ON DIOPHANTINE APPROXIMATION AND TRIGONOMETRIC POLYNOMIALS

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The usefulness of Diophantine approximation in achieving both positive and negative results in the subject of trigonometric interpolating polynomials is well established (cf. e.g. [1], [4]). The trigonometric polynomials, hereafter called simply polynomials, which we shall consider mainly and designate by $I_{n,u}(x;f)$ are those of order n taking on the values of a given function f at the points $u + 2\pi k/(2n+1)$, $k = 0, 1, \cdots, 2n$. Thus

$$I_{n,u}(x;f) = \frac{2}{2n+1}\sum_{k=0}^{2n} f(u + x_k^{(n)})D_n(x - u - x_k^{(n)}),$$

$$D_n(x) = \frac{\sin(2n+1)x/2}{2\sin(x/2)},\ x_k^{(n)} = \frac{2\pi k}{2n+1}\ .$$

It is assumed that f is periodic and defined almost everywhere so that for almost every u, $I_{n,u}(x;f)$ is defined for all n. Marcinkiewicz and Zygmund [4] have shown that each p, $1 \leqq p < 2$, there is a function f of class L^p such that for almost every point of the square $0 \leqq x \leqq 2\pi$, $0 \leqq u \leqq 2\pi$, $I_{n,u}(x;f)$ diverges. They made strong use of the following classical result of Diophantine approximation: for each x there are infinitely many rationals p/q such that $|x - p/q| \leqq 1/q^2$.

Our aim in this paper is to generalize the result of Marcinkiewicz and Zygmund. The chief tool of proof is a result proved in the next section, concerning the approximation of reals by rationals in which the range of the denominators is restricted. In the third section we give our main theorem to the effect that for any increasing function ψ defined on $(0, \infty)$ there is an f such that $\psi(|f|)$ is integrable over $0 \leqq x \leqq 2\pi$ and such that $I_{n,u}(x;f)$ diverges for almost every (x, u). In the last section we show this result holds for Jackson polynomials.

2. We begin with a preliminary lemma. If F is a measurable set, $|F|$ will denote its measure. We shall let C, C_1, and C_2 denote constants, independent of the values of the integers N, M, and m.

LEMMA 1. *Let N, M, and m be three integers such that $0 \leqq N < M \leqq m/2$. Let F be the subset of $(0,1)$ such that for each x in F there is an irreducible rational p/q, $0 < p < q$, $N < q \leqq M$ satisfying $|x - p/q| \leqq$*

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$1/qm$. *Then*

$$\frac{12(M-N)}{\pi^2 m} - \frac{C}{m}\log^2(M+1) \leqq |F| \leqq \frac{12(M-N)}{\pi^2 m} + \frac{C}{m}\log^2(M+1)\,.$$

If only $0 \leqq N < M \leqq m$, *then the second inequality above holds.*

F is the union of intervals of the form $(p/q - 1/qm, p/q + 1/qm)$. The number of irreducible rationals with denominator q of the above form is $\phi(q)$ where ϕ is the Euler function. The contribution to the measure of F from a given q is no more that $2\phi(q)/qm$ so that the measure of F does not exceed

$$\frac{2}{m}\sum_{q=N+1}^{M}\frac{\phi(q)}{q}$$

Let $\psi(0) = 0, \psi(n) = \sum_{q=1}^{n}\phi(q)$. Applying Abel's transformation to the above sum, we obtain

$$(1) \qquad |F| \leqq \frac{2}{m}\sum_{q=N+1}^{M}\frac{\psi(q)}{q(q+1)} + \frac{2}{m}\left\{\frac{\psi(M)}{M+1} - \frac{\psi(N)}{(N+1)}\right\}.$$

By a known theorem (cf. e.g. [3, p. 120])

$$(2) \qquad \frac{3q^2}{\pi^2} - C_1 q\log(q+1) \leqq \psi(q) \leqq \frac{3q^2}{\pi^2} + C_1 q\log(q+1)\,.$$

Substitution of (2) into (1) gives

$$|F| \leqq \frac{6}{\pi^2 m}\sum_{q=N+1}^{M}\frac{q}{q+1} + \frac{6M}{\pi^2 m} - \frac{6N^2}{\pi^2 m(N+1)} + \frac{C_2}{m}\log^2(M+1)\,.$$

