STATISTICAL METRIC SPACES

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Introduction. The concept of an abstract metric space, introduced by M. Fréchet in 1906 [2], furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance" appears. The objects under consideration may be most varied. They may be points, functions, sets, and even the subjective experiences of sensations. What matters is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain conditions. However, in numerous instances in which the theory of metric spaces is applied, this very association of a *single* number with a pair of elements is, realistically speaking, an over-idealization. This is so even in the measurement of an ordinary length, where the number given as the distance between two points is often not the result of a single measurement, but the average of a series of measurements. Indeed, in this and many similar situations, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. More precisely, instead of associating a number—the distance $d(p, q)$—with every pair of elements p, q, one should associate a distribution function F_{pq} and, for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x. When this is done one obtains a generalization of the concept of a metric space—a generalization which was first introduced by K. Menger in 1942 [5] and, following him, is called a statistical metric space.

The history of statistical metric spaces is brief. In the original paper, Menger gave postulates for the distribution functions F_{pq}. These included a generalized triangle inequality. In addition, he constructed a theory of betweeness and indicated possible fields of application.

In 1943, shortly after the appearance of Menger's paper, A. Wald published a paper [14] in which he criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality Wald constructed a theory of betweeness having certain advantages over Menger's theory [15].

In 1951 Menger continued his study of statistical metric spaces in a paper [7] devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory. In this paper Menger adopted Wald's version

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of the triangle inequality.[1]

Statistical metric spaces are also considered by Menger in the last chapter of his book *Géométrie Générale* [9]; and references to these spaces are scattered throughout his other works, e.g., [6], [8].[2]

In the present paper we continue the study of statistical metric spaces. Our paper is divided into three parts, each devoted to one main topic. They are:

(I) The axiomatics of statistical metric spaces, with particular emphasis on the triangle inequality;

(II) The construction and study of particular spaces;

(III) A consideration of topological notions in statistical metric spaces and a study of the continuity properties of the distance function.[3]

In concluding this introduction, we wish to express our thanks to Professor K. Menger for his never-failing interest and encouragement, and to our colleagues, Professors T. Erber and M. McKiernan, for their many valuable comments and suggestions.

I. Definitions and Preliminaries

1. Statistical metric spaces. As is customary, we call a real-valued function defined on the entire real line a *distribution function* if it is non-decreasing, left-continuous and has inf 0 and sup 1. We shall use various symbols for distribution functions. However, in the sequel, H will always denote the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leqq 0\,, \\ 1, & x > 0\,. \end{cases}$$

We shall also, for convenience, adhere to the convention that, for any distribution function F, and any $x > 0$, $F(x/0) = 1$, while $F(0/0) = 0$.

For purposes of reference and comparison, we list here the postulates, due originally to Fréchet, for an ordinary metric space. A metric space (briefly, an *M-space*) is an ordered pair (S, d), where S is an abstract set and d a mapping of $S \times S$ into the real numbers—i.e., d associates

[1] However, as Prof. Menger informs us, even before the paper was written, both he and Wald, in a number of conversations, had come to feel that the Wald inequality was in some respects too stringent a requirement to impose on all statistical metric spaces. Some support for this is furnished in the present paper (Theorems 5.4 and 6.4).

[2] In addition, in a note on Menger's paper [**7**], A. Špaček [**13**] has considered the question of determining the probability that a random function defined on every pair of points in a set is an ordinary metric on that set. In particular, he has established necessary and sufficient conditions for such a random function to be a metric with probability one. The connection between the concepts of Špaček and that of a statistical metric is considered in [**10**].

[3] Some of the results of this paper have been presented in [**12**].

a real number $d(p, q)$ with every pair (p, q) of elements of S. The mapping d is assumed to satisfy the following conditions:

(i) $d(p, q) = 0$ if, and only if, $p = q$. (Identity)

(ii) $d(p, q) \geqq 0$. (Positivity)

(iii) $d(p, q) = d(q, p)$. (Symmetry)

(iv) $d(p, r) \leqq d(p, q) + d(q, r)$. (Triangle Inequality)

DEFINITION 1.1. *A statistical metric space* (briefly, an *SM-space*) is an ordered pair $(S, \mathscr{F})$ where S is an abstract set (whose elements will be called *points*) and $\mathscr{F}$ is a mapping of $S \times S$ into the set of distribution functions—i.e., $\mathscr{F}$ associates a distribution function $\mathscr{F}(p, q)$ with every pair (p, q) of points in S. We shall denote the distribution function $\mathscr{F}(p, q)$ by F_{pq}, whence the symbol $F_{pq}(x)$ will denote the value of F_{pq} for the real argument x. The functions F_{pq} are assumed to satisfy the following conditions:

I. $F_{pq}(x) = 1$ for all $x > 0$ if, and only if, $p = q$.

II. $F_{pq}(0) = 0$.

III. $F_{pq} = F_{qp}$.

IV. If $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$, then $F_{pr}(x + y) = 1$.

In view of Condition II, which evidently implies that $F_{pq}(x) = 0$ for all $x \leqq 0$, Condition I is equivalent to the statement: $p = q$ if, and only if, $F_{pq} = H$.

Every M-space may be regarded as an SM-space of a special kind. One has only to set $F_{pq}(x) = H(x - d(p, q))$ for every pair of points (p, q) in the M-space. Furthermore, with the interpretation of $F_{pq}(x)$ as the probability that the distance from p to q is less than x, one sees that Conditions I, II, and III are straightforward generalizations of the corresponding conditions i, ii, iii. Condition IV is a 'minimal' generalization of the triangle inequality iv which may be interpreted as follows: If it is certain that the distance of p and q is less than x, and likewise certain that the distance of q and r is less than y, then it is certain that the distance of p and r is less than $x + y$.

Condition IV is always satisfied in M-spaces, where it reduces to the ordinary triangle inequality. However, in those SM-spaces in which the equality $F_{pq}(x) = 1$ does not hold (for $p \neq q$) for any finite x, IV will be satisfied only vacuously. It is therefore of interest to have 'stronger' versions of the generalized triangle inequality. We shall consider two such versions in detail. Before doing so, it is convenient to make the following:

DEFINITION 1.2. A triangle inequality will be said to hold *universally* in an *SM*-space if and only if it holds for all triples of points, distinct or not, in that space.

2. Menger spaces. In his original formulation [5], Menger gave as a generalized triangle inequality the following:

IVm. $$F_{pr}(x+y) \geqq T(F_{pq}(x), F_{qr}(y)) \text{ for all } x, y \geqq 0 ,$$

where T is a 2-place function on the unit square satisfying:

(a) $0 \leqq T(a, b) \leqq 1$,
(b) $T(c, d) \geqq T(a, b)$ for $c \geqq a, d \geqq b$,
(c) $T(a, b) = T(b, a)$,
(d) $T(1, 1) = 1$,
(e) $T(a, 1) > 0$ for $a > 0$.

