \qquad n = 1, 2, 3, \cdots ,
$$

provided $\bar{\Delta}_{n-2} \neq 0$. If the sequence $\{c_n\}$ is a monotone Hausdorff moment sequence, then the quantities $\underline{c}_n$ and $\bar{c}_n$ have been interpreted as the "downward" and "upward" projections, respectively, of c_n on the boundary of the corresponding convex body [1].

We can now state the following theorem:

THEOREM 2.1. *If the sequence* $\{c_n\}$ *is a monotone Hausdorff moment sequence, then the elements and the minimal parameters in the continued fraction* (1.2) *can be written in the forms*

$$a_n = (1 - m_{n-1})m_n = \frac{c_n - \underline{c}_n}{c_{n-1} - \underline{c}_{n-1}}, \qquad n = 1, 2, 3, \cdots, (\underline{c}_0 = 0) , \tag{2.7}$$

and

$$m_n = \frac{c_n - \underline{c}_n}{\bar{c}_n - \underline{c}_n}, \quad 1 - m_n = \frac{\bar{c}_n - c_n}{\bar{c}_n - \underline{c}_n}, \qquad n = 1, 2, 3, \cdots . \tag{2.8}$$

From the proof it will be clear that a more general theorem is true. If $\{c_n\}$, $(c_0 = 1)$, is an arbitrary sequence of real numbers and its corresponding Stieltjes-type continued fraction is written in the form (1.2), where no longer it is necessary that $0 \leqq m_n \leqq 1$, $n = 1, 2, 3, \cdots$, the relations (2.7) and (2.8) are still valid.

If $\{c_n\}$ is a monotone Hausdorff moment sequence, then the m_n can be interpreted as the ratio of the "distance" of c_n to the lower boundary to the "distance" between the upper and lower boundaries of the corresponding convex body. Similar interpretations can be given to the a_n and $(1 - m_n)$. By Theorem 2.1 the equivalence of the condition in terms of Hankel forms and Wall's characterization in terms of the continued fraction (1.2), for the existence of a monotone Hausdorff moment sequence, is apparent.

Proof. The proof depends upon the following lemma:

LEMMA 2.1. *The determinants in* (2.1), (2.2), (2.3), *and* (2.4) *satisfy the relation*

$$\underline{\Delta}_k \bar{\Delta}_k = \bar{\Delta}_{k+1} \underline{\Delta}_{k-1} + \underline{\Delta}_{k+1} \bar{\Delta}_{k-1}, \qquad k = 1, 2, 3, \cdots . \tag{2.9}$$

We shall indicate two proofs to this lemma.

Proof (1). By a substitution and an equivalence transformation, we write the continued fraction (1.2) in the form

$$\frac{1}{z} - \frac{a_1}{1} - \frac{a_2}{z} - \frac{a_3}{1} - \cdots , \tag{2.10}$$

where $a_k = (1 - m_{k-1})m_k$, $k = 1, 2, 3, \cdots$, $(m_0 = 0)$. The recurrence formulas for the denominators of the continued fraction (2.10) are given by

$$B_{2k}(z) = B_{2k-1}(z) - a_{2k-1}B_{2k-2}(z), \qquad k = 1, 2, 3, \cdots, \quad (B_0(z) = 1) , \tag{2.11}$$

and

$$B_{2k+1}(z) = zB_{2k}(z) - a_{2k}B_{2k-1}(z), \quad k = 0, 1, 2, \cdots, \quad (a_0 = 1,\ B_{-1}(z) = 0,\ B_0(z) = 1) . \tag{2.12}$$