This implies the second statement of the lemma. In case $M \leqq m/2$, there is no overlapping of the (open) intervals $(p/q - 1/qm, p/q + 1/qm)$. For otherwise, there are distinct rational $r/s, p/q$ (let us say $r/s > p/q$) of the required form such that

$$0 < \frac{r}{s} - \frac{p}{q} < \frac{1}{sm} + \frac{1}{qm} \text{ and } 0 < rq - ps < \frac{q+s}{m} \leqq 1\,.$$

This contradicts the fact that $rq - ps$ is an integer. Thus

$$|F| = \frac{2}{m}\sum_{q=N+1}^{M}\frac{\phi(q)}{q}\,.$$

Now the inequality (2) implies the lemma.

THEOREM 1. (i) *Let* m *be a sufficiently large positive integer, and let* γ *be a real number such that* $0 < \gamma < \pi^2/12$. *Let* E *be the subset of*

(0, 1) *such that for each* x *in* E *there exists an irreducible rational* $p/q, 0 < p < q, \gamma m < q \leqq m$ *for which* $|x - p/q| \leqq 1/\gamma m^2$. *Then there is an absolute constant* C *such that*

$$|E| \geqq 1 - \frac{12\gamma}{\pi^2} - Cm^{-1}\log^2 m \, .$$

(ii) *Let* γ *be a real number such that* $0 < \gamma < \pi^2/24$. *Let* E_1 *be the subset of* (0, 1) *such that for each* x *in* E_1 *there exists an irreducible rational* $p/q, 0 < p < q, \gamma m < q \leqq m$, *with* q *odd for which* $|x - p/q| \leqq 2/\gamma^2 m^2$. *Then there is an absolute constant* C *such that*

$$|E_1| \geqq 1 - \frac{24\gamma}{\pi^2} - Cm^{-1}\log^2 m \, .$$

As in the proof of the theorem mentioned in the introduction (cf. [6, p. 43]) we may find for each x in (0, 1) an irreducible rational p/q such that

$$|x - p/q| \leqq \frac{1}{qm}, \; 0 < q \leqq m \, . \tag{3}$$

If x is restricted to the (open) interval $I = (1/m, 1 - 1/m)$, then $0 < p < q$. We shall say q and x are associated if (3) holds with x in I and with p/q irreducible, $0 < p < q, 0 < q \leqq m$. Let F_1 be the subset of I for which all q associated with x do not exceed γm. Since each x is associated with some q, the set F_1 is a subset of the set F of Lemma 1 for which $N = 0$ and $M = [\gamma m]$, the greatest integer not exceeding γm. We may assume without loss of generality that $\gamma m > 1$. Let E be the complement of F_1 with respect to I. Since the measure of F does not exceed $12\gamma/\pi^2 + Cm^{-1}\log^2 m$, part (i) follows from (3) and the inequality $q > \gamma m$.

Let F_2 be the subset of I for which all q associated with an x in F_2 are such that $(1 - \gamma)m < q \leqq m$. F_2 is a subset of the set F of Lemma 1 for which $M = m, N = [(1 - \gamma)m]$. Let E_1 be the complement of $F_1 \cup F_2$ with respect to I. Then $|E_1| \geqq 1 - 24\gamma/\pi^2 - Cm^{-1}\log^2 m$. If x belongs to E_1, there is a q associated with x such that $\gamma m \leqq q \leqq m(1 - \gamma)$. If q is even, we may find integers η and ξ such that

$$\eta p - \xi q = 1 \tag{4}$$

where η must be odd, and automatically ξ/η is irreducible. Let η_0 be the least positive solution of (4) (cf. [1] for a similar argument). If $\eta_0 \geqq \gamma m$, it follows that

$$\left|\frac{p}{q} - \frac{\xi_0}{\eta_0}\right| = \frac{1}{q\eta_0} \leqq \frac{1}{\gamma^2 m^2}$$

and

$$(5)\qquad \left|x-\frac{\xi_0}{\eta_0}\right| \leqq \left|x-\frac{p}{q}\right| + \left|\frac{p}{q}-\frac{\xi_0}{\eta_0}\right| \leqq \frac{1}{qm}+\frac{1}{\gamma^2 m^2} \leqq \frac{2}{\gamma^2 m^2}$$

If $\eta_0 < \gamma m$, let $\eta_1 = \eta_0 + q$. Then $\gamma m \leqq q \leqq \eta_1 \leqq \gamma m + q \leqq m$, and (5) holds with ξ_0/η_0 replaced by ξ_1/η_1. We may assume that $\gamma^2 > 1/m$ so that $0 < \xi < \eta \leqq m$ as required.