In view of condition (d), it follows that IVm contains IV as a special case. Because of the rather general nature of the function T, about the most that can be given as an interpretation of IVm is a statement such as: Our knowledge of the third side of a triangle depends in a symmetric manner on our knowledge of the other two sides and increases, or at least does not decrease, as our knowledge of these other two sides increases. The interpretation can, however, be made precise by choosing T to be a specific function. There are numerous possible choices for T. We list here six of the simplest:

T_1: $T(a, b) = \text{Max}(a+b-1, 0)$, i.e., $T = \text{Max}(\text{Sum}-1, 0)$;
T_2: $T(a, b) = ab$, ,, $T = \text{Product}$;
T_3: $T(a, b) = \text{Min}(a, b)$, ,, $T = \text{Min}$;
T_4: $T(a, b) = \text{Max}(a, b)$, ,, $T = \text{Max}$;
T_5: $T(a, b) = a + b - ab$, ,, $T = \text{Sum-Product}$;
T_6: $T(a, b) = \text{Min}(a+b, 1)$, ,, $T = \text{Min}(\text{Sum}, 1)$.

The six functions are listed in order of increasing 'strength', where T'' is said to be *stronger* than T' (and T' *weaker* than T'') if $T''(a, b) \geqq T'(a, b)$ for all (a, b) on the unit square with strict inequality for at least one pair (a, b). Evidently, if IVm holds for any given T, it will hold a fortiori for all weaker T's. For $T = \text{Product}$, IVm may be interpreted as follows: The probability that the distance of p and r is less than $x + y$ is not less than the joint probability that, independently, the distance of p and q is less than x, and the distance of q and r is less than y. For $T = \text{Min}$ (Max), the interpretation is: The probability that the distance of p and r is less than $x + y$ is not less than the smaller (larger) of the probabilities that the distance of p and q is less than x and the distance of q and r is less than y. Similar interpretations may be given to the other choices of T. However as the following lemmas indicate, the three functions T_4, T_5, T_6 are actually too strong for most purposes.

LEMMA 2.1. *If an SM-space contains two distinct points, then* IVm *cannot hold universally in the space under the choice* $T = \text{Max}$.

Proof. Let p and q be two distinct points of the space and let x and y satisfy $0 < y < x$. Suppose IVm holds universally with $T = \text{Max}$. Then,

$$F_{pq}(x) \geqq \text{Max}\,(F_{pq}(x - y), F_{qq}(y)) = 1 .$$

But x can be *any* positive number, which by Condition I means $p = q$ and contradicts the assumption $p \neq q$.

LEMMA 2.2. *If an SM-space is not an M-space, and if* IVm *holds universally in the space for some choice of T satisfying* (a)—(e), *then the function T has the property that there exists a number a, $0 < a < 1$, such that* $T(a, 1) \leqq a$.

Proof. If an *SM*-space is not an *M*-space, then there is at least one pair p, q of (necessarily distinct) points for which F_{pq} assumes values other than 0 or 1. By the left-continuity and monotonicity of F_{pq}, this means that there is, not merely one point, but an open interval (x, y) on which we have $0 < F_{pq} < 1$. Now assume that $T(a, 1) = a + \phi(a)$, where $\phi(a) > 0$ for $0 < a < 1$. Let z be any point in (x, y) and take $t > 0$. Then

$$\begin{aligned} F_{pq}(z + t) &\geqq T(F_{pq}(z), F_{qq}(t)) \\ &= T(F_{pq}(z), 1) \\ &= F_{pq}(z) + \phi(F_{pq}(z)) . \end{aligned}$$

Letting $t \to 0+$, we have:

$$F_{pq}(z+) \geqq F_{pq}(z) + \phi(F_{pq}(z)) > F_{pq}(z) .$$

Thus F_{pq} is discontinuous at z, and therefore at every point of (x, y). But this is a contradiction, since a non-decreasing function can be discontinuous at only denumerably many points.

LEMMA 2.3. *If* IVm *holds universally in an SM-space and if T is continuous, then, for any $x > 0$,* $T(F_{pq}(x), 1) \leqq F_{pq}(x)$.

Proof. Let p, q and $x > 0$ be given and choose y such that $0 < y < x$. Then,

$$F_{pq}(x) \geqq T(F_{pq}(x-y), F_{qq}(y)) = T(F_{pq}(x-y), 1) .$$

Letting $y \to 0+$, we obtain

$$F_{pq}(x) \geqq \lim_{y\to 0+} T(F_{pq}(x-y), 1) .$$

But, by the assumed continuity of T,

$$\lim_{y\to 0+} T(F_{pq}(x-y), 1) = T(\lim_{y\to 0+} F_{pq}(x-y), 1) ,$$

while, by the left-continuity of F_{pq},

$$\lim_{y\to 0+} F_{pq}(x-y) = F_{pq}(x) .$$

This completes the proof.

Motivated by these lemmas, and noticing that the three weaker functions in our list of T's satisfy $T(a, 1) = a$, we are led to replace conditions (a), (d) and (e) by the condition,

(a′) $T(a, 1) = a$, $T(0, 0) = 0$.

This new condition implies that $T \leqq \mathrm{Min}$, for we have the inequalities

$$T(a, b) \leqq T(a, 1) = a ,$$
$$T(a, b) = T(b, a) \leqq T(b, 1) = b ,$$

whence $T(a, b) \leqq \mathrm{Min}\,(a, b)$. Thus, under (a′), Min becomes the strongest possible universal T. Similarly, the weakest possible T satisfying (a′), (b) and (c) is the function, henceforth denoted by T_w, which is given by,

$$T_w(x, y) = \begin{cases} a\ , & x = a, y = 1 \text{ or } y = a, x = 1\ , \\ 0\ , & \text{otherwise.} \end{cases}$$

It must not be construed, however, that functions stronger than Min or weaker than T_w thereby lose all interest; in fact, on numerous occasions, we shall find it of value to determine under what conditions —i.e., for which points p, q, r and for which numbers x, y—IVm holds for a function stronger than Min or weaker than T_w.

To the conditions on T considered thus far we also add the associativity condition,

(d′) $T[T(a, b), c] = T[a, T(b, c)]$,

which permits the extension of IVm to a polygonal inequality. Accordingly, we make the following:

DEFINITION 2.1. A *Menger space* is an *SM*-space in which IVm holds universally for some choice of T satisfying conditions (a′), (b), (c) and (d′).