Furthermore, we have

$$(2.13) \qquad B_{2k}(z) = \frac{\Delta_{2k}(z)}{\Delta_{2k-2}}, \qquad k = 1, 2, 3, \cdots,$$

and

$$(2.14) \qquad B_{2k+1}(z) = \frac{z\Delta_{2k+1}(z)}{\Delta_{2k-1}}, \qquad k = 0, 1, 2, \cdots,$$

where Δ_{2k-2} and Δ_{2k-1} are obtained from (2.1) and (2.2), respectively, and we define

$$(2.15) \qquad \Delta_{2k}(z) = \begin{vmatrix} 1 & c_1 & \cdots & c_{k-1} & 1 \\ c_1 & c_2 & \cdots & c_k & z \\ c_2 & c_3 & \cdots & c_{k+1} & z^2 \\ & & \cdots & & \\ c_k & c_{k+1} & \cdots & c_{2k-1} & z^k \end{vmatrix}, \qquad k = 1, 2, 3, \cdots,$$

and

$$(2.16) \qquad \Delta_{2k+1}(z) = \begin{vmatrix} c_1 & c_2 & \cdots & c_k & 1 \\ c_2 & c_3 & \cdots & c_{k+1} & z \\ c_3 & c_4 & \cdots & c_{k+2} & z^2 \\ & & \cdots & & \\ c_{k+1} & c_{k+2} & \cdots & c_{2k} & z^k \end{vmatrix}, \qquad k = 1, 2, 3, \cdots, \quad (\Delta_1(z) = 1).$$

By a sequence of elementary operations on $\Delta_{2k}(z)$ and $\Delta_{2k+1}(z)$ it is seen that $\Delta_k(1) = \bar{\Delta}_{k-1}$, $k = 1, 2, 3, \cdots$.
Substituting this result in (2.13) and (2.14) we have

$$(2.17) \qquad B_k(1) = \frac{\bar{\Delta}_{k-1}}{\Delta_{k-2}}, \qquad k = 1, 2, 3, \cdots.$$

We also note that

$$(2.18) \qquad a_k = \frac{\Delta_{k-3}\,\Delta_k}{\Delta_{k-2}\,\Delta_{k-1}}, \qquad k = 1, 2, 3, \cdots, \quad (\Delta_{-2} = 1).$$

Substituting the results of (2.17) and (2.18) in (2.11) and (2.12), the relation (2.9) follows immediately.

Proof (2). By Laplace's Development and a sequence of elementary operations, Lemma 2.1 can be established directly. We shall omit the details.

The proof of Theorem 2.1 now follows. Using (2.5) and (2.18), the relation (2.7) is immediate.

The relation (2.8) is established by induction. Assume that

$0 \leqq m_n < 1$, $n = 1, 2, 3, \cdots$. Using (2.5), (2.6), and Lemma 2.1 it is clear that

$$m_1 = a_1 = \frac{c_1 - \underline{c}_1}{c_0 - \underline{c}_0} = \frac{c_1 - \underline{c}_1}{\bar{c}_1 - \underline{c}_1} ,$$

where $c_0 = 1$ and we define $\underline{c}_0$ to be zero, and $m_2 = \dfrac{c_2 - \underline{c}_2}{\bar{c}_2 - \underline{c}_2}$. Now assume that $m_k = \dfrac{c_k - \underline{c}_k}{\bar{c}_k - \underline{c}_k}$. Again using (2.5), (2.6), and Lemma 2.1 in the relation $m_{k+1} = \dfrac{a_{k+1}}{1 - m_k}$, the definition for the minimal parameters in a chain sequence [3], the induction is completed. If $m_k = 1$ then m_{k+1} is defined to be zero. In this case the corresponding moments fall on the upper and lower boundaries of their respective convex bodies.

3. Some results from the theory of chain sequences. Regarding the uniqueness of the parameters g_p in the continued fraction (1.1) and the location of the moments in the convex bodies we have the following theorem:

THEOREM 3.1. *Given a monotone Hausdorff moment sequence,* $\{c_n\}$, *let*

$$\lim \frac{c_k - \underline{c}_k}{\bar{c}_k - c_k} = q . \tag{3.1}$$

If $q > 1$ the parameters g_p in (1.1) are uniquely determined, and if $q < 1$ the parameters are not uniquely determined. In case $q = 1$ the parameters may or may not be unique.

Proof. Wall [3] proved that the parameters in a chain sequence are uniquely determined if and only if the series

$$1 + \sum_{k=1}^{\infty} \frac{m_1 m_2 \cdots m_k}{(1 - m_1)(1 - m_2) \cdots (1 - m_k)}$$

diverges. Making use of this result and Theorem 2.1 our proof is immediate.