3. We begin this section with a lemma which is related to the results of the preceding section, but it contains only as much information as will be used in the proof of the next theorem.

LEMMA 2. *Let m be a sufficiently large integer, A_m a real satisfying $1 \leqq A_m \leqq \log m$, and $d \log m$ an integer with $8 < d < 10$. Let $\mathcal{N}$ be the set of odd positive integers $2n+1$ not exceeding m and such that*

$$(6)\qquad \left|\frac{\mu m}{\nu} - (2n+1)\right| \leqq \frac{4A_m^{1/2}}{\nu} \text{ for some } (\mu, \nu) \text{ such that}$$

$$0 < \mu \leqq \nu \leqq d \ \log \mathrm{m}.$$

Let G be the subset of $(0, 1)$ such that for x in G, there is a $2n+1$ in $\mathcal{N}$ and a k, $0 < k < 2n+1$ for which $|x - k/(2n+1)| \leqq 2A_m^{1/2}/m^2$. Then

$$|G| \leqq \frac{36d^2 \log^3 m}{m}.$$

For a given μ and ν, no more than $1 + 8A_m^{1/2}/\nu$ integers $2n+1$ satisfy (6). For a given ν, no more than $\nu + 8A_m^{1/2}$ integers may satisfy (6) for some $\mu \leqq \nu$. Hence N, the number of distinct integers in $\mathcal{N}$, does not exceed $d^2 \log^2 m + 8dA_m^{1/2} \log m$. If x belongs to G, x is contained in an interval of length $4A_m^{1/2}/m^2$ centered about some point $k/(2n+1)$. For each $2n+1$, the total length of the intervals is no more than $4A_m^{1/2}/m$. Thus,

$$|G| \leqq \frac{4NA_m^{1/2}}{m} \leqq \frac{36d^2 \log^3 m}{m}.$$

THEOREM 2. *Let ψ be a monotone increasing function defined on $(0, \infty)$. There exists a function f such that $\psi(|f|)$ is integrable on $(0, 2\pi)$ and such that the sequence $I_{n,u}(x; f)$ diverges for almost all points of the square $0 \leqq x \leqq 2\pi$, $0 \leqq u \leqq 2\pi$.*

Let A_m be a positive number satisfying the inequality $16 \leqq A_m \leqq (\log m)^{1/2}$. A more exact specification of A_m will be given at a later point. The function f will be a sum of periodic, step functions f_m of the following form. When x belongs to one of the intervals

$$|x - 2\pi j/m| \leqq 4\pi A_m^{1/2}/m^2, \qquad j = 0, 1, \cdots, m-1$$

let $f_m(x) = A_m$; when x belongs to one of the complementary intervals of $(0, 2\pi)$ let $f_m(x) = 0$. Let E_1 be the set of Theorem 1, part (ii), corresponding to m and $\gamma = A_m^{-1/4}$, and expanded to the interval $(0, 2\pi)$ on the u axis. For m sufficiently large, $|E_1| \geqq 2\pi(1 - 25/\pi^2 A_m^{1/4})$. Let G be the set of Lemma 2 expanded to the interval $(0, 2\pi)$ on the u axis. Let E_m be the difference set $E_1 - (G \cup G_1)$ where G_1 is the set of u such that $|u| \leqq 2\pi/(\log m)^{1/2}$. By our above estimates

$$(7) \qquad |E_m| \geqq 2\pi\left[1 - \frac{25}{\pi^2 A_m^{1/4}} - \frac{36d^2 \log^3 m}{m} - \frac{2}{(\log m)^{1/2}}\right].$$