The following lemma shows that, in determining whether or not an SM-space is a Menger space, only triples of distinct points need be considered.

LEMMA 2.4. If the points p, q, r are not all distinct, then IVm holds for the triple p, q, r under any choice of T satisfying (a′), (b), (c) and (d′).

Proof. We need only consider the choice $T = \mathrm{Min}$. If $p = r$, then $F_{pr} = H$ and the conclusion is immediate. If $p = q \neq r$, then for $x, y \geqq 0$,

$$\mathrm{Min}\,(F_{pq}(x), F_{qr}(y)) = \mathrm{Min}\,(H(x), F_{qr}(y)) \leqq F_{qr}(y) \leqq F_{qr}(x+y) = F_{pr}(x+y)\ .$$

3. **Wald spaces.** The other generalized triangle inequality that we consider is the one due to A. Wald [14, 15]. It is:

IVw. $$F_{pr}(x) \geqq [F_{pq} * F_{qr}](x)\ , \quad \text{for all } x \geqq 0\ ,$$

where $*$ denotes convolution, i.e.,

$$[F_{pq} * F_{qr}](x) = \int_{-\infty}^{\infty} F_{pq}(x-y)dF_{qr}(y)\ .$$

Since $F_{pq}(x-y) = 0$ for $y \geqq x$, and $F_{qr}(y) = 0$ for $y \leqq 0$, we may evidently write

$$[F_{pq} * F_{qr}](x) = \int_{0}^{x} F_{pq}(x-y)dF_{qr}(y)\ .$$

Since the convolution of the distribution functions of two independent random variables gives the distribution function of their sum, the interpretation of IVw is: The probability that the distance of p and r is less than x is not less than the probability that the sum of the distance of p and q and the distance of q and r (regarded as independent) is less than x.

DEFINITION 3.1. A *Wald space* is an SM-space in which IVw holds universally.

THEOREM 3.1. *A Wald space is a Menger space under the choice $T = Product$.*

Proof. In a Wald space, for any $x, y \geqq 0$, we have

$$\begin{aligned} F_{pr}(x+y) &\geqq \int_0^{x+y} F_{pq}(x+y-z)dF_{qr}(z) \\ &= \int_0^{x+y}\left[\int_0^{x+y-z} dF_{pq}(t)\right]dF_{qr}(z) \\ &= \iint\limits_{\substack{t,z\geqq 0 \\ t+z\leqq x+y}} dF_{pq}(t)dF_{qr}(z)\ . \end{aligned}$$

Now,

$$\iint_{\substack{t,z\geqq 0\\ t+z\leqq x+y}} dF_{pq}(t)dF_{qr}(z) \geqq \iint_{\substack{0\leqq t\leqq x\\ 0\leqq z\leqq y}} dF_{pq}(t)dF_{qr}(z)$$

since the rectangle $\{(t, z)\,;0 \leqq t \leqq x, 0 \leqq z \leqq y\}$ is contained in the triangle $\{(t, z)\,; t, z \geqq 0, t + z \leqq x + y\}$ and the F's are non-decreasing. But,

$$\iint_{\substack{0\leqq t\leqq x\\ 0\leqq z\leqq y}} dF_{pq}(t)dF_{qr}(z) = \int_0^x\int_0^y dF_{pq}(t)dF_{qr}(z)$$

$$= \int_0^x dF_{pq}(t)\int_0^y dF_{qr}(z) = F_{pq}(x)F_{qr}(y)\ .$$

Combining the various inequalities we obtain

$$F_{pr}(x + y) \geqq F_{pq}(x)F_{qr}(y) \tag{1}$$

which is IVm with $T =$ Product, and completes the proof of the theorem.

COROLLARY. *If the Wald inequality holds, then so does the inequality* IV.

Proof. By (1), if $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$ then $F_{pr}(x + y) = 1$.

The following lemma is a counterpart to Lemma 2.4:

LEMMA 3.1. *If the points p, q, r are not all distinct, then* IVw *holds for the triple p, q, r.*

Proof. If $p = r$, this is immediate, since in this case, $F_{pr} = H$.

Otherwise, if $p = q \neq r$, then, for $x \geqq 0$,

$$F_{pr}(x) = F_{qr}(x) = \int_0^x dF_{qr}(y) = \int_0^x H(x - y)dF_{qr}(y)$$

$$= \int_0^x F_{pq}(x - y)dF_{qr}(y)\ .$$

The case $p \neq q = r$ follows on interchanging r and p.

THEOREM 3.2. *If in an SM-space,* IVm *holds under $T =$ Max for all triples of distinct points, then the space is a Wald space.*

Proof. Let p, q, r be distinct. Then for any $x \geqq 0$,

$$F_{pr}(x) \geqq \mathrm{Max}\,(F_{pq}(0), F_{qr}(x)) = F_{qr}(x) = \int_0^x dF_{qr}(y)$$

$$\geqq \int_0^x F_{pq}(x - y)dF_{qr}(y)\ ,$$

since $0 \leqq F_{pq}(x - y) \leqq 1$. Therefore IVw holds for all triples of distinct points in the space. But, by the preceding lemma, IVw holds automatically for triples of non-distinct points. Consequently, IVw holds for all triples of points in the space.

This theorem is, in a sense, a partial converse to Theorem 3.1. As will be seen later, $T = \text{Max}$ in the theorem cannot be weakened to $T = \text{Min}$, let alone $T = \text{Product}$. Thus the true converse of Theorem 3.1 is false.

II. Particular Spaces

4. Equilateral spaces. The simplest metric spaces are the equilateral spaces in which

$$d(p, q) = \begin{cases} a\,, & \text{if } p \neq q \\ 0\,, & \text{if } p = q\,, \end{cases}$$

where a is positive.

Accordingly, we call an *SM*-space *equilateral* if, for some distribution function G satisfying $G(0) = 0$,

$$F_{pq}(x) = \begin{cases} G(x)\,, & \text{if } p \neq q\,, \\ H(x)\,, & \text{if } p = q\,, \end{cases} \tag{1}$$

where H is the distribution function defined in § 1. From (1) it follows that the Conditions I–IV defining an *SM*-space are satisfied.

Theorem 4.1. *The means, medians, etc., of the statistical distances in an equilateral SM-space form an equilateral M-space.*

Proof. Any one of these quantities is zero when $p = q$ and a fixed positive number for any p, q when $p \neq q$.

Theorem 4.2. *In an equilateral SM-space, the Menger triangle inequality* IVm *holds for any triple of distinct points under* $T = \text{Max}$, *and universally under* $T = \text{Min}$.