We designate the maximal parameters of the chain sequence in the continued fraction (1.1) by M_p. The maximal parameters can be interpreted in terms of "distances" in the convex bodies by the following theorem:

THEOREM 3.2. *The maximal parameters* M_n *in the continued fraction* (1.1) *can be written in the form*

$$M_n = \frac{c_n - \underline{c}_n}{\bar{c}_n - \underline{c}_n} + \frac{\bar{c}_n - c_n}{\bar{c}_n - \underline{c}_n}(1/T_n), \qquad n = 1, 2, 3, \cdots , \tag{3.2}$$

where

$$T_n = 1 + \sum_{r=n+1}^{\infty} \frac{c_{n+1} - \underline{c}_{n+1}}{\bar{c}_{n+1} - c_{n+1}} \frac{c_{n+2} - \underline{c}_{n+2}}{\bar{c}_{n+2} - c_{n+2}} \cdots \frac{c_r - \underline{c}_r}{\bar{c}_r - c_r} , \tag{3.3}$$

in the case that the a_{n+r}, $r = 1, 2, 3, \cdots$, of (2.9) are positive. If $a_{n+1}, a_{n+2}, \cdots, a_{n+r}$ are positive, $a_{n+k+1} = 0$, $(k > 0)$, and $m_{n+k} < 1$, then the summation in (3.3) runs only to $n + k$.

Proof. Wall [3] introduced an expression of the form (3.3) in discussing maximal parameters. Using his results and Theorem 2.1 our proof is immediate.

II. JACOBI-TYPE CONTINUED FRACTIONS AND CONVEX SETS

4. The "extended" monotone Hausdorff moment problem. A sequence of real number $\{c_n\}(n = 0, 1, 2, \cdots)$ shall be referred to as an "extended" monotone Hausdorff moment sequence if there exists a monotone nondecreasing real function $\phi(u)$, $-1 \leqq u \leqq 1$, such that

$$c_n = \int_{-1}^{1} u^n d\phi(u), \qquad n = 0, 1, 2, \cdots .$$

The problem of determining such a function $\phi(u)$ shall be referred to as the "extended" monotone Hausdorff moment problem. Again we shall assume throughout part II unless otherwise designated that $c_0 = 1$.

The work of Riesz [2] can be applied to the "extended" monotone Hausdorff moment problem. A sequence $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence if and only if the point $(c_1, c_2, \cdots, c_n)$, $n = 1, 2, 3, \cdots$, is in the convex closure of the arc whose parametric equations are given by (1.3) where $-1 \leqq t \leqq 1$.

Let

$$\frac{1}{b_1 z + 1} - \frac{a_1 z^2}{b_2 z + 1} - \frac{a_2 z^2}{b_3 z + 1} - \cdots \tag{4.1}$$

be the Jacobi-type continued fraction expansion of the power series

$$P(z) = \sum_{n=0}^{\infty} c_n z^n.$$

We shall agree that the continued fraction terminates with the first identically vanishing partial quotient. We shall show that if the sequence $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence, then the

a_p and b_p of (4.1) have the form of a generalized chain sequence and the parameters can again be represented in terms of "distances" in certain convex bodies.

5. The connecting theorem. As in §2, it is necessary to indicate some special notations for the Hankel determinants corresponding to an "extended" monotone Hausdorff moment sequence. We set

$$(5.1)\qquad \underline{\Delta}_{2n} = \begin{vmatrix} 1 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ & & \cdots & \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \cdots ,$$

$$(5.2)\qquad \underline{\Delta}_{2n+1} = \begin{vmatrix} 1 + c_1 & c_1 + c_2 & \cdots & c_n + c_{n+1} \\ c_1 + c_2 & c_2 + c_3 & \cdots & c_{n+1} + c_{n+2} \\ & & \cdots & \\ c_n + c_{n+1} & c_{n+1} + c_{n+2} & \cdots & c_{2n} + c_{2n+1} \end{vmatrix}, \qquad \begin{matrix} n = 0, 1, 2, \cdots , \\ (\underline{\Delta}_{-1} = 1) , \end{matrix}$$