Let $E_{m,j}$ be the set E_m translated by $-2\pi j/m$, $j = 0, 1, \cdots, m-1$: *i.e.* u belongs to $E_{m,j}$ if and only if $u + 2\pi j/m$ (modulo 2π) belongs to E_m. Let $-u$ belong to $E_{m,j}$ We may assume that $-u + 2\pi j/m$ belongs to the interval $(0, 2\pi)$. Since E_m is a subset of E_1, there exists, according to Theorem 1, part (ii), an odd integer, $2n+1$, $m/A_m^{1/4} \leqq 2n+1 \leqq m$, and an integer k, $0 < k < 2n+1$, such that

$$(8) \qquad \left|u - \frac{2\pi j}{m} + \frac{2\pi k}{2n+1}\right| \leqq \frac{4\pi A_m^{1/2}}{m^2}.$$

This inequality implies that $f_m(u + 2\pi k/(2n+1)) = A_m$. Since $-u + 2\pi j/m$ does not belong to the set G, the integer $2n+1$ cannot belong to the set $\mathscr{N}$ defined by (6). If $f_m(u + 2\pi(k+\mu)/(2n+1)) = A_m$ for some nonzero integer μ, then there must be a nonzero integer ν such that

$$(9) \qquad \left|u - \frac{2\pi(j+\nu)}{m} + \frac{2\pi(k+\mu)}{2n+1}\right| \leqq \frac{4\pi A_m^{1/2}}{m^2}.$$

We may assume that $\mu > 0$, $\nu > 0$. The inequalities (8) and (9) imply that

$$(10) \qquad \left|\frac{\mu}{2n+1} - \frac{\nu}{m}\right| \leqq \frac{4A_m^{1/2}}{m^2}$$

and (10) implies that $\mu \leqq \nu$. For if $\mu > \nu$, then

$$\frac{\mu}{2n+1} - \frac{\nu}{m} \geqq \frac{1}{2n+1} > \frac{4A_m^{1/2}}{m^2}.$$

It also follows from (10) that

$$\left|\frac{\mu m}{\nu} - (2n+1)\right| \leqq \frac{4A_m^{1/2}(2n+1)}{\nu m} \leqq \frac{4A_m^{1/2}}{\nu} .$$

Comparison of this inequality with (6) shows that $|\nu| > d(\log m)$. Our analysis shows, in fact, that if $f_m(u + x_i^{(n)}) = A_m$, then $f_m(u + x_{i+\nu}^{(n)}) = 0$ when $|\nu| \leqq d(\log m)$ and $2n+1$ does not belong to $\mathscr{N}$. For each $j = 0, 1, \cdots, m-1$, let I_j be the set of the x axis defined by

$$\frac{\pi}{mA_m^{1/4}} \leqq \left|x - \frac{2\pi j}{m}\right| \leqq \frac{\pi}{m} .$$

If x belongs to I_j, and if $-u$ belongs to $E_{m,j}$, then we find from (8) that

$$\left|x - u - \frac{2\pi k}{2n+1}\right| \leqq \left|x - \frac{2\pi j}{m}\right| + \left|u + \frac{2\pi k}{2n+1} - \frac{2\pi j}{m}\right|$$

$$\leqq \frac{\pi}{m} + \frac{4\pi A_m^{1/2}}{m^2} < \frac{3\pi}{2m}$$

for some k and for some n for which $m/A_m^{1/4} \leqq 2n+1 \leqq m$. Furthermore

$$\left|x - u - \frac{2\pi k}{2n+1}\right| \geqq \left|x - \frac{2\pi j}{m}\right| - \left|u + \frac{2\pi k}{2n+1} - \frac{2\pi j}{m}\right|$$

$$\geqq \frac{\pi}{mA_m^{1/4}} - \frac{4\pi A_m^{1/2}}{m^2} > \frac{\pi}{2mA_m^{1/4}} .$$

These inequalities imply that

(11) $$\frac{1}{\left|\sin\frac{1}{2}\left(x - u - \frac{2\pi k}{2n+1}\right)\right|} > \frac{m}{3}; \left|\sin\left(n + \frac{1}{2}\right)(x-u)\right|$$

$$\geqq \sin\frac{\pi}{4A_m^{1/2}} \geqq \frac{1}{2A_m^{1/2}} .$$

Now we are ready to estimate $I_{n,u}(x; f_m)$ with x in I_j and $-u$ in $E_{m,j}$.