Proof. Since G is non-decreasing,

$$G(x + y) \geqq \text{Max}\,(G(x), G(y)) \geqq \text{Min}\,(G(x), G(y))\,,$$

and

$$G(x + y) \geqq \text{Min}\,(G(x), 1)\,.$$

Corollary. *An equilateral SM-space is a Wald space.*

Proof. This is a direct consequence of Theorem 3.2.

There are also equilateral *SM*-spaces in which IVm holds under a stronger choice of T.

EXAMPLE 1.

$$G(x) = \begin{cases} 0\,, & x \leqq 0\,, \\ x\,, & 0 \leqq x \leqq 1\,, \\ 1\,, & 1 \leqq x\,. \end{cases}$$

For any triple of distinct points in this space, IVm holds under $T = \mathrm{Min}\,(\mathrm{Sum}, 1)$,since in all cases we have $G(x + y) \geqq \mathrm{Min}\,(G(x) + G(y), 1)$.

EXAMPLE 2.

$$G(x) = \begin{cases} 0\,, & x \leqq 0\,, \\ 1 - e^{-x}\,, & x \geqq 0\,. \end{cases}$$

For any triple of distinct points in this space, IVm holds under $T = \mathrm{Sum—Product}$. This follows from the fact that $e^{-x}e^{-y} = e^{-(x+y)}$.

However, even through, as Examples 1 and 2 show, there are equilateral SM-spaces in which the generalized triangle inequality IVm holds under a stronger T than Max, the result of Theorem 4.2 is the best possible. This is shown by:

EXAMPLE 3.

$$G(x) = \begin{cases} 0\,, & x \leqq 0\,, \\ a\,, & 0 < x \leqq k\,, \\ b\,, & k < x \leqq 3k\,, \\ 1\,, & 3k < x\,, \end{cases}$$

where $0 < a \leqq b < 1$ and k is any positive number. Then for $0 < x \leqq k$, $k < y \leqq 2k$, we have $G(x + y) = b = \mathrm{Max}\,(a, b)$, whence IVm cannot hold under any choice of T which is stronger than Max.

5. Simple spaces. A class of SM-spaces, more interesting than the equilateral, may be obtained as follows:

Let (S, d) be an M-space and G a distribution function, different from H, satisfying $G(0) = 0$. For every pair of points p, q in S, define the distribution function F_{pq} as follows:

$$F_{pq}(x) = \begin{cases} G[x/d(p, q)]\,, & p \neq q\,, \\ H(x)\,, & p = q\,. \end{cases} \tag{2}$$

DEFINITION 5.1. An SM-space $(S, \mathscr{F})$ is said to be a *simple space* if and only if there exists a metric d on S and a distribution function G satisfying $G(0) = 0$, such that, for every pair of points p, q in S, $\mathscr{F}(p, q) = F_{pq}$ is given by (2). Furthermore, we say that $(S, \mathscr{F})$ is the simple space *generated* by the M-space (S, d) and the distribution function G.

THEOREM 5.1. *A simple space is a Menger space under any choice of T satisfying* (a′), (b), (c) *and* (d′).

Proof. It is sufficient to show that IVm holds universally under $T = \text{Min}$, since this is the strongest choice of T possible. Thus, in view of Lemma 2.4, we have only to show that for p, q, r distinct,

$$G\left(\frac{x+y}{d(p, r)}\right) \geqq \text{Min}\,[G(x/d(p, q)), G(y/d(q, r))] \ . \tag{3}$$

Now, since d is an ordinary metric,

$$d(p, r) \leqq d(p, q) + d(q, r) \ .$$

Thus,

$$\frac{x+y}{d(p, r)} \geqq \frac{x+y}{d(p, q) + d(q, r)} \ . \tag{4}$$

Furthermore, since $d(p, q)$ and $d(q, r)$ are positive,

$$\begin{aligned} \text{Max}\,[x/d(p, q), y/d(q, r)] &\geqq \frac{x+y}{d(p, q) + d(q, r)} \\ &\geqq \text{Min}\,[x/d(p, q), y/d(q, r)] \end{aligned} \tag{5}$$

with equality on either side if and only if $x/d(p, q) = y/d(q, r)$. Consequently, on combining (4) and the right-hand inequality in (5) we have,

$$\frac{x+y}{d(p, r)} \geqq \text{Min}\,[x/d(p, q), y/d(q, r)] \ ,$$

which, since G is non-decreasing, implies (3) and completes the proof.

COROLLARY 1. *An equilateral M-space generates an equilateral SM-space.*

COROLLARY 2. *If $G(x) = H(x - 1)$, the generated SM-space reduces to the generating M-space.*

Proof.

$$F_{pq}(x) = H\left(\frac{x}{d(p, q)} - 1\right) = H(x - d(p, q)) \ .$$

In most simple spaces $T = \text{Max}$ will be too strong since the left-hand inequality in (5) shows that, for a triple of distinct points p, q, r

such that $d(p, r) = d(p, q) + d(q, r)$, IVm under $T = \text{Max}$ fails. Indeed, in simple spaces having sufficient structure the choice $T = \text{Min}$ implicit in Theorem 5.1 is the best possible.

THEOREM 5.2. *If (S, d) is a finite-dimensional Euclidean space, G a continuous distribution function such that $G(0) = 0$ and $0 < G(x) < 1$ for all $x > 0$, then* Min *is the strongest T under which* IVm *holds for all triples of distinct points.*

Proof. Suppose T is stronger than Min. Then there exists at least one pair of numbers, $a, b\,(0 < a, b < 1)$, such that $T(a, b) > \text{Min}\,(a, b)$. We distinguish two cases:

(i) If $a = b$, choose $x = y$ such that $G(x) = a$, and choose $d(p, q) = d(q, r) = 1$, $d(p, r) = 2$. Then, since equality is attained in (3), we cannot have $T(a, a) > \text{Min}\,(a, a) = a$.