$$(5.3)\qquad \bar{\Delta}_{2n} = \begin{vmatrix} 1 - c_2 & c_1 - c_3 & \cdots & c_{n-1} - c_{n+1} \\ c_1 - c_3 & c_2 - c_4 & \cdots & c_n - c_{n+2} \\ & & \cdots & \\ c_{n-1} - c_{n+1} & c_n - c_{n+2} & \cdots & c_{2n-2} - c_{2n} \end{vmatrix}, \qquad \begin{matrix} n = 1, 2, 3, \cdots , \\ (\bar{\Delta}_0 = 1), \text{ and} \end{matrix}$$

$$(5.4)\qquad \bar{\Delta}_{2n+1} = \begin{vmatrix} 1 - c_1 & c_1 - c_2 & \cdots & c_n - c_{n+1} \\ c_1 - c_2 & c_2 - c_3 & \cdots & c_{n+1} - c_{n+2} \\ & & \cdots & \\ c_n - c_{n+1} & c_{n+1} - c_{n+2} & \cdots & c_{2n} - c_{2n+1} \end{vmatrix}, \qquad \begin{matrix} n = 0, 1, 2, \cdots , \\ (\bar{\Delta}_{-1} = 1) . \end{matrix}$$

The sequence $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence if and only if the Hankel forms

$$\sum_{i,j=0}^{n} c_{i+j}x_i x_j, \quad \sum_{i,j=0}^{n} (c_{i+j} + c_{i+j+1})x_i x_j ,$$
$$\sum_{i,j=0}^{n-1} (c_{i+j} - c_{i+j+2})x_i x_j , \quad \sum_{i,j=0}^{n} (c_{i+j} - c_{i+j+1})x_i x_j$$

are all positive semidefinite. As in part I replace c_{2n} by $\underline{c}_{2n}$ in (5.1) and c_{2n+1} by $\underline{c}_{2n+1}$ in (5.2). Setting $\underline{\Delta}_{2n}$ and $\underline{\Delta}_{2n+1}$ equal to zero, we have the single relation

$$(5.5)\qquad \underline{c}_n = c_n - \frac{\underline{\Delta}_n}{\underline{\Delta}_{n-2}} , \qquad n = 1, 2, 3, \cdots .$$

Similarly, (5.3) and (5.4) yield

$$\bar{c}_n = c_n + \frac{\bar{\Delta}_n}{\bar{\Delta}_{n-2}}, \qquad n = 1, 2, 3, \cdots . \tag{5.6}$$

The methods of Karlin and Shapley [1] can be applied so that if the sequence $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence then the quantities $\underline{c}_n$ and $\bar{c}_n$ of (5.5) and (5.6) are again interpreted as the "downward" and "upward" projections, respectively, of c_n on the boundary of the corresponding convex body.

We can now state the following theorem:

THEOREM 5.1. *If the sequence* $\{c_n\}$ *is an "extended" monotone Hausdorff moment sequence, then the elements* a_n *and* b_n *in the continued fraction* (4.1) *can be written in the forms*

$$a_n = 4m_n(1 - m_{n-1})l_n(1 - l_n), \quad n = 1, 2, 3, \cdots, \quad 0 \leqq m_n \leqq 1, \ (m_0 = 0), \ 0 \leqq l_n \leqq 1 , \tag{5.7}$$

$$= \frac{c_{2n} - \underline{c}_{2n}}{c_{2n-2} - \underline{c}_{2n-2}}, \qquad n = 1, 2, 3, \cdots, (c_0 = 0) , \tag{5.8}$$

where

$$m_n = \frac{c_{2n} - \underline{c}_{2n}}{\bar{c}_{2n} - \underline{c}_{2n}} , \qquad l_n = \frac{c_{2n-1} - \underline{c}_{2n-1}}{\bar{c}_{2n-1} - \underline{c}_{2n-1}} , \tag{5.9}$$

$$1 - m_n = \frac{\bar{c}_{2n} - c_{2n}}{\bar{c}_{2n} - \underline{c}_{2n}}, \quad 1 - l_n = \frac{\bar{c}_{2n-1} - c_{2n-1}}{\bar{c}_{2n-1} - \underline{c}_{2n-1}} , \ n=1, 2, 3, \cdots ,$$