(12) $$I_{n,u}(x; f_m) = \frac{f_m(u + x_k^{(n)})}{2n+1} \frac{(-1)^k \sin(n+1/2)(x-u)}{\sin\frac{1}{2}(x - u - x_k^{(n)})}$$

$$+ \frac{\sin(n+1/2)(x-u)}{2n+1} \sum_{i \neq k} \frac{f_m(u + x_i^{(n)})(-1)^i}{\sin\frac{1}{2}(x - u - x_i^{(n)})} .$$

Denote the first and second terms on the right by D_1 and D_2 respectively.

By (8) and (11)

$$(13)\qquad |D_1| \geqq \frac{2|\sin(n+1/2)(x-u)|mA_m}{3(2n+1)} \geqq |\sin(n+1/2)(x-u)|\frac{A_m}{3}\ .$$

We may assume that for the terms of D_2, $|x_i^{(n)} - x_k^{(n)}| \leqq \pi$ so that except possibly for one term of the sum which can be ignored, $|x - u - x_i^{(n)}| \leqq \pi$. Hence for the terms of D_2, $|\sin 2^{-1}(x-u-x_i^{(n)}| \geqq |x-u-x_i^{(n)}|/\pi$, and

$$|D_2| \leqq \frac{\pi|\sin(n+1/2)(x-u)|}{2n+1}\sum_{i\neq k}\frac{f_m(u+x_i^{(n)})}{|x-u-x_i^{(n)}|}\ .$$

The denominator of the terms in the sum increases with $|i-k|$. Furthermore if i and i' are distinct values of the index for which the numerator is nonzero, then $|i-k| > d\log m$, $|i'-k| > d\log m$, and $|i-i'| > d\log m$. Thus we find that

$$\begin{aligned}|D_2| &\leqq \frac{2\pi|\sin(n+1/2)(x-u)|A_m}{2n+1}\sum_{r=1}^{M}\frac{2n+1}{2\pi r d\log m}\\ &\leqq \frac{|\sin(n+1/2)(x-u)|}{d\log m}A_m\log(M+1),\ M=\left\langle\frac{2n+1}{2d\log m}\right\rangle\ .\end{aligned}$$

We denote by $\langle y\rangle$ the least integer $\geqq y$. From this inequality and from (11), (12), and (13), we deduce that if x belongs to I_j, and if $-u$ belongs to $E_{m,j}$, there exists an integer, $2n+1$, and a positive constant C such that

$$(14)\qquad |I_{n,u}(x;f_m)| \geqq C|\sin(n+1/2)(x-u)|A_m \geqq \frac{CA_m^{1/2}}{2}\ .$$

The product set $I_j \times E_{m,j}$ of the xu-plane has two dimensional measure equal to $2\pi|E_m|(1-A_m^{-1/4})/m$. There are m such mutually disjoint sets, and the total measure of their union, H_m, is $2\pi|E_m|(1-A_m^{-1/4})$. Thus if $(x, -u)$ belongs to H_m, then (14) holds for the proper n. We note here that $\lim|H_m| = 4\pi^2$ if $\lim A_m = \infty$.

Let

$$(15)\qquad f(x) = \sum_{i=1}^{\infty} f_{m(i)}(x)\ .$$

We shall impose various conditions on the sequence of positive integers $m(i)$, all related to the rapidity of its growth. Let $\{B_i\}$ be a sequence of reals going to ∞ so that $\sum_{j<i}B_j \leqq B_i^{1/4}$. Let $m(i)$ increase so rapidly that $\log m(i) \geqq B_i^2$ and that

$$(16)\qquad \psi(2B_i)\frac{(\log m(i))^{1/4}}{m(i)} \leqq 2^{-i}\ .$$

Let $A_{m(i)} = B_i$ so that $A_{m(i)} \leqq (\log m(i))^{1/2}$ as required. Now $f_m(x)$ is 0 except on a set of measure not exceeding $4\pi A_m^{1/2}/m \leqq 4\pi(\log m)^{1/4}/m$. Let

$$\rho_j = \sum_{i=j}^{\infty} \frac{(\log m(i))^{1/4}}{m(i)}, \sum_{j=1}^{\infty} \rho_j < \infty .$$