(ii) If $a \neq b$, we may suppose that $a < b$. Let $\varepsilon = T(a, b) - \text{Min}\,(a, b)$ and choose u, v so that $a = G(u)$ and $b = G(v)$. Such numbers u, v clearly exist since G is a continuous distribution function; moreover $u < v$. Also, since G is continuous, there exists an $h > 0$ such that

$$G(u + h) < G(u) + \varepsilon = a + \varepsilon\,.$$

Now let $d(q, r) = t$ be fixed and choose

$$\begin{aligned} &d(p, q) = s \text{ such that } \frac{t}{s + t} < \frac{h}{v - u}\,, \\ &d(p, r) = d(p, q) + d(q, r) = s + t\,, \\ &x = ud(p, q) \text{ and } y = vd(q, r)\,. \end{aligned}$$

Then,

$$\text{Min}\,[G(x/d(p, q)), G(y/d(q, r))] = \text{Min}\,[G(u), G(v)] = \text{Min}\,(a, b) = a\,.$$

Furthermore,

$$G\Big(\frac{x + y}{d(p, r)}\Big) = G\Big(\frac{us + vt}{s + t}\Big) = G\Big(u + \frac{(v - u)t}{s + t}\Big) \leqq G(u + h) < a + \varepsilon\,,$$

which contradicts the hypothesis

$$G\Big(\frac{x + y}{d(p, r)}\Big) \geqq T(a, b) = \text{Min}\,(a, b) + \varepsilon = a + \varepsilon\,.$$

THEOREM 5.3. *In a simple space, the means (if they exist), medians, modes (if unique) each form an M-space homothetic*[4] *to the original M-space.*

[4] Two M-spaces, M_1 and M_2, with distance functions d_1 and d_2, respectively are said to be homothetic if there exists a number $a > 0$ and a one-to-one mapping, f, from M_1 to M_2 such that, for every $p, q \in M_1$, $d_1(p, q) = ad_2(f(p), f(q))$.

Proof. Let $E[G] = \int_0^\infty x dG(x) = \mu$.

If $p = q$, then

$$E[F_{pq}] = \int_0^\infty x dH(x) = 0 .$$

If $p \neq q$, then

$$E[F_{pq}] = \int_0^\infty x dG(x/d(p, q))$$

which on substituting $t = x/d(p, q)$ becomes

$$E[F_{pq}] = d(p, q)\int_0^\infty t dG(t) = \mu d(p, q) .$$

The other cases are similar.

In Theorem 3.1 it was shown that every Wald space is a Menger space under $T =$ Product and in the corollary to Theorem 4.2 that every equilateral space is a Wald space. However, as the following examples show, there exist simple spaces which are not Wald spaces. Thus the converse of Theorem 3.1 is false. For, if in a Menger space IVm holds under $T =$ Min, it holds a fortiori under $T =$ Product.

THEOREM 5.4. *There exist simple spaces which are not Wald spaces.*

Proof. We give two counter-examples:

EXAMPLE 4.

$$F_{pq}(x) = 1 - e^{-x/d(p,q)} .$$

With $d(p, q) = R, d(q, r) = S$ and $d(p, r) = T$, one obtains (taking into account the fact that the lower limit of the convolution integral is 0 and the upper limit is x),

$$(6) \qquad [F_{pq} * F_{qr}](x) = \begin{cases} 1 - \dfrac{1}{R - S}(Re^{-x/R} - Se^{-x/S}) , & R \neq S , \\ 1 - \left(1 + \dfrac{x}{R}\right)e^{-x/R} , & R = S . \end{cases}$$

In order that the Wald inequality IVw be satisfied, we must have

$$(7) \qquad F_{pr}(x) \geqq [F_{pq} * F_{qr}](x) , \text{ for every } x \geqq 0 ,$$

Suppose $R \neq S$, say $R > S$. Then, keeping x fixed and applying the mean value theorem to the second term on the upper right-hand side of (6), we have

$$[F_{pq} * F_{qr}](x) = 1 - \left(1 + \frac{x}{t}\right)e^{-x/t}\ , \text{ where } S < t < R\ . \tag{8}$$

Furthermore, if $R = S$, we observe, on comparing (8) with (6), that (8) holds with $t = R$. Thus, in both cases, in order that (7) hold, it is necessary that

$$\left(1 + \frac{x}{t}\right)e^{-x/t} \geqq e^{-x/T}\ , \text{ for all } x \geqq 0\ ;$$

that is,

$$1 + \frac{x}{t} \geqq e^{x(1/t-1/T)}\ , \text{ for all } x \geqq 0\ .$$

This will be true if and only if $(1/t - 1/T) \leqq 0$, i.e., $T \leqq t$. In particular, therefore, it is necessary that $T \leqq R$. But this means that the side of the triangle pqr whose length is T certainly cannot be the longest side of that triangle. Thus we conclude: If $d(p, r) \geqq \text{Max}\,(d(p, q), d(q, r))$, then the Wald inequality will fail for sufficiently large x.

EXAMPLE 5.

$$F_{pq}(x) = \begin{cases} 0\ , & x \leqq 0\ , \\ x/d(p, q)\ , & 0 \leqq x \leqq d(p, q)\ , \\ 1\ , & x \geqq d(p, q)\ . \end{cases}$$

In this simple space, one can construct an example in which IVw holds for x sufficiently small and again for x sufficiently large, but fails in an intermediate range. For instance, if, $d(p, q) = 1$, $d(q, r) = 3$, and $d(p, r) = 3.75$, then for $2.50 < x < 3.68$, we have $[F_{pq} * F_{qr}](x) > F_{pr}(x)$.

6. Normal spaces. Statistical metric spaces also arise very naturally in the following manner: Let $p, q, \cdots$, be random variables on a common M-space E with distance function d. Then $d(p, q)$ is a random variable on the Cartesian product $E \times E$. Let F_{pq} be the distribution function of $d(p, q)$. Then the ordered pair $(S, \mathscr{F})$, where S is the set $\{p, q, \cdots\}$ and $\mathscr{F}$ the class of ordered pairs $\{(p, q), F_{pq}\}$ is an SM-space.

Particularly interesting SM-spaces of this type result when S is taken to be a set of mutually independent spherically-symmetric Gaussian random variables on an n-dimensional Euclidean space. We have investigated these at some length. However, reproducing the details here would take us too far afield. We shall therefore restrict ourselves to a brief presentation, without proof, of the main results.[5]

Let S_n^* be the set of all mutually independent, spherically-symmetric, n-dimensional Gaussian random variables, $p, q, \cdots$, etc., on a Euclidean n-space E^n. Let the n-dimensional mean of p (which is a point in E^n) be $\boldsymbol{m}_p$ and the common standard deviation of the one-dimensional marginal distributions of p be σ_p; and let the corresponding objects for q be $\boldsymbol{m}_q$ and σ_q, respectively. Then, upon setting

$$r = r(p, q) = d(\boldsymbol{m}_p, \boldsymbol{m}_q) ,$$

where d is the metric in E^n, and

$$\sigma = \sigma(p, q) = (\sigma_p^2 + \sigma_p^2)^{1/2} ,$$

the distribution function, F_{pq}, of $d(p, q)$ is given by:

$$(9) \qquad F_{pq}(x) = R_n(\sigma, r; x) = \begin{cases} 0 , & x \leqq 0 , \\ \left(\dfrac{r}{\sigma}\right)^{1-n/2} e^{-r^2/2\sigma^2} \displaystyle\int_0^{x/\sigma} y^{n/2} I_{(n/2)-1}\left(\frac{r}{\sigma} y\right) e^{-y^2/2} dy , & x \geqq 0 , \end{cases}$$

if $\sigma > 0$, where I_ν is the modified Bessel function of order ν; and by

$$F_{pq}(x) = R_n(0, r; x) = H(x - r) ,$$

if $\sigma = 0$. The distribution function R_n has been found in different contexts, first by Bose [1][6], and more recently, by K. S. Miller, R. I. Bernstein and L. E. Blumenson [**11**].