and

$$b_n = 1 - 2m_{n-1}(1 - l_{n-1}) - 2(1 - m_{n-1})l_n, \qquad n = 1, 2, 3, \cdots , \ (l_0 = m_0 = 0) , \tag{5.10}$$

$$= 1 - \frac{c_{2n-1} - \underline{c}_{2n-1}}{c_{2n-2} - \underline{c}_{2n-2}} - \frac{c_{2n-2} - \underline{c}_{2n-2}}{c_{2n-3} - \underline{c}_{2n-3}}, \qquad n = 2, 3, 4, \cdots . \tag{5.11}$$

As in part I it will be clear that a more general theorem is true. If $\{c_n\}$, $(c_0 = 1)$, is an arbitrary sequence of real numbers and its corresponding Jacobi-type continued fraction (4.1) is written in the form that the a_n and b_n are given by (5.7) and (5.10), respectively, where $l_0 = m_0 = 0$ but it is no longer necessary that $0 \leqq l_n \leqq 1$ and $0 \leqq m_n \leqq 1$, $n = 1, 2, 3, \cdots$, then the relations (5.8) and (5.11) with (5.9) holding are still valid.

If $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence, the geometric interpretations of the a_n, b_n, l_n, and m_n are apparent.

Proof. The proof depends upon the following lemma.

LEMMA 5.1. *The determinants in* (5.1), (5.2), (5.3), *and* (5.4) *satisfy the relations*

$$(5.12)\qquad \begin{aligned} \underline{\Delta}_{2k+1}\overline{\Delta}_{2k+1} &= \overline{\Delta}_{2k+2}\underline{\Delta}_{2k} + \underline{\Delta}_{2k+2}\overline{\Delta}_{2k}, \\ 2\underline{\Delta}_{2k}\overline{\Delta}_{2k} &= \overline{\Delta}_{2k+1}\underline{\Delta}_{2k-1} + \underline{\Delta}_{2k+1}\overline{\Delta}_{2k-1}, \end{aligned} \qquad k = 0, 1, 2, \cdots .$$

Proof. By Laplace's Development and a sequence of elementary operations, Lemma 5.1 can be established directly. We shall omit the details.

The proof to the theorem now follows. A well known formula for the a_k is given by

$$(5.13)\qquad a_k = \frac{\underline{\Delta}_{2k}\underline{\Delta}_{2k-4}}{\underline{\Delta}_{2k-2}\underline{\Delta}_{2k-2}}, \qquad k = 2, 3, 4, \cdots, \left(a_1 = \frac{\underline{\Delta}_2}{\underline{\Delta}_0^2}\right).$$

The formulas (5.5) and (5.13) yield (5.8).

By a substitution and an equivalence transformation, we write the continued fraction (4.1) in the form

$$(5.14)\qquad \frac{1}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_2 + z} - \cdots .$$

The recurrence formula for the denominators of the continued fraction (5.14) is given by

$$(5.15)\qquad \begin{aligned} &B_k(z) = (b_k + z)B_{k-1}(z) - a_{k-1}B_{k-2}(z), \\ &\qquad k = 1, 2, 3, \cdots, \ (a_0 = 1,\ B_{-1}(z) = 0,\ B_0(z) = 1) . \end{aligned}$$

Furthermore, we have

$$(5.16)\qquad B_k(z) = \frac{\underline{\Delta}_{2k}(z)}{\underline{\Delta}_{2k-2}}, \qquad k = 1, 2, 3, \cdots ,$$

where $\underline{\Delta}_{2k-2}$ is obtained from (5.1) and we define $\underline{\Delta}_{2k}(z)$ the same as in (2.15). By a sequence of elementary operations on $\underline{\Delta}_{2k}(z)$ it is seen that $\underline{\Delta}_{2k}(-1) = (-1)^k\underline{\Delta}_{2k-1}$, $k = 1, 2, 3, \cdots$. Substituting this result in (5.16) we have

$$(5.17)\qquad B_k(-1) = \frac{(-1)^k\underline{\Delta}_{2k-1}}{\underline{\Delta}_{2k-2}}, \qquad k = 1, 2, 3, \cdots .$$