It follows that the series in (15) converges almost everywhere and that $\sum_{j=i}^{\infty} f_{m(j)}(x)$ is 0 outside a set of measure $4\pi\rho_i$. Let K_i be the set of x values for which $f_{m(i)}(x) \neq 0$, and $f_{m(j)}(x) = 0$ when $j > i$. The K_i's are mutually disjoint, and their union is, except for a set of measure 0, the set where $f(x) \neq 0$. Moreover, $|K_i| \leqq 4\pi(\log m(i))^{1/4}/m(i)$. When x is in K_i,

$$\psi(|f(x)|) \leqq \psi\Big(\sum_{j=1}^{i} f_{m(j)}(x)\Big) \leqq \psi(2B_i) .$$

Thus by (16)

$$\int_0^{2\pi} \psi(|f(x)|)dx \leqq \sum_{i=1}^{\infty} \psi(2B_i)|K_i| < \infty .$$

In the estimation of the interpolating polynomials, we shall require certain other conditions. Thus we assume that f belongs to L^p for some $p > 1$ and that

$$|K_j|(2B_j)^p \leqq \frac{2^{-j}}{m(j-1)}, \qquad j > 1 .$$

From this it follows that

$$\int_0^{2\pi} |\sum_{j>i} f_{m(j)}(x)|^p dx \leqq \sum_{j>i} |K_j|(2B_j)^p \leqq \frac{2^{-i}}{m(i)} . \tag{17}$$

Furthermore we note that $\sum_{j=1}^{i-1} f_{m(j)}(x)$ is a function of bounded variation so that, for each u, the interpolating polynomials converge to the function at every point of continuity, *i.e.* outside a finite set [6; p. 36]. Thus, given $m(1), m(2), \cdots, m(i-1)$, we choose $m(i)$ so large that for $2n + 1 \geqq m(i)/B_i^{1/4}$,

$$\Big|I_{n,u}(x; \sum_{j=1}^{i-1} f_{m(j)})\Big| \leqq 2 \max_x \Big(\sum_{j=1}^{i-1} f_{m(j)}(x)\Big) \leqq 2A_{m(i)}^{1/4} \tag{18}$$

for (x, u) outside a set of two dimensional measure not exceeding 2^{-i}. Finally since $\lim |H_m| = 4\pi^2$, the $m(i)$ can be spread out so sparsely that

$$\sum_{i=1}^{\infty} |H'_{m(i)}| < \infty \tag{19}$$

where H'_m is the complement of H_m.

To estimate $I_{n,u}(x;f)$, we let $m(i)/B_i^{1/4} \leqq 2n+1 \leqq m(i)$. Then

$$I_{n,u}(x;f) = I_{n,u}(x;\sum_{j<i} f_{m(j)}) + I_{n,u}(x;f_{m(i)}) + I_{n,u}(x;\sum_{j>i} f_{m(j)}) \,. \tag{20}$$

Let $g(x, u)$ be the maximum of the absolute value of the last term on the right for $2n+1 \leqq m(i)$. A result of Marcinkiewicz and Zygmund [4] implies that

$$\int_0^{2\pi}\int_0^{2\pi} |g(x, u)|^p dxdu \leqq \sum_{2n+1\leqq m(i)} \int_0^{2\pi}\int_0^{2\pi} |I_{n,u}(x;\sum_{j>i} f_{m(j)})|^p dxdu$$
$$\leqq C_p m(i)\int_0^{2\pi} |\sum_{j>i} f_{m(j)}(x)|^p dx$$

and the last term on the right does not exceed $C_p 2^{-i}$ by (17). C_p is a constant depending only on p. Thus

$$\max_{2n+1\leqq m(i)} |I_{n,u}(x;\sum_{j>i} f_{m(j)})| \leqq C_p^{1/p}$$

outside a set of measure 2^{-i}. This, together with (18) and (20), implies

$$|I_{n,u}(x;f)| \geqq |I_{u,u}(x;f_{m(i)})| - 2A_{m(i)}^{1/4} - C_p^{1/p}$$

outside a set of measure 2^{-i+1}. Combining the above with (14) implies that for each $(x, -u)$ outside a set of measure $|H'_{m(i)}| + 2^{-i+1}$, there exists an n and a positive constant C such that

$$|I_{n,u}(x;f)| \geqq CA_{m(i)}^{1/2} \,.$$

From (19) this inequality is true for almost every $(x, -u)$ with sufficiently large i and appropriate n, and the theorem follows.