An SM-space $(S_n, \mathscr{F})$ will be called an (n-dimensional) *normal space* if: (a) S_n is a subset of S_n^* having the property that, for every point $\boldsymbol{m}$ in E^n, there is at least one p in S_n whose expectation is $\boldsymbol{m}$; (b) For p, q in S_n, $\mathscr{F}(p, q) = F_{pq}$ is given by (9). A normal space is called *homogeneous* if $\sigma_p = \sigma_q$ for all p, q in S_n; otherwise, *inhomogeneous*.

Normal spaces have the following properties:

1. For all n, the means[7] of the F_{pq}'s form a metric space. This metric space is 'asymptotic in the large' to E^n in the following sense: If the distance between the means of p and q is r, then (for σ fixed) the mean of F_{pq} as a function of r is asymptotic to r for large r. In the 'small', the metric is definitely non-Euclidean. Furthermore, for $\sigma > 0$, there is a *positive minimum distance* between distinct points. That is to say, if $p \neq q$, and $r = d(\boldsymbol{m}_p, \boldsymbol{m}_q) > 0$, then the mean of F_{pq}, which is the expected value of the random variable $d(p, q)$, is greater than $\sqrt{2}\,\sigma[\Gamma(n/2 + 1/2)/\Gamma(n/2)]$. When $r = 0$ equality is attained.

[5] The detailed discussion will be the subject of another paper.

[6] See also Mahalanobis [**3**].

[7] These always exist; indeed, the F_{pq}'s in a normal space have moments of all orders.

2. All normal spaces are Menger spaces under the choice $T = T_w$.

3. A homogeneous normal space is a Menger space under the choice $T = \text{Max}(\text{Sum} - 1, 0)$. For some triples of points, p_1, p_2, p_3, and for some numbers, x, y—subject to certain restrictions—IVm will hold under a stronger choice of T. However, other than T_w and Max (Sum-1, 0), none of the T's listed in § 2 will hold universally. In particular, there are triples of points for which IVm does not hold for all x, y under $T =$ Product. Consequently,

4. No normal space is a Wald space.

Whether IVm will hold in a normal space under a T that is weaker than Product, yet stronger than Max (Sum-1, 0) is not known.

III. Topology, Convergence, Continuity

7. In the theory of metric spaces, the concept of a neighborhood can be introduced and defined with the aid of the distance function. A similar procedure applies in the theory of statistical metric spaces. In fact, neighborhoods in *SM*-spaces may be defined in several non-equivalent ways. Here we shall consider only one of these—the one which seems to be the strongest, in that its consequences most nearly resemble the classical results on *M*-spaces.

Definition 7.1. Let p be a point in the *SM*-space $(S, \mathscr{F})$. By an ε, λ-neighborhood of p, $\varepsilon > 0, \lambda > 0$, we mean the set of all points q in S for which $F_{pq}(\varepsilon) > 1 - \lambda$. We write:

$$N_p(\varepsilon, \lambda) = \{q ; F_{pq}(\varepsilon) > 1 - \lambda\} .$$

The interpretation is: $N_p(\varepsilon, \lambda)$ is the set of all points q in S for which the probability of the distance from p to q being less than ε is greater than $1 - \lambda$. Observe that this neighborhood of a point in an *SM*-space depends on two parameters.

Theorem 7.1. *In a simple space, $N_p(\varepsilon, \lambda)$ is an ordinary spherical neighborhood of p in the generating M-space.*

Proof. For any p, q, we have

$$F_{pq}(\varepsilon) = G(\varepsilon/d(p, q)) ,$$

which will be greater than $1 - \lambda$ provided only that $d(p, q)$ is sufficiently small.

Lemma 7.1. *If $\varepsilon_1 \leqq \varepsilon_2$ and $\lambda_1 \leqq \lambda_2$, then $N_p(\varepsilon_1, \lambda_1) \subset N_p(\varepsilon_2, \lambda_2)$.*

Proof. Suppose $q \in N_p(\varepsilon_1, \lambda_1)$ so that $F_{pq}(\varepsilon_1) > 1 - \lambda_1$. Then, $F_{pq}(\varepsilon_2) \geqq F_{pq}(\varepsilon_1) > 1 - \lambda_1 \geqq 1 - \lambda_2$, whence, by definition, $q \in N_p(\varepsilon_2, \lambda_2)$.

THEOREM 7.2. *If $(S, \mathscr{F})$ is a Menger space and T is continuous then $(S, \mathscr{F})$ is a Hausdorff space in the topology induced by the family of ε, λ-neighborhoods $\{N_p\}$.*

Proof. We have to show that the following four properties are satisfied:

(A) For every p in S, there exists at least one neighborhood, N_p, of p; every neighborhood of p contains p.

(B) If N_p^1 and N_p^2 are neighborhoods of p, then there exists a neighborhood of p, N_p^3, such that $N_p^3 \subset N_p^1 \cap N_p^2$.

(C) If N_p is a neighborhood of p, and $q \in N_p$, then there exists a neighborhood of q, N_q, such that $N_q \subset N_p$.

(D) If $p \neq q$, then there exist disjoint neighborhoods, N_p and N_q, such that $p \in N_p$ and $q \in N_q$.

Proof of (A). For every $\varepsilon > 0$ and every $\lambda > 0$, $p \in N_p(\varepsilon, \lambda)$ since $F_{pp}(\varepsilon) = 1$ for any $\varepsilon > 0$.

Proof of (B). Let

$$N_p^1(\varepsilon_1, \lambda_1) = \{q\,; F_{pq}(\varepsilon_1) > 1 - \lambda_1\}$$

and

$$N_p^2(\varepsilon_2, \lambda_2) = \{q\,; F_{pq}(\varepsilon_2) > 1 - \lambda_2\}$$

be the given neighborhoods of p, and consider

$$N_p^3 = \{q\,; F_{pq}(\operatorname{Min}(\varepsilon_1, \varepsilon_2)) > 1 - \operatorname{Min}(\lambda_1, \lambda_2)\} .$$

Clearly $p \in N_p^3$; and since $\operatorname{Min}(\varepsilon_1, \varepsilon_2) \leqq \varepsilon_1$ and $\operatorname{Min}(\lambda_1, \lambda_2) \leqq \lambda_1$, by Lemma 7.1, $N_p^3 \subset N_p^1$. Similarly, $N_p^3 \subset N_p^2$, whence $N_p^3 \subset N_p^1 \cap N_p^2$.