Setting z equal to -1 in (5.15), using the formulas (5.13) and (5.17), we can solve for b_k and obtain (5.11). We note that if we had set z equal to 1 and followed a similar procedure, we would have obtained the formula

$$(5.18)\qquad b_n = \frac{\bar{c}_{2n-1} - c_{2n-1}}{c_{2n-2} - \underline{c}_{2n-2}} + \frac{c_{2n-2} - \underline{c}_{2n-2}}{\bar{c}_{2n-3} - c_{2n-3}} - 1, \qquad n = 2, 3, 4, \cdots .$$

Assume that $0 \leqq m_n < 1$, $0 \leqq l_n < 1$, $n = 1, 2, 3, \cdots$. Using (5.5), (5.6), and (5.9) it can be shown directly that $b_1 = 1 - 2l_1$, and $a_1 = 4m_1l_1(1 - l_1)$. Now by using (5.5), (5.6), (5.9), and Lemma 5.1, (5.11) reduces to (5.10) for $n = k$, $k = 2, 3, 4, \cdots$. A similar statement applies to (5.8). If $m_k = 1$ then m_{k+1} is defined to be zero. A similar statement applies to l_k. In either case the corresponding moments fall on the upper and lower boundaries of their respective convex bodies.

If (5.18) had been used in place of (5.11) we note that (5.10) would have been obtained in the form

$$b_n = 2(1 - l_n)(1 - m_{n-1}) + 2m_{n-1}l_{n-1} - 1, \quad n = 1, 2, 3, \cdots, (l_0 = m_0 = 0) . \tag{5.19}$$

By Theorem 5.1 and the condition in terms of Hankel forms, we can now state a theorem which characterizes the existence of an "extended" monotone Hausdorff moment sequence in terms of continued fractions. This theorem is analogous to Wall's solution [3], [4] for the regular monotone Hausdorff moment sequence. By Theorem 5.1 and an extension of the work of Karlin and Shapley, the equivalence of the continued fraction solution and the condition in terms of Hankel forms, and hence convex bodies, is apparent.

THEOREM 5.2. *The sequence* $\{c_n\}$ *is an "extended" monotone Hausdorff moment sequence if and only if the power series*

$$P(z) = \sum_{n=0}^{\infty} c_n z^n$$

has a Jacobi-type continued fraction (4.1) *expansion where the* a_n *and* b_n *are given by* (5.7) *and* (5.10), *respectively, and* $l_0 = m_0 = 0$, *and* $0 \leqq l_n \leqq 1$, $0 \leqq m_n \leqq 1$, $n = 1, 2, 3, \cdots$.

It should be pointed out that $P(z) = \sum_{m=0}^{\infty} c_m z^m$ is a moment generating function for the "extended" monotone Hausdorff moment problem if and only if $Q(w) = (1 + z)P(z)$, where $w = \dfrac{2z}{1 + z}$, is a moment generating function for the regular monotone Hausdorff moment problem. From these relations it is observed that the l_n and m_n of Theorem 5.1 are equal to m_{2n-1} and m_{2n}, $n = 1, 2, 3, \cdots$, respectively, of Theorem 2.1. These results are obtained by contraction.

It can also be noted that $\{c_n\}$ is an "extended" monotone Hausdorff moment sequence if and only if

$$\{d_n/2^n\}, \quad d_n = \sum_{j=0}^{n} \binom{n}{n-j} c_j ,$$

is a regular monotone Hausdorff moment sequence. This result can be obtained by comparing coefficients in $P(z)$ and $Q(w)$ under the indicated transformation.