4. That Theorem 2 holds for Jackson polynomials is relatively easy to prove. We have

$$J_{n,u}(x;f) =$$
$$\frac{1}{(n+1)^2}\sum_{i=0}^{n} f(u+t_i^{(n)})\left\{\frac{\sin 2^{-1}(n+1)(x-u-t_i^{(n)})}{\sin 2^{-1}(x-u-t_i^{(n)})}\right\}^2, \; t_i^{(n)} = \frac{2\pi i}{n+1} \,.$$

If $f(x) \geqq f_m(x) \geqq 0$,

$$J_{n,u}(x;f) \geqq \frac{f_m(u+t_k^{(n)})}{(n+1)^2}\left\{\frac{\sin 2^{-1}(n+1)(x-u)}{\sin 2^{-1}(x-u-t_k^{(n)})}\right\}^2 \,. \tag{21}$$

Thus all of the previous proof devoted to showing that there was not undue interference with one dominant term is now unnecessary. The rest of the proof is very much like the previous one. With some adjustments in the function, we gain additional information.

THEOREM 3. *Given ψ as before, there exists f such that $\psi(|f|)$ is integrable over $(0, 2\pi)$ and such that the sequence $J_{n,u}(x; f)$ diverges for almost every point of the square $0 \leqq x \leqq 2\pi, 0 \leqq u \leqq 2\pi$. Furthermore for any $p \geqq 1$ and $\varepsilon, 0 < \varepsilon < 1$, there is a function f of class L^p such that for almost every point (x, u)*

$$\overline{\lim_n} \frac{|J_{n,u}(x; f)|}{n^{1-\epsilon}} > 0 .$$

Let α, β, and A_m be positive reals to be specified at a later point. Let f_m be a periodic step function of the following form. When x belongs to one of the intervals

$$\left|x - \frac{2\pi j}{m}\right| \leqq \frac{2\pi A_m^\alpha}{m^2}, \qquad j = 0, 1, \cdots, m-1$$

let $f_m(x) = A_m$; when x belongs to one of the complementary intervals of $(0, 2\pi)$, let $f_m(x) = 0$. Let E_m be the set E of Theorem 1, part (i), corresponding to m and $\gamma_m = A_m^{-\alpha}$, expanded to $(0, 2\pi)$ of the u axis. Let $E_{m,j}$ be the translation of E_m by $-2\pi j/m$; and let I_j be the set of the x axis such that for some j satisfying $0 \leqq j \leqq m-1$,

$$\frac{2\pi}{mA_m^\beta} \leqq \left|x - \frac{2\pi j}{m}\right| \leqq \frac{\pi}{m} .$$

Given $-u$ in $E_{m,j}$ and x in I_j, there exists an n, $mA_m^{-\alpha} \leqq n+1 \leqq m$, and a k such that

$$\left|u - \frac{2\pi j}{m} + t_k^{(n)}\right| \leqq \frac{2\pi A_m^\alpha}{m^2} .$$

For proper choice of A_m, we have as before

$$\frac{\pi}{mA_m^\beta} \leqq |x - u - t_k^{(n)}| \leqq \frac{3\pi}{2m}$$

so that from (21) $J_{n,u}(x; f)$ exceeds $A_m^{1-2\beta}/10$. Since $||f_m||_p^p = 4\pi A_m^{p+\alpha}/m$, we need only have $A_m^{p+\alpha} = o(m)$ to write $f(x) = \sum_{i=1}^\infty f_{m(i)}(x)$ with the $m(i)$ spread out sufficiently. If α and β are small, the result follows.

Since the sequence of Jackson polynomials corresponding to a continuous function converges uniformly to that function [6; p. 47], it is essentially only for the class of bounded functions that the question of the behaviour of the Jackson polynomials on the square $0 \leqq x \leqq 2\pi$, $0 \leqq u \leqq 2\pi$ is unresolved. However this is no longer true for the ordinary polynomials $I_{n,u}(x; f)$ which may act in a quite irregular way (cf. [2], [5]); and the behaviour of $I_{n,u}(x; f)$ for f continuous still presents a problem of considerable interest.

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