Proof of (C). Let $N_p = \{r\,; F_{pr}(\varepsilon_1) > 1 - \lambda_1\}$ be the given neighborhood of p. Since $q \in N_p$,

$$F_{pq}(\varepsilon_1) > 1 - \lambda_1 .$$

Now, F_{pq} is left-continuous at ε_1. Hence, there exists an $\varepsilon_0 < \varepsilon_1$ and a $\lambda_0 < \lambda_1$, such that

$$F_{pq}(\varepsilon_0) > 1 - \lambda_0 > 1 - \lambda_1 .$$

Let $N_q = \{r\,; F_{qr}(\varepsilon_2) > 1 - \lambda_2\}$, where $0 < \varepsilon_2 < \varepsilon_1 - \varepsilon_0$, and λ_2 is chosen such that

$$T(1-\lambda_0, 1-\lambda_2) > 1-\lambda_1 .$$

Such a λ_2 exists since, by hypothesis, T is continuous, $T(a, 1) = a$, and $1-\lambda_0 > 1-\lambda_1$. Now suppose $s \in N_q$, so that

$$F_{qs}(\varepsilon_2) > 1-\lambda_2 .$$

Then

$$\begin{aligned} F_{ps}(\varepsilon_1) \geqq T(F_{pq}(\varepsilon_0), F_{qs}(\varepsilon_1-\varepsilon_0)) &\geqq T(F_{pq}(\varepsilon_0), F_{qs}(\varepsilon_2)) \\ &\geqq T(1-\lambda_0, 1-\lambda_2) > 1-\lambda_1 . \end{aligned}$$

But this means $s \in N_p$, whence $N_q \subset N_p$.

Proof of (*D*). Let $p \neq q$. Then there exists an $x > 0$ and an a, $0 \leqq a < 1$, such that, $F_{pq}(x) = a$. Let

$$N_p = \{r; F_{pr}(x/2) > b\} \text{ and } N_q = \{r; F_{qr}(x/2) > b\} ,$$

where b is chosen so that $0 < b < 1$ and $T(b, b) > a$. Such a number b exists, since T is continuous and $T(1, 1) = 1$. Now suppose there is a point s in $N_p \cap N_q$, so that $F_{ps}(x/2) > b$ and $F_{qs}(x/2) > b$. Then

$$a = F_{pq}(x) \geqq T(F_{ps}(x/2), F_{qs}(x/2)) \geqq T(b, b) > a ,$$

which is a contradiction. Thus N_p and N_q are disjoint.

It should be noted that the function T appeared only in the proofs of (C) and (D). Also, the ε, λ-neighborhoods defined at various stages in the proof may consist of only a single point. The situation here is analogous to the one that arises in connection with isolated points in M-spaces.

8. As is well known, in an M-space the notion of convergence of a sequence of points $\{p_n\}$ to a point p may be introduced with the aid of the neighborhood concept. There is also a well known theorem which states that the distance function d is continuous on S: that is to say, if $p_n \to p$ and $q_n \to q$, then $d(p_n, q_n) \to d(p, q)$. The proof of this theorem depends strongly on the triangle inequality.[8] Having defined neighborhoods in SM-spaces, it is natural to consider the above questions in this more general setting. In so doing, a significant difference arises, for there are now two distinct types of questions regarding convergence and continuity that must be considered: (a) Those relating to the distance function, $\mathscr{F}$, considered as a function on $S \times S$—either for a fixed value of x, or for a range of values; (*b*) Those relating to the individual distribution functions F_{pq}—either for a fixed pair of points (p, q) or for

[8] In fact, as Menger has shown [**4**, pp. 142-145], the triangle inequality is, in a certain sense, the simplest condition implying continuity that can be imposed on a semi-metric.

a set of pairs of points. As is to be expected, these questions are not independent.

DEFINITION 8.1. A sequence of points $\{p_n\}$ in an *SM*-space is said to converge to a point p in S (and we write $p_n \to p$) if and only if, for every $\varepsilon > 0$ and every $\lambda > 0$, there exists an integer $M_{\varepsilon,\lambda}$, such that $p_n \in N_p(\varepsilon, \lambda)$, i.e., $F_{pp_n}(\varepsilon) > 1 - \lambda$, whenever $n > M_{\varepsilon,\lambda}$.

LEMMA 8.1. *If $p_n \to p$, then $F_{pp_n} \to F_{pp} = H$, i.e., for every x, $F_{pp_n}(x) \to F_{pp}(x) = H(x)$, and conversely.*

Proof. (a) If $x > 0$, then for every $\lambda > 0$, there exists an integer $M_{x,\lambda}$ such that $F_{pp_n}(x) > 1 - \lambda$ whenever $n > M_{x,\lambda}$. But this means that $\lim_{n\to\infty} F_{pp_n}(x) = 1 = F_{pp}(x)$.

(b) If $x = 0$, then for every n, $F_{pp_n}(0) = 0$ and hence $\lim_{n\to\infty} F_{pp_n}(0) = 0 = F_{pp}(0)$.

The converse is immediate.

COROLLARY. *The convergence is uniform on any closed interval $[a, b]$ such that $a > 0$, i.e., $M_{x,\lambda}$ is independent of x for $a \leqq x \leqq b$.*

Proof. For any x, $a \leqq x \leqq b$, $F_{pp_n}(x) \geqq F_{pp_n}(a)$.