6. The continued fraction of the first differences. We prove the following theorem:

THEOREM 6.1. *If*

$$(6.1)\qquad 1 + c_1z + c_2z^2 + \cdots \sim \frac{1}{b_1z+1} - \frac{a_1z^2}{b_2z+1} - \frac{a_2z^2}{b_3z+1} - \cdots ,$$

where

$$(6.2)\qquad b_n = 1 - 2m_{n-1}(1-l_{n-1}) - 2(1-m_{n-1})l_n,\ n = 1, 2, 3, \cdots ,$$

$$(6.3)\qquad a_n = 4m_n(1-m_{n-1})l_n(1-l_n),\ n = 1, 2, 3, \cdots,\ (l_0 = m_0 = 0) ,$$

then

$$(6.4)\qquad \Delta c_0 + \Delta c_1z + \Delta c_2z^2 + \cdots \sim \frac{a_0^*}{b_1^*z+1} - \frac{a_1^*z^2}{b_2^*z+1} - \frac{a_2^*z^2}{b_3^*z+1} - \cdots ,$$

where $\Delta c_n = c_{n+1} - c_n$, $n = 1, 2, 3, \cdots$, $(\Delta c_0 = 1 - c_1)$, and

$$(6.5)\qquad \begin{aligned} b_1^* &= 1 - 2l_1(1-m_1) , \\ b_n^* &= 1 - 2m_{n-1}(1-l_n) - 2l_n(1-m_n), \qquad n = 2, 3, 4, \cdots , \end{aligned}$$

$$(6.6)\qquad \begin{aligned} a_0^* &= 2(1-l_1) , \\ a_n^* &= 4l_n(1-l_{n+1})m_n(1-m_n), \qquad n = 1, 2, 3, \cdots . \end{aligned}$$

Proof. In order to prove the theorem it is necessary to note some determinants for the sequence $\{\Delta c_n\}$ corresponding to $\underline{\Delta}_{2n}$ and $\underline{\Delta}_{2n+1}$ of (5.1) and (5.2), respectively, for the sequence $\{c_n\}$. Noting (5.3) and (5.4) we observe that

$$(6.7)\qquad \underline{\Delta}_{2k}^* = \bar{\Delta}_{2k+1},\ \underline{\Delta}_{2k+1}^* = \bar{\Delta}_{2k+2}, \qquad k = 0, 1, 2, \cdots .$$

We observe directly that

$$a_0^* = 1 - c_1 = \Delta c_1 = 2(1-l_1) .$$

Using (5.13) and (6.7) we note that

$$(6.8)\qquad a_k^* = \frac{\underline{\Delta}_{2k}^*\ \underline{\Delta}_{2k-4}^*}{\underline{\Delta}_{2k-2}^*\underline{\Delta}_{2k-2}^*} = \frac{\bar{\Delta}_{2k+1}\bar{\Delta}_{2k-3}}{\bar{\Delta}_{2k-1}\bar{\Delta}_{2k-1}} .$$

The relations in (6.6) can now be established by (5.5) (5.6), (5.9), and Lemma 5.1.

Now, by (5.5), (5.10), (5.11), and (6.7),

$$(6.9)\qquad b_k^* = 1 - \frac{\underline{\Delta}_{2k-1}^*\ \underline{\Delta}_{2k-4}^*}{\underline{\Delta}_{2k-2}^*\ \underline{\Delta}_{2k-3}^*} - \frac{\underline{\Delta}_{2k-2}^*\ \underline{\Delta}_{2k-5}^*}{\underline{\Delta}_{2k-3}^*\ \underline{\Delta}_{2k-4}^*}$$

$$= 1 - \frac{\bar{\Delta}_{2k} \quad \bar{\Delta}_{2k-3}}{\bar{\Delta}_{2k-1} \bar{\Delta}_{2k-2}} - \frac{\bar{\Delta}_{2k-1} \bar{\Delta}_{2k-4}}{\bar{\Delta}_{2k-2} \bar{\Delta}_{2k-3}} ,$$

$$k = 1, 2, 3, \cdots , \; (\underline{\Delta}^*_{-1} = \underline{\Delta}^*_{-2} = 1, \underline{\Delta}^*_{-3} = 0) .$$

Now again by (5.5), (5.6), (5.9), and Lemma 5.1, the relations in (6.5) follow.

We note that a similar proof could be given for the corresponding theorem for a regular monotone Hausdorff moment sequence, thereby giving another proof to this well known result [4].

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