THEOREM 8.1. *If $(S, \mathscr{F})$ is a Menger space and T is continuous, then the statistical distance function, $\mathscr{F}$, is a lower semi-continuous function of points, i.e., for every fixed x, if $q_n \to q$ and $p_n \to p$, then,*

$$\liminf_{n\to\infty} F_{p_nq_n}(x) = F_{pq}(x) .$$

Proof. If $x = 0$, this is immediate, since for every n, $F_{p_nq_n}(0) = 0 = F_{pq}(0)$. Suppose then that $x > 0$, and let $\varepsilon > 0$ be given. Since F_{pq} is left-continuous at x, there is an h, $0 < 2h < x$, such that

$$F_{pq}(x) - F_{pq}(x - 2h) < \varepsilon/3 .$$

Set $F_{pq}(x - 2h) = a$. Since T is continuous, and $T(a, 1) = a$, there is a number t, $0 < t < 1$, such that

$$T(a, t) > a - \varepsilon/3$$

and

$$T(a - \varepsilon/3, t) > a - 2\varepsilon/3 .$$

Since $q_n \to q$ and $p_n \to p$, by Lemma 8.1 there exists an integer $M_{h,t}$ such that $F_{qq_n}(h) > t$ and $F_{pp_n}(h) > t$ whenever $n > M_{h,t}$. Now,

$$F_{p_n q_n}(x) \geqq T(F_{p_n q}(x-h), F_{qq_n}(h))$$

and

$$F_{p_n q}(x-h) \geqq T(F_{pq}(x-2h), F_{pp_n}(h)) .$$

Thus, on combining the various inequalities, we obtain

$$F_{p_n q}(x-h) \geqq T(a, t) > a - \varepsilon/3 ,$$

whence

$$F_{p_n q_n}(x) \geqq T(a-\varepsilon/3, t) > a - 2\varepsilon/3 > F_{pq}(x) - \varepsilon .$$

COROLLARY 1. *Let p be a fixed point and suppose $q_n \to q$. Then*

$$\liminf_{n\to\infty} F_{pq_n}(x) = F_{pq}(x) .$$

COROLLARY 2. *If $(S, \mathscr{F})$ is a Wald space, then $\mathscr{F}$ is a lower semi-continuous function of points.*

Proof. By Theorem 3.1, in a Wald space, the Menger inequality holds under $T =$ Product, which is continuous.

THEOREM 8.2. *Let $(S, \mathscr{F})$ be a Menger space. Suppose that T is continuous*[9] *and at least as strong as* Max (Sum-1, 0). *Suppose further that $p_n \to p$, $q_n \to q$, and that F_{pq} is continuous at x. Then $F_{p_n q_n}(x) \to F_{pq}(x)$, i.e., the distance function $\mathscr{F}$ is a continuous function of points at (p, q, x); or, expressed in another way, the sequence of functions $\{F_{p_n q_n}\}$ converges weakly to F_{pq}.*

Proof. In view of Theorem 8.1, it suffices to prove upper semi-continuity, i.e., that for $\varepsilon > 0$ and n sufficiently large,

$$F_{p_n q_n}(x) < F_{pq}(x) + \varepsilon . \tag{1}$$

Suppose then that $\varepsilon > 0$ is given. Since F_{pq} is continuous, and in particular therefore right-continuous at x, there exists an $h > 0$, such that

$$F_{pq}(x+2h) - F_{pq}(x) < \varepsilon/3 . \tag{2}$$

By Lemma 8.1, there is an integer M such that the conditions,

$$F_{pp_n}(h) > 1 - \varepsilon/3 , \tag{3}$$

$$F_{qq_n}(h) > 1 - \varepsilon/3 , \tag{4}$$

are simultaneously satisfied for all $n > M$. And from IVm, we have

[9] Here, as well as in Theorems 7.2 and 8.1, the continuity of T may be replaced by the weaker condition, $\lim_{x\to 1} T(a, x) = a$.

$$(5)\qquad F_{pq}(x+2h) \geqq T(F_{pq_n}(x+h),\ F_{qq_n}(h))$$

and

$$(6)\qquad F_{pq_n}(x+h) \geqq T(F_{p_nq_n}(x),\ F_{pp_n}(h))\ .$$

Now, by hypothesis, T is at least as strong as Max (Sum$-1, 0$), so that on combining (3) and (6) we obtain

$$(7)\qquad F_{pq_n}(x+h) \geqq F_{p_nq_n}(x) + F_{pp_n}(h) - 1 > F_{p_nq_n}(x) - \varepsilon/3\ ;$$

and, on combining (7) with (4) and (5), we obtain

$$F_{pq}(x+2h) \geqq F_{pq_n}(x+h) + F_{qq_n}(h) - 1 > F_{p_nq_n}(x) - 2\varepsilon/3\ .$$

Finally, combining (8) with (2) yields (1) and completes the proof.

COROLLARY 1. *Under the hypotheses of Theorem* 8.2, *if* $q_n \to q$, *then* $F_{pq_n}(x) \to F_{pq}(x)$.

COROLLARY 2. *If the functions* F_{pq} *are each continuous functions for all* p, q *in* S, *then* $\mathscr{F}$ *is a continuous function of points.*

COROLLARY 3. *If* $(S, \mathscr{F})$ *is a Wald space and if the functions* F_{pq} *are each continuous, then* $\mathscr{F}$ *is a continuous function of points.*

BIBLIOGRAPHY

1. R. Bose, *On the exact distribution and moment-coefficients of the D^2-statistics*, Science and Culture, **1** (1935), p. 205.
2. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti de Circolo Matematico di Palermo, **22** (1906), 1-74.
3. P. C. Mahalanobis, *On the generalized distance in statistics*, Proc. Nat. Inst. of Sci. India, **2** (1936), 49-55.
4. K. Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, **100** (1928), 75-163.
5. ———, *Statistical metrics*, Proc. Nat. Acad. of Sci., U. S. A., **28** (1942), 535-537.
6. ———, *Probabilistic theories of relations*, Ibid., **37** (1951), 178-180.
7. ———, *Probabilistic geometry*, Ibid., **37** (1951), 226-229.
8. ———, *Ensembles flous et fonctions aléatoires*, Comptes Rendus Acad. Sci. Paris, **232** (1951), 2001-2003.
9. ———, *Géométrie générale*, Mémorial des Sciences Mathématiques, no. 124, Paris 1954.
10. K. Menger, B. Schweizer and A. Sklar, *On probabilistic metrics and numerical metrics with probability 1*, Czechoslovak Math. Journal, **9** (84), (1959), 459-466.
11. K. S. Miller, R. I. Bernstein and L. E. Blumenson, *Generalized Rayleigh processes*, Quarterly of Applied Math., **16** (1958), 137-145.
12. B. Schweizer and A. Sklar, *Espaces métriques aléatoires*, Comptes Rendus Acad. Sci. Paris, **247** (1958), 2092-2094.
13. A. Špaček, *Note on K. Menger's probabilistic geometry*, Czechoslovak Math. Journal **6** (81), (1956), 72-74.

14. A. Wald, *On a statistical generalization of metric spaces,* Proc. Nat. Acad. Sci., U. S. A., **29** (1943), 196-197.

15. ———, *On a statistical generalization of metric spaces,* Reports of a Mathematical Colloquium, Second Series, 5-6, (1944), 76-79. Reprinted in: Abraham Wald: Selected Papers in Statistics and Probability, New York 1955, pp. 413-416.

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